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A Formula for Computation of the Real Stability Radius*

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A readily computable formula for the real stability radius is presented.

Key Words—Stability; robustness; perturbation techniques; matrix analysis; linear systems; singular values.

Abstract—This paper presents a readily computable formula for the real stability radius with respect to an arbitrary stability region in the complex plane.

1. INTRODUCTION

In many engineering applications it is required that a square matrix have all of its eigenvalues in a prescribed area in the complex plane. We shall use the word stability to describe such an eigenvalue clustering property. Furthermore, it is often desired that the matrix should maintain this stability property when its elements are subject to certain perturbations. The real stability radius measures the ability of a matrix to preserve its stability under a certain class of real perturbations.

Let us partition the complex plane $\mathbb{C}$ into two disjoint subsets $\mathbb{C}_g$ and $\mathbb{C}_b$, i.e. $\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b$, such that $\mathbb{C}_g$ is open. A matrix is said to be stable if its eigenvalues are contained in $\mathbb{C}_g$. Denote the singular values of $M \in \mathbb{C}^{p\times m}$, ordered nonincreasingly, by $\sigma_i(M)$, $i = 1, 2, \ldots, \min\{p, m\}$. Also denote $\sigma_1(M)$ by $\bar{\sigma}(M)$ and $\sigma_{\min\{p,m\}}(M)$ by $\underline{\sigma}(M)$. Let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Following Hinrichsen and Pritchard (1986b) we define the (structured) stability radius of a matrix triple $(A, B, C) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n}$ as

$$r_{\mathbb{F}}(A, B, C) := \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times p} \text{ and } A + B\Delta C \text{ is unstable}\}.$$

We abbreviate $r_{\mathbb{F}}(A, I, I)$ by $r_{\mathbb{F}}(A)$ and call it the (unstructured) stability radius of A. For real (A, B, C), $r_{\mathbb{R}}(A, B, C)$ is called the real stability radius, and for complex (A, B, C), $r_{\mathbb{C}}(A, B, C)$ is called the complex stability radius. The stability radius problem concerns the computation of $r_{\mathbb{F}}(A, B, C)$ when (A, B, C) is given.

Let $\partial\mathbb{C}_g$ denote the boundary of $\mathbb{C}_g$. By continuity, we can easily show that for stable A,

$$\begin{aligned} r_{\mathbb{F}}(A, B, C) &= \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times p} \text{ and } A + B\Delta C \\ &\qquad \text{has an eigenvalue on } \partial\mathbb{C}_g\} \\ &= \inf_{s\in\partial\mathbb{C}_g} \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times p} \text{ and} \\ &\qquad \det(sI - A - B\Delta C) = 0\} \\ &= \inf_{s\in\partial\mathbb{C}_g} \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times p} \text{ and} \\ &\qquad \det[I - \Delta C(sI - A)^{-1}B] = 0\} \end{aligned}$$

Hence the key issue in the computation of the stability radius is to solve the following linear algebra problem: given $M \in \mathbb{C}^{p\times m}$, compute

$$\mu_{\mathbb{F}}(M) := [\inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times p} \text{ and } \det(I - \Delta M) = 0\}]^{-1}.$$

Simple singular value arguments show that $\mu_{\mathbb{C}}(M) = \bar{\sigma}(M)$. Hence

$$r_{\mathbb{C}}(A, B, C) = \left\{\sup_{s\in\partial\mathbb{C}_g} \bar{\sigma}[C(sI - A)^{-1}B]\right\}^{-1}, \quad (1)$$

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which was essentially obtained by Doyle and Stein (1981), Chen and Desoer (1982) and Hinrichsen and Pritchard (1986b). Equation (1) relates the complex stability radius to the concept of the $\mathcal{H}_\infty$ norm.

This paper concerns the computation of $r_{\mathbb{R}}(A, B, C)$. As we have seen,

$$r_{\mathbb{R}}(A, B, C) = \left\{ \sup_{s \in \partial\mathbb{C}_g} \mu_{\mathbb{R}}[C(sI - A)^{-1}B] \right\}^{-1}. \quad (2)$$

Our main result is a simple formula for $\mu_{\mathbb{R}}$ that allows computation of the real stability radius using (2). Let us denote the real and imaginary parts of a complex matrix M by $\operatorname{Re} M$ and $\operatorname{Im} M$ respectively, i.e. $\operatorname{Re} M$ and $\operatorname{Im} M$ are real matrices such that $M = \operatorname{Re} M + \mathrm{j} \operatorname{Im} M$.

Main result

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma \in (0,1]} \sigma_2\left(\begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix}\right). \quad (3)$$

The function to be minimized is a unimodal function on $(0, 1]$.

Since the function to be minimized in (3) is unimodal, any local minimum is a global minimum. Many standard search algorithms, such as golden section search, can be used with guaranteed convergence to a global minimum.

In a sense, the stability radius problem, although not having been called so, has been studied for decades. It is difficult to trace the exact history, partly because it has been treated by several authors in different fields independently. A theorem in Rudin (1973, p. 239) and its proof immediately leads to

$$r_{\mathbb{F}}(A) \geq \left\{ \sup_{s \in \partial\mathbb{C}_g} \bar{\sigma}[(sI - A)^{-1}] \right\}^{-1}.$$

Various versions of this inequality have appeared in many textbooks. The fact that this inequality is actually an equality when $\mathbb{F} = \mathbb{C}$ follows from the classical Schmidt/Mirsky theorem (often also attributed to Eckart and Young) of approximating a matrix by one of lower rank (see e.g. Stewart and Sun, 1990, p. 208, Theorem 4.18). For contributions to various aspects of the complex unstructured stability radius $r_{\mathbb{C}}(A)$, see also Van Loan (1985), Hinrichsen and Pritchard (1986a), Martin (1987) and Byers (1988).

The stability radius $r_{\mathbb{F}}(A, B, C)$ has been motivated from several different viewpoints. It arises in the stability robustness analysis of a feedback loop consisting of a fixed linear time-invariant system and a norm-bounded uncertain gain representing uncertain parameters. It can also be posed from a pure matrix perturbation point of view, in which the matrices B and C reflect the structural information of the perturbation matrix $B\Delta C$, as in Hinrichsen and Pritchard (1986b). The solution to the structured complex stability radius problem, again, is a simple application of the Schmidt/Mirsky lower-rank matrix approximate theorem.

When the stability radius is used to analyze the stability of a linear time-invariant system under parametric perturbation, the real stability radius is more natural than its complex counterpart. This turns out, however, to be a much more difficult problem. Obviously, $r_{\mathbb{R}}(A, B, C) \geq r_{\mathbb{C}}(A, B, C)$. The ratio $r_{\mathbb{R}}(A, B, C)/r_{\mathbb{C}}(A, B, C)$ can actually be arbitrarily large (Hinrichsen and Motscha, 1988). Hinrichsen, Pritchard and associates studied various properties of the real stability radius, and surveyed their results in Hinrichsen and Pritchard (1990). Several lower bounds on $r_{\mathbb{R}}(A)$ were obtained by Qiu and Davison (1991) using tensor product techniques. Conditions under which $r_{\mathbb{R}}(A) = r_{\mathbb{C}}(A)$ were investigated by Hinrichsen and Pritchard (1986a) and Lewkowicz (1992), though the conditions obtained were in general difficult to verify.

The specialization of the right-hand sides of (2) and (3) to the case where $B = C = I$ was shown to be a lower bound on $r_{\mathbb{R}}(A)$ by Qiu and Davison (1992), and was also conjectured to be actually equal to $r_{\mathbb{R}}(A)$. Our main result stated above completely solves the general real structured stability radius problem. In particular, it shows that the conjecture of Qiu and Davison is indeed true.

It is well known that $\mu_{\mathbb{R}}(M)$ is easy to compute if M is either a row vector or a column vector. Formulas obtained using Euclidean space geometry that do not involve the minimization over γ were given in Biernacki *et al.* (1987), Hinrichsen and Pritchard (1988) and Qiu and Davison (1989). In Section 2 we shall show that this special advantage can be generalized a bit further: if $\operatorname{rank}(\operatorname{Im} M) \leq 1$ then the minimization over γ can be eliminated and $\mu(M)$ can be computed according to a simple explicit formula, which reduces to the formulas in the literature when specialized to the case where M is either a row vector or a column vector.

The paper is organized in the following way. Section 2 gives a proof of the main result. It also gives a more complete statement of that result and a procedure to construct a smallest real matrix Δ such that $I - \Delta M$ is singular. Section 3 addresses the sensitivity of $\mu_{\mathbb{R}}(M)$ to the changes in M. In Section 4 we specialize the results to the unstructured real stability radius and also generalize the definition of the structured stability radius so that it covers linear fractional perturbations. Section 5 presents several examples that illustrate different possible behaviors of the function on the right-hand side of (3) at its minimum and also illustrate the extra sweep over $\partial\mathbb{C}_g$ needed for the real stability radius computation. Section 6 presents some concluding remarks.

2. PROOF OF THE MAIN RESULT

The proof is long and involved. The idea is to rewrite the mixed problem involving a complex matrix and a realness constraint into a purely real problem. It is then easy to prove using the Schmidt/Mirsky theorem that the left-hand side of (3) is less than or equal to the right-hand side. To prove the opposite inequality, we construct a specific real Δ such that $I - \Delta M$ is singular and $[\bar{\sigma}(\Delta)]^{-1}$ is equal to the right-hand side of (3). To this end, we first investigate the properties possessed by the singular vectors of the matrix on the right-hand side of (3) corresponding to its second singular value at a minimum; then we construct the required Δ, separately for three different cases, in terms of these singular vectors.

Let $M \in \mathbb{C}^{p\times m}$ be given. Introduce $X := \operatorname{Re} M$ and $Y := \operatorname{Im} M$. The case when $Y = 0$ is trivial; we then have $\mu_{\mathbb{R}}(M) = \mu_{\mathbb{C}}(M) = \bar{\sigma}(X)$. Hence we assume $Y \neq 0$ in the following proof. For $\Delta \in \mathbb{R}^{m\times p}$, the matrix $I - \Delta M$ is singular if and only if there are $v_1, v_2 \in \mathbb{R}^m$ with $(v_1, v_2) \neq (0, 0)$ such that

$$[I - \Delta(X + jY)](v_1 + jv_2) = 0. \tag{4}$$

An equivalent form of (4) is

$$\left(I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right)\begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix} = 0. \tag{5}$$

The advantage of (5) is that only real numbers are involved. Since $(v_1, v_2) \neq (0, 0)$, the columns of $\begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix}$ are linearly independent; therefore

$$\operatorname{rank}\left(I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right) \leq 2m - 2. \tag{6}$$

To proceed, we need a version of the Schmidt/Mirsky theorem, tailored according to our need.

Lemma 1. Let $E \in \mathbb{F}^{m\times p}$ and $F \in \mathbb{F}^{p\times m}$. Then for $i = 1, \ldots, \min\{p, m\}$,

$$\inf\{\bar{\sigma}(E) : \operatorname{rank}(I_m - EF) \leq m - i\} = [\sigma_i(F)]^{-1}.$$

Proof. If $\bar{\sigma}(E) < [\sigma_i(F)]^{-1}$ then $\sigma_i(EF) \leq \bar{\sigma}(E)\sigma_i(F) < 1$ (Stewart and Sun, 1990, p. 34, Theorem 4.5). By the Schmidt/Mirsky theorem, $\operatorname{rank}(I_m - EF) > m - i$. This shows '$\geq$'. Now let a singular value decomposition of F by $U\Sigma V^*$, where U and V are unitary matrices over $\mathbb{F}$ and

$$\Sigma = \operatorname{diag}\{\sigma_1(F), \sigma_2(F), \ldots, \sigma_{\min\{p,m\}}(F)\} \in \mathbb{R}^{p\times m}.$$

Define

$$E = V \operatorname{diag}\{[\sigma_1(F)]^{-1}, \ldots, [\sigma_i(F)]^{-1}, 0, \ldots, 0\}U^*.$$

Then $\bar{\sigma}(E) = [\sigma_i(F)]^{-1}$ and $\operatorname{rank}(I_m - EF) = m - i$. This shows '$\leq$'. □

To reduce the conservatism caused by applying Lemma 1 directly to (6), we resort to the widely used technique of scaling. It turns out that this scaling completely eliminates the conservatism. Let $\gamma \in \mathbb{R}\backslash\{0\}$. From (6), we get

$$\begin{aligned} &\operatorname{rank}\left(I - \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \times \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}\right) \\ &= \operatorname{rank}\left(I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X \end{bmatrix}\right) \\ &\leq 2m - 2. \end{aligned} \tag{7}$$

Let us introduce the notation

$$P(\gamma) = \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X \end{bmatrix}.$$

Lemma 1 and the inequality (7) imply that

$$\begin{aligned} \bar{\sigma}(\Delta) &= \bar{\sigma}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \\ &\geq \sigma_2^{-1}[P(\gamma)] \quad \forall \gamma \neq 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \mu_{\mathbb{R}}(M) &\leq \inf_{\gamma\neq 0} \sigma_2[P(\gamma)] \\ &= \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)]. \end{aligned}$$

Here the search over γ has been restricted to $(0, 1]$ because $P(\gamma)$, $P(-\gamma)$ and $P(\gamma^{-1})$ all have the same singular values.

The rest of this section is devoted to the proof of the reverse inequality:

$$\mu_{\mathbb{R}}(M) \geq \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)] =: \sigma^*, \tag{8}$$

which is significantly more difficult. We only need to prove this for the case when $\sigma^* > 0$. The proof is done by an explicit construction of a real Δ such that $I - \Delta M$ is singular and $\bar{\sigma}(\Delta) = \sigma^{*-1}$. Let us use $(\cdot)^\dagger$ to denote the Moore–Penrose generalized inverse. The following lemma is needed in the construction.

Lemma 2. Let $U \in \mathbb{R}^{p\times k}$ and $V \in \mathbb{R}^{m\times k}$. If $U^{\mathrm{T}}U = V^{\mathrm{T}}V \neq 0$ then $\bar{\sigma}(VU^{\dagger}) = 1$ and $VU^{\dagger}U = V$.

Proof. Since $U^{\mathrm{T}}U = V^{\mathrm{T}}V$, it follows that there exists an orthogonal matrix W such that $V = WU$. Hence

$$VU^{\dagger}U = WUU^{\dagger}U = WU = V$$

and

$$\bar{\sigma}(VU^{\dagger}) = \bar{\sigma}(WUU^{\dagger}) = \bar{\sigma}(UU^{\dagger}) = 1.$$

Here we have used the fact that $UU^{\dagger}$ is a nonzero orthogonal projection. □

First we shall treat the case when $\inf_{\gamma\in(0,1]}\sigma_2[P(\gamma)]$ is attained for some $\gamma^* \in (0, 1]$. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a pair of left and right singular vectors of $P(\gamma^*)$ corresponding to σ^*, with $u_1, u_2 \in \mathbb{R}^p$ and $v_1, v_2 \in \mathbb{R}^m$, and set

$$\Delta = \sigma^{*-1}[v_1 \quad v_2][u_1 \quad u_2]^{\dagger}. \tag{9}$$

If u and v can be chosen so that

$$[u_1 \quad u_2]^{\mathrm{T}}[u_1 \quad u_2] = [v_1 \quad v_2]^{\mathrm{T}}[v_1 \quad v_2] \tag{10}$$

then it follows from Lemma 2 that $\bar{\sigma}(\Delta) = \sigma^{*-1}$ and that

$$\begin{aligned}[I - \Delta(X + \mathrm{j}Y)](v_1 + \mathrm{j}\gamma^* v_2) \\ = v_1 + \mathrm{j}\gamma^* v_2 - \Delta\sigma^* u_1 - \Delta \mathrm{j}\gamma^*\sigma^* u_2 = 0,\end{aligned}$$

which means that $I - \Delta M$ is singular. Hence Δ given by (9) is the desired construction. What follows is a long elaboration showing that the singular vectors u and v can always be chosen so that (10) is satisfied when $\gamma^* \in (0, 1]$.

The proof for the case when $\inf_{\gamma\in(0,1]}\sigma_2[P(\gamma)]$ is attained as $\gamma \to 0$, which occurs if and only if rank $(Y) = 1$, is carried out in a different way, in which an explicit formula for $\mu_{\mathbb{R}}(M)$, involving no minimization, and a more direct construction of Δ are available.

We start with several claims on the singular vectors of $P(\gamma)$. The first is of a purely algebraic nature.

Claim 1. Let $\gamma \in \mathbb{R}\backslash\{-1, 0, 1\}$ and let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a pair of left and right singular vectors of $P(\gamma)$ corresponding to some nonzero singular value σ. Then $u_1^{\mathrm{T}}u_2 = v_1^{\mathrm{T}}v_2$.

Proof. The singular vectors satisfy

$$\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{11}$$

$$\begin{bmatrix} X^{\mathrm{T}} & \gamma^{-1}Y^{\mathrm{T}} \\ -\gamma Y^{\mathrm{T}} & X^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sigma\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{12}$$

The difference between $[u_2^{\mathrm{T}} \ u_1^{\mathrm{T}}]$ times (11) and $[v_2^{\mathrm{T}} \ v_1^{\mathrm{T}}]$ times (12) gives

$$(\gamma + \gamma^{-1})(u_1^{\mathrm{T}}Yv_1 - u_2^{\mathrm{T}}Yv_2) = 2\sigma(u_1^{\mathrm{T}}u_2 - v_1^{\mathrm{T}}v_2). \tag{13}$$

Similarly, the sum of $[u_2^{\mathrm{T}} \ -u_1^{\mathrm{T}}]$ times (11) and $[v_2^{\mathrm{T}} \ -v_1^{\mathrm{T}}]$ times (12) gives

$$(\gamma - \gamma^{-1})(u_1^{\mathrm{T}}Yv_1 - u_2^{\mathrm{T}}Yv_2) = 0. \tag{14}$$

Since $\sigma \neq 0$ and $\gamma \neq 0$ or ± 1, the claim follows from (13) and (14). □

The second claim concerns the singular vectors of $P(\gamma)$ corresponding to singular values at extrema. We need several lemmas.

Lemma 3. Let $F(\gamma) \in \mathbb{R}^{p\times m}$ be a (real) analytic matrix function on an open set $\Gamma \subset \mathbb{R}$. Then there exist an analytic diagonal matrix function $\tilde{\Sigma}(\gamma) = \mathrm{diag}\,(\tilde{\sigma}_1(\gamma), \ldots, \tilde{\sigma}_{\min\{p,m\}}(\gamma)) \in \mathbb{R}^{p\times m}$ and analytic orthogonal matrix functions $\tilde{U}(\gamma) = [\tilde{u}_1(\gamma) \ \ldots \ \tilde{u}_p(\gamma)] \in \mathbb{R}^{p\times p}$ and $\tilde{V}(\gamma) = [\tilde{v}_1(\gamma) \ \ldots \ \tilde{v}_m(\gamma)] \in \mathbb{R}^{m\times m}$, all of which are defined on Γ, such that

$$\tilde{\Sigma}(\gamma) = \tilde{U}^{\mathrm{T}}(\gamma)F(\gamma)\tilde{V}(\gamma).$$

Furthermore,

$$\frac{\mathrm{d}\tilde{\sigma}_i(\gamma)}{\mathrm{d}\gamma} = \tilde{u}_i^{\mathrm{T}}(\gamma)\frac{\mathrm{d}F(\gamma)}{\mathrm{d}\gamma}\tilde{v}_i(\gamma), \tag{15}$$

for $i = 1, \ldots, \min\{p, m\}$.

Proof. The first statement above follows from a similar result for Hermitian matrices in Baumgärtel (1985, p. 149, Corollary 3), see also Kato (1966, Section II.6.2). To prove (15), differentiate $F(\gamma)\tilde{v}_i(\gamma) = \tilde{\sigma}_i(\gamma)\tilde{u}_i(\gamma)$. This gives

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}\gamma}(\gamma)\tilde{v}_i(\gamma) + F(\gamma)\frac{\mathrm{d}\tilde{v}_i}{\mathrm{d}\gamma}(\gamma) \\ = \frac{\mathrm{d}\tilde{\sigma}_i}{\mathrm{d}\gamma}(\gamma)\tilde{u}_i(\gamma) + \tilde{\sigma}_i(\gamma)\frac{\mathrm{d}\tilde{u}_i}{\mathrm{d}\gamma}(\gamma).\end{aligned}$$

Multiplying both sides by $\tilde{u}_i^{\mathrm{T}}(\gamma)$ from the left and noting that $\tilde{u}_i^{\mathrm{T}}(\gamma)F(\gamma) = \tilde{\sigma}_i(\gamma)\tilde{v}_i^{\mathrm{T}}(\gamma)$, we obtain

$$\begin{aligned}\tilde{u}_i^{\mathrm{T}}(\gamma)\frac{\mathrm{d}F}{\mathrm{d}\gamma}(\gamma)\tilde{v}_i(\gamma) + \sigma_i(\gamma)\tilde{v}_i^{\mathrm{T}}(\gamma)\frac{\mathrm{d}\tilde{v}_i}{\mathrm{d}\gamma}(\gamma) \\ = \frac{\mathrm{d}\tilde{\sigma}_i}{\mathrm{d}\gamma}(\gamma) + \tilde{\sigma}_i(\gamma)\tilde{u}_i^{\mathrm{T}}(\gamma)\frac{\mathrm{d}\tilde{u}_i}{\mathrm{d}\gamma}(\gamma).\end{aligned}$$

From $\tilde{u}_i^{\mathrm{T}}(\gamma)\tilde{u}_i(\gamma)=1$ and $\tilde{v}_i^{\mathrm{T}}(\gamma)\tilde{v}_i(\gamma)=1$, it follows that

$$\tilde{u}_i^{\mathrm{T}}(\gamma)\frac{\mathrm{d}\tilde{u}_i}{\mathrm{d}\gamma}(\gamma)=0,\quad \tilde{v}_i^{\mathrm{T}}(\gamma)\frac{\mathrm{d}\tilde{v}_i}{\mathrm{d}\gamma}(\gamma)=0.$$

Hence (15) follows. □

Apparently, $|\tilde{\sigma}_i(\gamma)|$, $i=1,\ldots,\min\{p,m\}$, are singular values of $F(\gamma)$. However, they are not in any particular order. In the following, we shall also use the ordered singular values $\sigma_1(\gamma)\geq\ldots\geq\sigma_{\min\{p,m\}}(\gamma)\geq 0$ of $F(\gamma)$. The differences between $\tilde{\sigma}_i(\gamma)$ and $\sigma_i(\gamma)$ are that the former are analytic whereas the latter are generally not, and the latter are nonnegative and ordered nonincreasingly whereas the former are generally not. Despite its lack of analyticity on the whole of Γ, $\sigma_i(\gamma)$ is continuous and piecewise-analytic.

Lemma 4. Let $F(\gamma)\in\mathbb{R}^{p\times m}$ be an analytic matrix function on an open set $\Gamma\subset\mathbb{R}$. Let $\sigma_1(\gamma)\geq\ldots\geq\sigma_{\min\{p,m\}}(\gamma)\geq 0$ be its ordered singular values. If $\sigma_i(\gamma)$ has a nonzero local extremum $\gamma^*\in\Gamma$ then there exists a pair of left and right singular vectors $u\in\mathbb{R}^p$ and $v\in\mathbb{R}^m$ of $F(\gamma^*)$ corresponding to $\sigma_i(\gamma^*)$ such that $u^{\mathrm{T}}\,\mathrm{d}F/\mathrm{d}\gamma(\gamma^*)v=0$.

Proof. If $\sigma_i(\gamma)$ is analytic at γ^* then we can assume, without loss of generality, that $\sigma_i(\gamma)=\tilde{\sigma}_1(\gamma)$ in an open neighborhood of γ^*. Thus γ^* is also a stationary point of $\tilde{\sigma}_1(\gamma)$. Let $\tilde{u}_1(\gamma)$ and $\tilde{v}_1(\gamma)$ be a pair of left and right analytic singular vectors corresponding to $\tilde{\sigma}_1(\gamma)$. The lemma then follows, since (15) gives

$$\tilde{u}_1^{\mathrm{T}}(\gamma^*)\frac{\mathrm{d}F}{\mathrm{d}\gamma}(\gamma^*)\tilde{v}_1(\gamma^*)=0.$$

If, instead, $\sigma_i(\gamma)$ is not analytic at γ^* then we can assume, without loss of generality, that in an open neighborhood of γ^*, $\sigma_i(\gamma)=\tilde{\sigma}_1(\gamma)$ for $\gamma\leq\gamma^*$ and $\sigma_i(\gamma)=\tilde{\sigma}_2(\gamma)$ for $\gamma\geq\gamma^*$. Let $\tilde{u}_k(\gamma)$ and $\tilde{v}_k(\gamma)$, $k=1,2$, be the pair of left and right analytic singular vectors corresponding to $\tilde{\sigma}_k(\gamma)$. Then (15) gives

$$\frac{\mathrm{d}\tilde{\sigma}_1}{\mathrm{d}\gamma}(\gamma^*)=\tilde{u}_1^{\mathrm{T}}(\gamma^*)\frac{\mathrm{d}F}{\mathrm{d}\gamma}(\gamma^*)\tilde{v}_1(\gamma^*),$$

$$\frac{\mathrm{d}\tilde{\sigma}_2}{\mathrm{d}\gamma}(\gamma^*)=\tilde{u}_2^{\mathrm{T}}(\gamma^*)\frac{\mathrm{d}F}{\mathrm{d}\gamma}(\gamma^*)\tilde{v}_2(\gamma^*).$$

Put $u_\alpha=\alpha\tilde{u}_1+(1-\alpha^2)^{1/2}\tilde{u}_2$ and $v_\alpha=\alpha\tilde{v}_1+(1-\alpha^2)^{1/2}\tilde{v}_2$ for $\alpha\in[0,1]$. Then $u_\alpha(\gamma^*)$ and $v_\alpha(\gamma^*)$ also form a pair of singular vectors of $F(\gamma^*)$ corresponding to the singular value $\sigma_i(\gamma^*)$. Define

$$f(\alpha)=u_\alpha^{\mathrm{T}}(\gamma^*)\frac{\mathrm{d}F}{\mathrm{d}\gamma}(\gamma^*)v_\alpha(\gamma^*).$$

Since γ^* is an local extremum of $\sigma_i(\gamma)$, we must have

$$f(0)f(1)=\frac{\mathrm{d}\tilde{\sigma}_1}{\mathrm{d}\gamma}(\gamma^*)\frac{\mathrm{d}\tilde{\sigma}_2}{\mathrm{d}\gamma}(\gamma^*)\leq 0.$$

By continuity, $f(\alpha)=0$ has a solution in $[0,1]$. This proves the lemma. □

For the matrix $P(\gamma)$, the singular vectors described in Lemma 4 satisfy some pleasant alignment conditions.

Claim 2. Let $\gamma\in\mathbb{R}\backslash\{0\}$ and let $u=\begin{bmatrix}u_1\\u_2\end{bmatrix}$ and $v=\begin{bmatrix}v_1\\v_2\end{bmatrix}$ be a pair of left and right singular vectors of $P(\gamma)$ corresponding to a nonzero singular value σ. If the extra condition $u^{\mathrm{T}}\,\mathrm{d}P/\mathrm{d}\gamma(\gamma)v=0$ is satisfied then $u_1^{\mathrm{T}}u_1=v_1^{\mathrm{T}}v_1$ and $u_2^{\mathrm{T}}u_2=v_2^{\mathrm{T}}v_2$.

Proof. The singular vectors satisfy (11), (12) and

$$[u_1^{\mathrm{T}}\ \ u_2^{\mathrm{T}}]\begin{bmatrix}0 & -Y\\-\gamma^{-2}Y & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=0. \tag{16}$$

Equation (16) gives

$$u_1^{\mathrm{T}}Yv_2+\gamma^{-2}u_2^{\mathrm{T}}Yv_1=0.$$

Multiplying (11) by $[u_1^{\mathrm{T}}\ -u_2^{\mathrm{T}}]$ from the left and (12) by $[v_1^{\mathrm{T}}\ -v_2^{\mathrm{T}}]$ from the left, we obtain

$$\begin{aligned}u_1^{\mathrm{T}}Xv_1-\gamma u_1^{\mathrm{T}}Yv_2-\gamma^{-1}u_2^{\mathrm{T}}Yv_1-u_2^{\mathrm{T}}Xv_2&\\=u_1^{\mathrm{T}}Xv_1-u_2^{\mathrm{T}}Xv_2&\\=\sigma(u_1^{\mathrm{T}}u_1-u_2^{\mathrm{T}}u_2)&\end{aligned}$$

and

$$\begin{aligned}v_1^{\mathrm{T}}X^{\mathrm{T}}u_1+\gamma^{-1}u_1^{\mathrm{T}}Y^{\mathrm{T}}u_2+\gamma v_2^{\mathrm{T}}Y^{\mathrm{T}}u_1-v_2^{\mathrm{T}}X^{\mathrm{T}}u_2&\\=v_1^{\mathrm{T}}X^{\mathrm{T}}u_1-v_2^{\mathrm{T}}X^{\mathrm{T}}u_2&\\=\sigma(v_1^{\mathrm{T}}v_1-v_2^{\mathrm{T}}v_2).&\end{aligned}$$

Since $\sigma>0$, we get

$$u_1^{\mathrm{T}}u_1-u_2^{\mathrm{T}}u_2=v_1^{\mathrm{T}}v_1-v_2^{\mathrm{T}}v_2.$$

Claim 2 now follows from $u_1^{\mathrm{T}}u_1+u_2^{\mathrm{T}}u_2=v_1^{\mathrm{T}}v_1+v_2^{\mathrm{T}}v_2=1$. □

We are now ready to show the inequality (8). We need to treat three different cases separately. Note that these three cases are not mutually exclusive.

Case 1: *$\sigma^*=\sigma_2[P(\gamma^*)]$ for some $\gamma^*\in$* $(0,1)$. Lemma 4 together with Claims 1 and 2, tells us that a pair of singular vectors $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ of $P(\gamma^*)$ corresponding to σ^* can be chosen to satisfy (10). Then (8) follows as discussed previously.

Case 2: $\sigma^* = \sigma_2[P(1)]$. We have to treat this case separately, since Claim 1 is not valid for $\gamma = 1$. We know, however, that the singular values of $P(1)$ are paired so that $\sigma_{2i-1}[P(1)] = \sigma_{2i}[P(1)] = \sigma_i(M)$ for all i. In particular, the largest and the second largest singular values of $P(1)$ are equal to σ^*. We need to consider two possibilities. The first is that the multiplicity of the largest singular value of $P(1)$ is two. Without loss of generality, assume $\sigma_1[P(1)] = \sigma_2[P(1)] = \tilde{\sigma}_1(1) = \tilde{\sigma}_2(1)$, where $\tilde{\sigma}_1(\gamma)$ and $\tilde{\sigma}_2(\gamma)$ are analytic singular values of $P(\gamma)$. Note also that if $\sigma(\gamma)$ is a singular value then so is $\sigma(\gamma^{-1})$. Since $\gamma = 1$ is a minimum of $\sigma_2(\gamma)$ that is equal to $\min\{\tilde{\sigma}_1(\gamma), \tilde{\sigma}_2(\gamma)\}$ locally around $\gamma = 1$, it follows that $\gamma = 1$ must be a local minimum of $\tilde{\sigma}_1(\gamma)$ and $\tilde{\sigma}_2(\gamma)$. Let $\begin{bmatrix} u_1(\gamma) \\ u_2(\gamma) \end{bmatrix}$ and $\begin{bmatrix} v_1(\gamma) \\ v_2(\gamma) \end{bmatrix}$ be a pair of analytic singular vectors of $P(\gamma)$ corresponding to $\tilde{\sigma}_2(\gamma)$. By Claim 1, we know that

$$u_1^{\mathrm{T}}(\gamma)u_2(\gamma) = v_1^{\mathrm{T}}(\gamma)v_2(\gamma), \quad \gamma \neq 0, \pm 1.$$

By continuity, we must therefore have

$$u_1^{\mathrm{T}}(1)u_2(1) = v_1^{\mathrm{T}}(1)v_2(1).$$

Using the fact that $\mathrm{d}\tilde{\sigma}_2/\mathrm{d}\gamma(1) = 0$, we conclude from the derivative relation (15) and Claim 2 that

$$u_1^{\mathrm{T}}(1)u_1(1) = v_1^{\mathrm{T}}(1)v_1(1),$$

$$u_2^{\mathrm{T}}(1)u_2(1) = v_2^{\mathrm{T}}(1)v_2(1).$$

Putting $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1(1) \\ v_2(1) \end{bmatrix}$, it follows that (10) holds.

We have completed the proof for the first possibility of Case 2. However, the construction above is not quite readily implementable numerically. Here we pause for an interesting observation that renders surprising numerical advantages.

Remark. If (10) holds for a pair of left and right singular vectors $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ of $P(1)$ corresponding to a nonzero singular value σ of multiplicity 2 then it holds for every such pair.

Proof. It is easy to check that $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ form a pair of singular vectors of $P(1)$ corresponding to a nonzero singular value σ of multiplicity 2 if and only if $u_1 + \mathrm{j}u_2$ and $v_1 + \mathrm{j}v_2$ form a pair of singular vectors of M corresponding to the same singular value of multiplicity 1. Now suppose $\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$ and $\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$ are another such pair. Then, since σ is a distinct nonzero singular value of M, we have

$$\tilde{u}_1 + \mathrm{j}\tilde{u}_2 = (u_1 + \mathrm{j}u_2)\mathrm{e}^{\mathrm{j}\theta}, \quad \tilde{v}_1 + \mathrm{j}\tilde{v}_2 = (v_1 + \mathrm{j}v_2)\mathrm{e}^{\mathrm{j}\theta}$$

for some $\theta \in [0, 2\pi)$. These can be rewritten as

$$[\tilde{u}_1 \quad \tilde{u}_2] = [u_1 \quad u_2]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

$$[\tilde{v}_1 \quad \tilde{v}_2] = [v_1 \quad v_2]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

We immediately see that if $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ satisfy (10) then so do $\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$ and $\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$. □

The second possibility of Case 2 is that the multiplicity of the largest singular value is greater than two. This means that the largest four singular values of $P(1)$ are equal to σ^*, i.e. the two (or more) largest singular values of $M = X + \mathrm{j}Y$ are equal to σ^*. This possibility is related to a problem considered by Lewkowicz (1992), which inspired our solution. Bring in a singular value decomposition

$$M = \sigma^*(\mu_1 v_1^{\mathrm{H}} + \mu_2 v_2^{\mathrm{H}}) + \sum_{i=3}^{\min\{p,m\}} \sigma_i(M)\mu_i v_i^{\mathrm{H}},$$

where $(\cdot)^{\mathrm{H}}$ means conjugate transpose. Introduce

$$\begin{bmatrix} \mu \\ v \end{bmatrix} := \begin{bmatrix} \mu_1 & \mu_2 \\ v_1 & v_2 \end{bmatrix}\xi, \tag{17}$$

where $\xi \in \mathbb{C}^2$ is a vector of unit length. Then μ and v also form a pair of singular vectors of M corresponding to σ^*. If ξ can be found so that $\mu^{\mathrm{T}}\mu - v^{\mathrm{T}}v = 0$ then $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} \mathrm{Re}\,\mu \\ \mathrm{Im}\,\mu \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := \begin{bmatrix} \mathrm{Re}\,v \\ \mathrm{Im}\,v \end{bmatrix}$ form a pair of left and right singular vectors of $P(1)$ corresponding to σ^*, and the condition (10), which is equivalent to $\mu^{\mathrm{T}}\mu - v^{\mathrm{T}}v = 0$, is satisfied. The inequality (8) now follows as discussed previously. To show the existence of such a desired ξ, one needs to study how $\mu^{\mathrm{T}}\mu - v^{\mathrm{T}}v$ varies with ξ. This can be done straightforwardly with elementary calculus as in Qiu *et al.* (1993), but an alternative is to introduce the Takagi factorization (Horn and Johnson, 1985, pp. 204–205, Corollary 4.4.4):

$$\begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix} = \Xi^{\mathrm{T}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Xi,$$

where Ξ is a unitary matrix and $\lambda_1 \geq \lambda_2 \geq 0$. Clearly, a desired ξ can be chosen as follows:

$$\xi = \begin{cases} \text{any vector of unit length in } \mathbb{C}^2 \\ \quad \text{if } \lambda_1 = 0, \\ \Xi^{\mathrm{H}} \begin{bmatrix} \mathrm{j}\sqrt{\lambda_2} \\ \sqrt{\lambda_1} \end{bmatrix} \dfrac{1}{\sqrt{\lambda_1 + \lambda_2}} \\ \quad \text{if } \lambda_1 \neq 0. \end{cases}$$

Case 3: $\sigma^* = \lim_{\gamma\to 0} \sigma_2[P(\gamma)]$. It follows from e.g. Stewart and Sun (1990, p. 33, Theorem 4.4) that $\sigma_2[P(\gamma)] \geq \sigma_2(\gamma^{-1}Y)$, so Case 3 is relevant only if rank $Y = 1$. We need a lemma to proceed.

Lemma 5. Let $F(\gamma) = G(\gamma) + \gamma^{-1}H \in \mathbb{R}^{p\times m}$, where $G(\gamma)$ is analytic on an open interval Γ around 0 and H is a constant matrix with rank $(H) =: r < \min\{p, m\}$. Let $\sigma_1(\gamma) \geq \ldots \geq \sigma_{\min\{p,m\}}(\gamma) \geq 0$ be the ordered singular values of $F(\gamma)$ defined on $\Gamma\backslash\{0\}$. Assume that a singular value decomposition of H is given by

$$H = [U_1 \quad U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1 \quad V_2]^{\mathrm{T}}$$

where $\Sigma_1 \in \mathbb{R}^{r\times r}$. Then

$$\lim_{\gamma\to 0} \sigma_{r+i}(\gamma) = \sigma_i[U_2^{\mathrm{T}} G(0) V_2]$$

for $i = 1, \ldots, \min\{p, m\} - r$.

Proof. Without loss of generality, assume that an analytic singular value decomposition of $\gamma F(\gamma)$ is

$$\gamma F(\gamma) = [\tilde{U}_1(\gamma) \quad \tilde{U}_2(\gamma)] \begin{bmatrix} \tilde{\Sigma}_1(\gamma) & 0 \\ 0 & \tilde{\Sigma}_2(\gamma) \end{bmatrix} \times [\tilde{V}_1(\gamma) \quad \tilde{V}_2(\gamma)]^{\mathrm{T}},$$

where $\tilde{\Sigma}_1(0) \in \mathbb{R}^{r\times r}$ and $\tilde{\Sigma}_2(0) = 0$. Then

$$\gamma^{-1}\tilde{\Sigma}_2(\gamma) = \tilde{U}_2^{\mathrm{T}}(\gamma) F(\gamma) \tilde{V}_2(\gamma) = \tilde{U}_2^{\mathrm{T}}(\gamma) G(\gamma) \tilde{V}_2(\gamma) + \gamma^{-1} \tilde{U}_2^{\mathrm{T}}(\gamma) H \tilde{V}_2(\gamma),$$

and

$$H = [\tilde{U}_1(0) \quad \tilde{U}_2(0)] \begin{bmatrix} \tilde{\Sigma}_1(0) & 0 \\ 0 & 0 \end{bmatrix} \times [\tilde{V}_1(0) \quad \tilde{V}_2(0)]^{\mathrm{T}}.$$

Since both $\tilde{U}_2^{\mathrm{T}}(\gamma)\tilde{U}_1(0)$ and $\tilde{V}_1^{\mathrm{T}}(0)\tilde{V}_2(\gamma)$ are analytic and vanishing at $\gamma = 0$, it follows that

$$\lim_{\gamma\to 0} \gamma^{-1} \tilde{U}_2^{\mathrm{T}}(\gamma) H \tilde{V}_2(\gamma) = \lim_{\gamma\to 0} \gamma^{-1} \tilde{U}_2^{\mathrm{T}}(\gamma) \tilde{U}_1(0) \tilde{\Sigma}_1(0) \tilde{V}_1^{\mathrm{T}}(0) \tilde{V}_2(\gamma) = 0.$$

Therefore

$$\lim_{\gamma\to 0} \gamma^{-1}\tilde{\Sigma}_2(\gamma) = \tilde{U}_2^{\mathrm{T}}(0) G(0) \tilde{V}_2(0).$$

Since all singular values of $\gamma^{-1}\tilde{\Sigma}_1(\gamma)$ go to infinity as $\gamma \to 0$,

$$\lim_{\gamma\to 0} \sigma_{r+i}(\gamma) = \lim_{\gamma\to 0} \sigma_i[\gamma^{-1}\tilde{\Sigma}_2(\gamma)] = \sigma_i[\tilde{U}_2^{\mathrm{T}}(0) G(0) \tilde{V}_2(0)] = \sigma_i[U_2^{\mathrm{T}} G(0) V_2].$$

Note that $\tilde{U}_2(0)$ and $\tilde{V}_2(0)$ can be replaced by U_2 and V_2, since they have the same ranges respectively. □

Following the notation in the lemma, put

$$G(\gamma) = \begin{bmatrix} X & -\gamma Y \\ 0 & X \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix},$$

and let a real singular value decomposition of Y be

$$U^Y \Sigma^Y (V^Y)^{\mathrm{T}} = [U_1^Y \quad U_2^Y] \begin{bmatrix} \sigma_1(Y) & 0 \\ 0 & 0 \end{bmatrix} \times [V_1^Y \quad V_2^Y]^{\mathrm{T}}.$$

Then a singular value decomposition of H is

$$H = \begin{bmatrix} 0 & I \\ U^Y & 0 \end{bmatrix} \begin{bmatrix} \Sigma^Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^Y & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}}.$$

Applying Lemma 5, we obtain

$$\lim_{\gamma\to 0} \sigma_2[P(\gamma)] = \bar{\sigma}\left(\begin{bmatrix} 0 & I \\ U_2^Y & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} V_2^Y & 0 \\ 0 & I \end{bmatrix} \right) = \max\{\bar{\sigma}[(U_2^Y)^{\mathrm{T}} X], \bar{\sigma}(X V_2^Y)\}.$$

Now we want to show that $\lim_{\gamma\to 0} \sigma_2[P(\gamma)] = \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)]$. If u and v are a pair of left and right real singular vectors of $(U_2^Y)^{\mathrm{T}}X$ corresponding to $\bar{\sigma}[(U_2^Y)^{\mathrm{T}}X]$ then the choice

$$\Delta = -\frac{v u^{\mathrm{T}} (U_2^Y)^{\mathrm{T}}}{\bar{\sigma}[(U_2^Y)^{\mathrm{T}} X]} \tag{18}$$

satisfies $[I + \Delta(X + \mathrm{j}Y)]v = 0$ and $\bar{\sigma}(\Delta)^{-1} = \bar{\sigma}[(U_2^Y)^{\mathrm{T}}X]$. Similarly, if u and v are a pair of left and right real singular vectors of XV_2^Y corresponding to $\bar{\sigma}(XV_2^Y)$ then the choice

$$\Delta = -\frac{V_2^Y v u^{\mathrm{T}}}{\bar{\sigma}(X V_2^Y)} \tag{19}$$

satisfies $u^{\mathrm{T}}[I + (X + \mathrm{j}Y)\Delta] = 0$ and $\bar{\sigma}(\Delta)^{-1} = \bar{\sigma}(XV_2^Y)$. Together, this shows that

$$\max\{\bar{\sigma}[(U_2^Y)^{\mathrm{T}} X], \bar{\sigma}(X V_2^Y)\} \leq \mu_{\mathbb{R}}(M),$$

so

$$\inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)] \leq \lim_{\gamma\to 0} \sigma_2[P(\gamma)] = \max\{\bar{\sigma}[(U_2^Y)^{\mathrm{T}} X], \bar{\sigma}(X V_2^Y)\} \leq \mu_{\mathbb{R}}(M) \leq \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)],$$

and therefore the inequalities above can be replaced by equalities. This also shows that $\sigma^* = \lim_{\gamma\to 0}\sigma_2[P(\gamma)]$ if and only if rank $Y = 1$.

Note that if $\min\{p, m\} = 1$ then U_2^Y or V_2^Y will be empty. We define the largest singular value of an empty matrix to be zero.

We have completed the proof of the equality (3). Now suppose that $\sigma_2[P(\gamma)]$ has a local extremum (either minimum or maximum) $\gamma^{**} \in (0, 1)$ such that $\sigma_2[P(\gamma^{**})] > \sigma^*$. Then, using exactly the same arguments as in Case 1, one can construct a real Δ such that $I - \Delta M$ is singular and $\bar\sigma(\Delta) = \{\sigma_2[P(\gamma^{**})]\}^{-1} < \sigma^{*-1}$. This contradicts (3) and therefore cannot happen. This shows that $\sigma_2[P(\gamma)]$ is a unimodal function on $(0, 1]$.

To recap, we summarize what we have proved in this section in the following theorem.

Theorem. If $X, Y \in \mathbb{R}^{p\times m}$ and $M = X + jY$ then

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X\end{bmatrix}\right). \quad (20)$$

The function to be minimized is a unimodal function on $(0, 1]$. If rank $Y = 1$ then, furthermore,

$$\begin{aligned}\mu_{\mathbb{R}}(M) &= \lim_{\gamma\to 0} \sigma_2\left(\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X\end{bmatrix}\right) \\ &= \max\{\bar\sigma[(U_2^Y)^{\mathrm{T}}X], \bar\sigma(XV_2^Y)\}\end{aligned}$$

where U_2^Y and V_2^Y come from any singular value decomposition of Y:

$$Y = [U_1^Y \quad U_2^Y]\begin{bmatrix}\bar\sigma(Y) & 0 \\ 0 & 0\end{bmatrix}[V_1^Y \quad V_2^Y]^{\mathrm{T}}.$$

Note that the theorem implies that $\mu_{\mathbb{R}}(M) = \mu_{\mathbb{C}}(M)$ if and only if the minimum value of $\sigma_2[P(\gamma)]$ is attained at $\gamma = 1$.

We also summarize a procedure to construct a worst Δ.

Construction of a worst Δ

1. If $Y = 0$, find a pair of left and right real singular vectors u and v of X corresponding to $\bar\sigma(X)$ and set $\Delta = vu^{\mathrm{T}}/\bar\sigma(X)$.
2. If rank $(Y) = 1$, compute a real singular value decomposition

$$Y = [U_1^Y \quad U_2^Y]\begin{bmatrix}\bar\sigma(Y) & 0 \\ 0 & 0\end{bmatrix}[V_1^Y \quad V_2^Y]^{\mathrm{T}}.$$

   - If $\bar\sigma[(U_2^Y)^{\mathrm{T}}X] \geq (XV_2^Y)$, find a pair of left and right real singular vectors u and v of $(U_2^Y)^{\mathrm{T}}X$ corresponding to $\mu_{\mathbb{R}}(M)$ and set

$$\Delta = -\frac{vu^{\mathrm{T}}(U_2^Y)^{\mathrm{T}}}{\bar\sigma[(U_2^Y)^{\mathrm{T}}X]}.$$

   - If $\bar\sigma(XV_2^Y) > \bar\sigma[(U_2^Y)^{\mathrm{T}}X]$, find a pair of left and right real singular vectors u and v of XV_2^Y corresponding to $\mu_{\mathbb{R}}(M)$ and set

$$\Delta = -\frac{V_2^Y vu^{\mathrm{T}}}{\bar\sigma(XV_2^Y)}.$$

3. If rank $(Y) > 1$, find a minimum $\gamma^* \in (0, 1]$ of $\sigma_2[P(\gamma)]$.
   - If $\gamma^* \in (0, 1)$ and $\sigma_2[P(\gamma^*)]$ has multiplicity 1 or of $\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity 2, find a pair of left and right real singular vectors $\begin{bmatrix}u_1\\u_2\end{bmatrix}$ and $\begin{bmatrix}v_1\\v_2\end{bmatrix}$ of $P(\gamma^*)$ corresponding to $\mu_{\mathbb{R}}(M)$.
   - If $\gamma^* \in (0, 1)$ and $\sigma_2[P(\gamma^*)]$ has multiplicity $r > 1$, find matrices $U \in \mathbb{R}^{2m\times r}$ and $V \in \mathbb{R}^{2p\times r}$ with orthonormal columns such that

$$P(\gamma^*)V = \sigma_2[P(\gamma^*)]U.$$

   Carry out a real Schur decomposition

$$\begin{aligned}U^{\mathrm{T}}\frac{\mathrm{d}P}{\mathrm{d}\gamma}(\gamma^*)V + V^{\mathrm{T}}\frac{\mathrm{d}P^{\mathrm{T}}}{\mathrm{d}\gamma}U \\ = W\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)W^{\mathrm{T}},\end{aligned}$$

   where W is orthogonal and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. Lemma 4 implies that $\lambda_1\lambda_r \leq 0$. Take

$$w = \begin{cases}\text{any unit length vector in } \mathbb{R}^r \\ \quad \text{if } \lambda_1 = \lambda_r = 0, \\ \dfrac{W[\sqrt{-\lambda_r} \quad 0 \quad \ldots \quad 0 \quad \sqrt{\lambda_1}]^{\mathrm{T}}}{\sqrt{\lambda_1 - \lambda_r}} \\ \quad \text{if } \lambda_1 \neq \lambda_r,\end{cases}$$

   and $\begin{bmatrix}u_1\\u_2\end{bmatrix} = Uw$ and $\begin{bmatrix}v_1\\v_2\end{bmatrix} = Vw$.
   - If $\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity $2r > 2$, find matrices $[\mu_1 \quad \mu_2] \in \mathbb{C}^{m\times 2}$ and $[v_1 \quad v_2] \in \mathbb{C}^{p\times 2}$ with orthonormal columns such that

$$M[v_1 \quad v_2] = \sigma_2[P(1)][\mu_1 \quad \mu_2].$$

   Carry out the Takagi factorization

$$\begin{aligned}\begin{bmatrix}\mu_1 & \mu_2 \\ v_1 & v_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}I & 0 \\ 0 & -I\end{bmatrix}\begin{bmatrix}\mu_1 & \mu_2 \\ v_1 & v_2\end{bmatrix} \\ = \Xi^{\mathrm{T}}\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}\Xi,\end{aligned}$$

   where Ξ is a unitary matrix and $\lambda_1 \geq \lambda_2 \geq 0$. Take

$$\xi = \begin{cases}\text{any unit length vector in } \mathbb{C}^2 \\ \quad \text{if } \lambda_1 = 0, \\ \Xi^H\begin{bmatrix}j\sqrt{\lambda_2} \\ \sqrt{\lambda_1}\end{bmatrix}\dfrac{1}{\sqrt{\lambda_1 + \lambda_2}} \\ \quad \text{if } \lambda_1 \neq 0,\end{cases}$$

and

$$\begin{bmatrix}\mu \\ \nu\end{bmatrix} = \begin{bmatrix}\mu_1 & \mu_2 \\ \nu_1 & \nu_2\end{bmatrix}\xi,$$

$$\begin{bmatrix}u_1 \\ u_2\end{bmatrix} = \begin{bmatrix}\operatorname{Re}\mu \\ \operatorname{Im}\mu\end{bmatrix}, \quad \begin{bmatrix}v_1 \\ v_2\end{bmatrix} = \begin{bmatrix}\operatorname{Re}\nu \\ \operatorname{Im}\nu\end{bmatrix}.$$

Finally set

$$\Delta = \sigma_2[P(\gamma^*)][v_1 \quad v_2][u_1 \quad u_2]^{\dagger}.$$

Note that the worst Δ constructed in the above procedure has rank no more than two.

3. CONTINUITY PROPERTIES

In computing the real stability radius, it is of interest to know how sensitive $\mu_{\mathbb{R}}(M)$ is to changes in M. The example

$$\mu_{\mathbb{R}}(1+\mathrm{j}\varepsilon) = \begin{cases}1 & \text{if } \varepsilon = 0, \\ 0 & \text{if } \varepsilon \neq 0\end{cases}$$

shows that $\mu_{\mathbb{R}}(M)$ can be discontinuous at certain M. Upper-semicontinuity of the map $M \mapsto \mu_{\mathbb{R}}(M)$, however, follows from the following general argument: if $I - \Delta M$ is invertible for all $\Delta \in \mathcal{K}$, where $\mathcal{K}$ is compact, then $I - \Delta\tilde{M}$ is invertible for all $\Delta \in \mathcal{K}$ and all $\tilde{M}$ in an open neighborhood of M.

In addition, $M \mapsto \mu_{\mathbb{R}}(M)$ is continuous at any M with $\operatorname{rank}(\operatorname{Im} M) > 1$, as the following relative error bound shows.

Proposition. If $\operatorname{rank}(\operatorname{Im} M) > 1$ then, for all $E \in \mathbb{C}^{p\times m}$,

$$\frac{|\mu_{\mathbb{R}}(M+E) - \mu_{\mathbb{R}}(M)|}{\mu_{\mathbb{R}}(M)} \leq \frac{\bar{\sigma}(E)}{\sigma_2(\operatorname{Im} M)}.$$

Proof. Let $\gamma^* \in (0, 1]$ be a minimum of $\sigma_2[P(\gamma)]$. Then

$$\begin{aligned}\mu_{\mathbb{R}}&(M+E) \\ &\leq \mu_{\mathbb{R}}(M) + \bar{\sigma}\left(\begin{bmatrix}\operatorname{Re} E & -\gamma^* \operatorname{Im} E \\ \gamma^{*-1}\operatorname{Im} E & \operatorname{Re} E\end{bmatrix}\right) \\ &\leq \mu_{\mathbb{R}}(M) + \gamma^{*-1}\bar{\sigma}\left(\begin{bmatrix}\operatorname{Re} E & -\operatorname{Im} E \\ \operatorname{Im} E & \operatorname{Re} E\end{bmatrix}\right) \\ &= \mu_{\mathbb{R}}(M) + \gamma^{*-1}\bar{\sigma}(E).\end{aligned}$$

Noting that $\gamma^{*-1}\sigma_2(\operatorname{Im} M) \leq \mu_{\mathbb{R}}(M)$ gives

$$\frac{\mu_{\mathbb{R}}(M+E)}{\mu_{\mathbb{R}}(M)} \leq 1 + \frac{\bar{\sigma}(E)}{\sigma_2(\operatorname{Im} M)}$$

To obtain the other half of the inequality, we exchange the roles of M and $M+E$ and invert:

$$\begin{aligned}\frac{\mu_{\mathbb{R}}(M+E)}{\mu_{\mathbb{R}}(M)} &\geq \left(1 + \frac{\bar{\sigma}(E)}{\sigma_2[\operatorname{Im}(M+E)]}\right)^{-1} \\ &= 1 - \frac{\bar{\sigma}(E)}{\sigma_2[\operatorname{Im}(M+E)] + \bar{\sigma}(E)} \\ &\geq 1 - \frac{\bar{\sigma}(E)}{\sigma_2(\operatorname{Im} M)}\end{aligned}$$

□

The only possible discontinuity points are therefore at M with $\operatorname{rank}(\operatorname{Im} M) \leq 1$. For $\operatorname{Im} M = 0$, we have shown the existence above, but we have not been able to find any example of a discontinuity at any M with $\operatorname{rank}(\operatorname{Im} M) = 1$.

Note added in proof. After the manuscript was sent to the press, we proved that actually $\mu_{\mathbb{R}}(M)$ are discontinuous only at real M.

4. A SPECIALIZATION AND A GENERALIZATION

It is well known that a more convenient formula for the complex unstructured stability radius is given by

$$r_{\mathbb{C}}(A) = \min_{s \in \partial\mathbb{C}_g} \underline{\sigma}(A - sI).$$

Analogously, an alternative formula for the real unstructured stability radius is available, which might sometimes be simpler to apply.

Corollary. Assume that $A \in \mathbb{R}^{n\times n}$ $(n > 1)$ is stable. Then

$$\begin{aligned}r_{\mathbb{R}}&(A) \\ &= \min_{s\in\partial\mathbb{C}_g} \max_{\gamma\in(0,1]} \sigma_{2n-1}\left(\begin{bmatrix}A - \operatorname{Re} sI & -\gamma \operatorname{Im} sI \\ \gamma^{-1}\operatorname{Im} sI & A - \operatorname{Re} sI\end{bmatrix}\right).\end{aligned}$$

For each fixed $s \in \partial\mathbb{C}_g$, the function to be maximized is quasiconcave.

We leave it to the reader to derive this from (2) and (3) and to justify the use of 'max' and 'min' instead of 'sup' and 'inf'. Note that, because of the proposition in the previous section, the only possible discontinuity of the function to be minimized occurs at the intersection of $\partial\mathbb{C}_g$ with the real axis.

In the definition of $r_{\mathbb{F}}(A, B, C)$, the perturbed matrix $A + B\Delta C$ depends on the perturbation matrix Δ in an affine way. In applications, however, a perturbed matrix may depend on the perturbation in a linear fractional way. This motivates a more general definition of the structured stability radius. For $(A, B, C, D) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m} \times \mathbb{F}^{p\times n} \times \mathbb{F}^{p\times m}$, introduce

$$\begin{aligned}r_{\mathbb{F}}&(A, B, C, D) \\ &:= \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m\times p}, \det(I - \Delta D) = 0 \\ &\qquad \text{or } A + B(I - \Delta D)^{-1}\Delta C \text{ is unstable}\}.\end{aligned}$$

Again we leave it to the reader to show that if $\mathbb{C}_b$ is unbounded and A is stable then

$$\begin{aligned}r_{\mathbb{F}}&(A, B, C, D) \\ &= \left\{\sup_{s\in\partial\mathbb{C}_g} \mu_{\mathbb{F}}[D + C(sI - A)^{-1}B]\right\}^{-1}.\end{aligned}$$

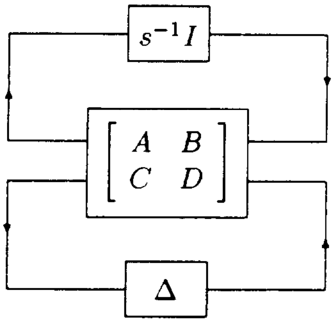


Fig. 1. Feedback interpretation of the generalized stability radius.

In the case when $\mathbb{C}_g = \{s \in \mathbb{C} : \mathrm{Re}(s) < 0\}$ or $\mathbb{C}_g = \{s \in \mathbb{C} : |s| < 1\}$, $r_{\mathbb{F}}(A, B, C, D)$ gives the smallest norm of a complex ($\mathbb{F} = \mathbb{C}$) or real ($\mathbb{F} = \mathbb{R}$) perturbation Δ that destabilizes the feedback system shown in Fig. 1.

5. EXAMPLES

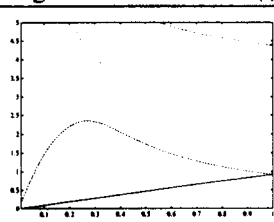
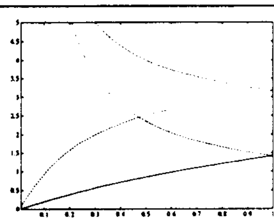
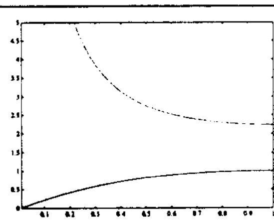
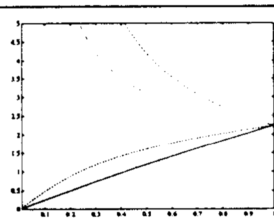
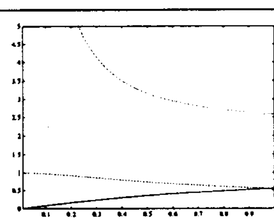
Example 1. Recall from Section 2 the notation

$$P(\gamma) = \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix}.$$

In this example we illustrate various behaviors of $\sigma_2[P(\gamma)]$ at its minimum γ^*. The data and computed results are listed in Table 1. There are essentially five possibilities:

(i) $\gamma^* \in (0, 1)$ and $\sigma_2[P(\gamma)]$ is smooth at γ^*;
(ii) $\gamma^* \in (0, 1)$ and $\sigma_2[P(\gamma)]$ is nonsmooth at γ^*;
(iii) $\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity 2;
(iv) $\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity greater than 2;
(v) $\inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)]$ is attained as $\gamma \to 0$.

Table 1. For Example 1: behavior of $\sigma_2[P(\gamma)]$

M	Singular values of $P(\gamma)$	$\mu_{\mathbb{R}}(M)$	Worst Δ	Remark
$\begin{bmatrix} 4+j & 1 \\ -1 & j \end{bmatrix}$		3.8042	$\begin{bmatrix} 0.1382 & -0.2236 \\ 0.2236 & 0.1382 \end{bmatrix}$	$\sigma_2[P(\gamma)]$ is smooth at γ^*
$\begin{bmatrix} 2+j & 1 \\ 1 & 2+j \end{bmatrix}$		2.4495	$\begin{bmatrix} 0.3333 & -0.2357 \\ 0.2357 & 0.3333 \end{bmatrix}$	$\sigma_2[P(\gamma)]$ is nonsmooth at γ^*
$\begin{bmatrix} 1+j & -1 \\ 1 & 1+j \end{bmatrix}$		2.2361	$\begin{bmatrix} 0.2000 & 0.4000 \\ -0.4000 & 0.2000 \end{bmatrix}$	$\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity 2
$\begin{bmatrix} 2+j & 0 \\ 0 & 1+j2 \end{bmatrix}$		2.2361	$\begin{bmatrix} 0.3333 & -0.2981 \\ 0.2981 & 0.3333 \end{bmatrix}$	$\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity greater than 2
$\begin{bmatrix} 1+j & 2 \\ 0 & 1 \end{bmatrix}$		2.2361	$\begin{bmatrix} 0.0000 & 0.0000 \\ 0.4000 & 0.2000 \end{bmatrix}$	$\inf_{\gamma\in(0,1]}\sigma_2[P(\gamma)]$ is attained as $\gamma \to 0$ since rank $(\mathrm{Im}\, M) = 1$

The construction of a smallest Δ such that $I-\Delta M$ is singular has to be carried out in different ways for these different possibilities, as was done in Section 2.

Example 2. Assume $\mathbb{C}_g=\{s\in\mathbb{C}:\operatorname{Re}s<0\}$. Find $r_{\mathbb{R}}(A,B,C)$ for

$$A=\begin{bmatrix}79 & 20 & -30 & -20\\ -41 & -12 & 17 & 13\\ 167 & 40 & -60 & -38\\ 33.5 & 9 & -14.5 & -11\end{bmatrix},$$

$$B=\begin{bmatrix}0.2190 & 0.9347\\ 0.0470 & 0.3835\\ 0.6789 & 0.5194\\ 0.6793 & 0.8310\end{bmatrix},$$

$$C=\begin{bmatrix}0.0346 & 0.5297 & 0.0077 & 0.0668\\ 0.0535 & 0.6711 & 0.3834 & 0.4175\end{bmatrix}.$$

We plot $\mu_{\mathbb{R}}[C(j\omega I-A)^{-1}B]$, computed using golden section search, and $\mu_{\mathbb{C}}[C(j\omega I-A)^{-1}B]$ in Fig. 2. Their maximum values are 1.9450 and 2.5546 respectively. These maxima occur at $\omega=1.38$ and $\omega=9.9$ respectively. We get $r_{\mathbb{R}}(A,B,C)=0.5141$ and $r_{\mathbb{C}}(A,B,C)=0.3914$. Note that the critical frequencies for $\mu_{\mathbb{R}}[C(j\omega I-A)^{-1}B]$ and $\mu_{\mathbb{C}}[C(j\omega I-A)^{-1}B]$ are dramatically different.

To obtain a smallest real perturbation Δ such that $A+B\Delta C$ is unstable, we need to find a smallest Δ such that $I-\Delta C(j\omega I-A)^{-1}B$ is singular at $\omega=1.38$. At this frequency, the minimum of the second singular value of

$$\begin{bmatrix}\operatorname{Re}C(j\omega I-A)^{-1}B & -\gamma\operatorname{Im}C(j\omega I-A)^{-1}B\\ \gamma^{-1}\operatorname{Im}C(j\omega I-A)^{-1}B & \operatorname{Re}C(j\omega I-A)^{-1}B\end{bmatrix}$$

occurs at $\gamma=0.2267$. Its corresponding left and right singular vectors are

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}0.3892\\ 0.9178\\ -0.0553\\ 0.0558\end{bmatrix},\quad \begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}-0.1615\\ 0.9837\\ 0.0669\\ 0.0411\end{bmatrix}.$$

A smallest real Δ is given by

$$\Delta=r_{\mathbb{R}}(A,B,C)[v_1\quad v_2][u_1\quad u_2]^{+}$$
$$=\begin{bmatrix}-0.4996 & 0.1214\\ 0.1214 & 0.4996\end{bmatrix}.$$

Example 3. Assume $\mathbb{C}_g=\{s\in\mathbb{C}:|s|<1\}$. Find $r_{\mathbb{R}}(A,B,C)$ for

$$A=\begin{bmatrix}4.7527\times10^{-1} & 7.5787\times10^{-1} & 7.9939\\ -4.1523\times10^{-2} & 8.9051\times10^{-1} & 7.5787\times10^{-1}\\ -7.5787\times10^{-2} & -4.1523\times10^{-2} & 4.7527\times10^{-1}\end{bmatrix},$$

$$B=\begin{bmatrix}8.0086\times10^{-2} & 4.2994\times10^{-2}\\ -1.4704\times10^{-3} & 9.4791\times10^{-2}\\ -4.2994\times10^{-3} & -1.4704\times10^{-3}\end{bmatrix},$$

$$C=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

We plot $\mu_{\mathbb{R}}[C(e^{j\theta}I-A)^{-1}B]$, computed using golden section search, and $\mu_{\mathbb{C}}[C(e^{j\theta}I-A)^{-1}B]$ in Fig. 3. Their maximum values are 9.6395×10^{-1} and 1.3384 respectively. Both maxima occur at $\theta=1.0053$. We get $r_{\mathbb{R}}(A,B,C)=1.0374$ and $r_{\mathbb{C}}(A,B,C)=7.4715\times10^{-1}$.

To obtain a smallest real perturbation Δ such that $A+B\Delta C$ is unstable, we need to find a smallest Δ such that $I-\Delta C(e^{j\theta}I-A)^{-1}B$ is

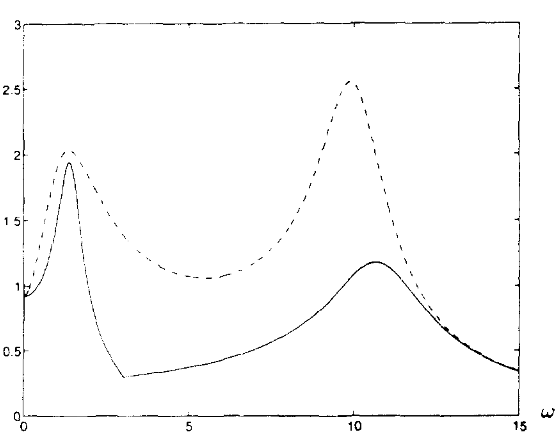


Fig. 2. For Example 2: the solid line is $\mu_{\mathbb{R}}[C(j\omega I-A)^{-1}B]$ and the dashed line is $\mu_{\mathbb{C}}[C(j\omega I-A)^{-1}B]$.

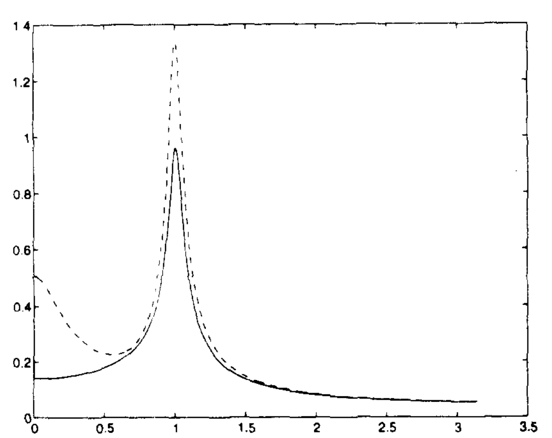


Fig. 3. For Example 3: the solid line is $\mu_{\mathbb{R}}[C(e^{j\theta}I-A)^{-1}B]$ and the dashed line is $\mu_{\mathbb{C}}[C(e^{j\theta}I-A)^{-1}B]$.

singular at $\theta = 1.0053$. At this frequency, the minimum of the second singular value of

$$\begin{bmatrix} \operatorname{Re} C(e^{j\theta}I - A)^{-1}B & -\gamma \operatorname{Im} C(e^{j\theta}I - A)^{-1}B \\ \gamma^{-1} \operatorname{Im} C(e^{j\theta}I - A)^{-1}B & \operatorname{Re} C(e^{j\theta}I - A)^{-1}B \end{bmatrix}$$

occurs at $\gamma = 2.5158 \times 10^{-1}$. Its corresponding left and right singular vectors are

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -9.6237 \times 10^{-1} \\ -1.1778 \times 10^{-1} \\ 3.0436 \times 10^{-2} \\ -2.4299 \times 10^{-1} \end{bmatrix},$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -8.5474 \times 10^{-1} \\ 4.5765 \times 10^{-1} \\ -1.1498 \times 10^{-1} \\ -2.1621 \times 10^{-1} \end{bmatrix}.$$

A smallest real Δ is given by

$$\begin{aligned} \Delta &= r_{\mathbb{R}}(A, B, C)[v_1 \quad v_2][u_1 \quad u_2]^{\dagger} \\ &= \begin{bmatrix} 8.4830 \times 10^{-1} & 5.9714 \times 10^{-1} \\ -5.9715 \times 10^{-1} & 8.4829 \times 10^{-1} \end{bmatrix}. \end{aligned}$$

6. CONCLUDING DISCUSSION

This paper has presented a formula for computation of the real stability radius. The basic problem is a pure linear algebra problem: given a complex matrix M, find the smallest real matrix Δ such that $I - \Delta M$ is singular. Our main result reduces this problem to the minimization of a unimodal function in the interval $(0, 1]$. Our proof also gives a way to construct a worst Δ such that $I - \Delta M$ is singular. This then gives a computationally efficient way to compute the real structured stability radius and to construct a smallest destabilizing Δ.

The real stability radius problem is only one application of the linear algebra problem solved in this paper. We expect more applications of our main result, which is of fundamental importance, in other scientific and engineering disciplines.

Finally, it is of interest to note that the linear algebra problem that we have considered in this paper has rather deep and rich connections to many other problems in linear algebra, in particular the theory of complex symmetric matrices (Horn and Johnson, 1985, Chapter 4). The first three authors have recently shown the following extension of Lemma 1 (the Schmidt/Mirsky approximation theorm): for $M \in \mathbb{C}^{p \times m}$, the smallest spectral norm of a real Δ such that $\operatorname{rank}(I - \Delta M) \leq m - k$ is given by (3) with σ_2 replaced by σ_{2k}. This result will be published elsewhere.

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