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A LQG Control Problem with Degree and Robustness Constraints

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Abstract: In this paper, we aim to obtain the optimal LQG controller under degree and robustness constraints. This is a mixed control problem of finding the full-order controller to provide the best nominal transient performance subject to a robust stability bound. Firstly, the set of desired controllers that meet both the degree and robust stability constraints are characterized over a convex set. Then, we carry out optimization in the convex parameter set for the controller that gives optimal transient performance. Examples and observations are presented, and comparisons with other approaches show that our approach excels.

Key Words: LQG control, Robustness, Degree, Controller parameterization, Optimization

1 INTRODUCTION

Consider the feedback system in Fig. 1. Let $T_{wz_i}, i = 1, 2$, denote the transfer functions from the exogenous input w to the controlled output z_i. One formulation of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem is to find the stabilizing controller $C(s)$ such that $\|T_{wz_1}\|_\infty \leqslant \alpha$ and $\|T_{wz_2}\|_2$ is minimized. This problem was posed over a decade ago and have remained mostly unsolved to this date.

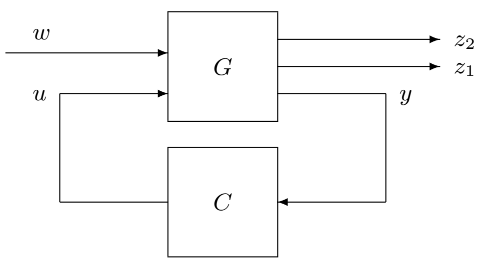

Figure 1: Feedback system for general mixed control

In such a mixed optimal control problem, the $\mathcal{H}_2$ norm is usually used to measure the transient performance with respect to a known exogenous signal, whereas the $\mathcal{H}_\infty$ norm is usually used to measure the attenuation of unknown disturbance or robustness against dynamic uncertainty. The pure $\mathcal{H}_2$ optimal control problem and the pure $\mathcal{H}_\infty$ problem, corresponding to the cases when one of z_1 and z_2 is absent, have well-known nice solutions. The optimal $\mathcal{H}_2$ controller has the same degree as the given plant, so does at least one of the controllers satisfying the $\mathcal{H}_\infty$ constraint.

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There have been quite many attempts to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9]. Bernstein and Haddad [1] gave necessary conditions for optimality of the controllers of a prespecified order. Doyle, Zhou et al. [4, 5] presented necessary and sufficient conditions for the existence of an optimal controller. However, all these conditions are given in terms of coupled Riccati equations and there are no effective procedures to solve them. Mustafa has shown that the maximum entropy $\mathcal{H}_\infty$ control problem [10] approximates a special mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem in which z_1 and z_2 are the same. In this case the so-called "central controller" minimizes an upper bound of the $\mathcal{H}_2$ norm [11]. Khargonekar and Rotea [2] used another upper bound of the $\mathcal{H}_2$ norm for the case when z_1 is not necessarily the same as z_2 and showed that minimizing this upper bound instead lead to a convex optimization problem over a finite dimensional space. Scherer et al. [6] presented a general framework for multi-objective optimal control based on the linear matrix inequality (LMI) technique and a very conservative approximation to the mixed performance specifications. Numerical experiences show that the "optimal" controller obtained in this way is far from the true optimal, even worse than the easily obtainable central controller. Moreover, the LMI formulation introduces a large number of auxiliary variables which makes the computation rather complicated. Method based on the finite dimensional approximation of the Youla parameter was proposed in [9]. The mixed control problem then reduces to a semi-definite programming (SDP) problem by using the LMI formulation of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms [6]. The drawbacks of this method are that the resulting controller is still conservative and that auxiliary variables are brought in by the LMI formulation. Also the obtained controller may have too high an order compared to the order of the given plant. Sznaier et al. [7] used the same idea and showed that when the Youla parameter is approximated by a higher and higher dimensional space, the controller obtained converges to the true optimal controller.

In [12], it has been shown that the solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is in general given by an infinite order controller and hence not practical for implementation. Therefore, it is reasonable to modify the problem by putting an explicit bound on the controller order. A reasonable such bound is the order of the given plant since the optimal $\mathcal{H}_2$ controller, the optimal $\mathcal{H}_\infty$ controller and the central $\mathcal{H}_\infty$ controllers for all pre-specified performance levels all meet this degree bound. None of the attempts mentioned above explicitly impose such a degree constraint in the controller, but some of them do result in controllers satisfying this constraint, either as the consequence of the approximation or due to the restricted class of controllers considered.

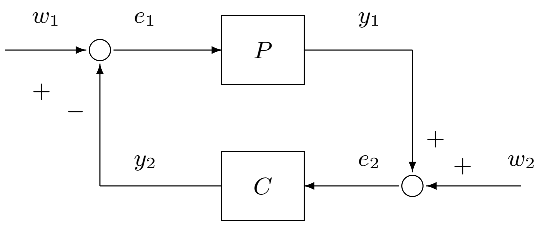

Figure 2: Feedback system concerned in this paper

In this paper, we will study a special and yet important mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem with the controller degree constraint explicitly spelled out. In particular, we will focus on the feedback system in Fig. 2. The problem to be dealt with in this paper is as follows:

LQG control problem with degree and robustness constraints: Given an n-th order plant $P(s)$, find the stabilizing controller $C(s)$ that

- $\deg C(s) \leqslant n$,
- $b_{P,C} \geqslant \beta$,
- $\|T_{wy}\|_2$ is minimized,

where

$$b_{P,C} = \left\| \begin{bmatrix} 1 \\ C(s) \end{bmatrix} [1+P(s)C(s)]^{-1} [\ 1 \quad P(s)\ ] \right\|_\infty^{-1} \quad (1)$$

$$T_{wy} = \begin{bmatrix} P(s) & P(s)C(s) \\ C(s)P(s) & C(s) \end{bmatrix} [1+C(s)P(s)]^{-1}. \quad (2)$$

Here, T_{wy} is the the closed-loop transfer function from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\|T_{wy}\|_2^2$ is the standard LQG cost function. The quantity $b_{P,C}$ is called *robust stability margin*, a measure of the system robust stability against gap metric [13], ν-gap metric [14], and normalized coprime factor [15] uncertainties. Thus, the requirement $b_{P,C} \geqslant \beta$ ensures certain level of closed-loop robustness.

The problem will be solved in several steps. We firstly show that the set of controllers that meet both the degree and the robust stability constraints can be parameterized by a set of $\mathcal{H}_\infty$ functions satisfying an interpolation condition and with a bounded norm and a bounded degree. We then borrow the ideas in [16, 17] to parameterize this set further by a finite dimensional convex set. Finally $\mathcal{H}_2$ optimization over this convex set will be performed to find the controller that minimizes $\|T_{wy}\|_2$.

2 CONTROLLER PARAMETERIZATION

In this section, we will parameterize the required stabilizing controller set $\mathcal{C}_{n,\beta}$ for a given plant $P(s)$:

$$\mathcal{C}_{n,\beta} \triangleq \{C(s) : \deg C(s) \leqslant n, \text{ and } b_{P,C} \geqslant \beta.\} \quad (3)$$

2.1 Degree and robustness constraints

By Equation (1),

$$\frac{1}{b_{P,C}} = \left\| \begin{bmatrix} 1 \\ C(s) \end{bmatrix} [1+P(s)C(s)]^{-1} [\ 1 \quad P(s)\ ] \right\|_\infty = \max_{\omega\in\mathbb{R}} \sqrt{\frac{[1+|P(j\omega)|^2][1+|C(j\omega)|^2]}{|1+P(j\omega)C(j\omega)|^2}}.$$

Since $|P(j\omega)|^2 = P(j\omega)P(-j\omega)$, it follows that

$$\frac{[1+|P(j\omega)|^2][1+|C(j\omega)|^2]}{|1+P(j\omega)C(j\omega)|^2} = 1+\left|\frac{P(-j\omega)-C(j\omega)}{1+P(j\omega)C(j\omega)}\right|^2.$$

So we have

$$b_{P,C} = \frac{1}{\sqrt{1+\max_{\omega\in\mathbb{R}}\left|\frac{P(-j\omega)-C(j\omega)}{1+P(j\omega)C(j\omega)}\right|^2}} = \left\{1+\left\|\frac{P(-s)-C(s)}{1+P(s)C(s)}\right\|_\infty^2\right\}^{-\frac{1}{2}}. \quad (4)$$

Suppose the plant and the controller have strictly proper transfer functions:

$$P(s) = \frac{b(s)}{a(s)}, \quad C(s) = \frac{h(s)}{g(s)}, \quad (5)$$

where $a(s)$ and $p(s)$ are both monic polynomials. So,

$$\frac{P(-s)-C(s)}{1+P(s)C(s)} = \frac{b(-s)g(s)-a(-s)h(s)}{a(s)g(s)+b(s)h(s)}\frac{a(s)}{a(-s)}.$$

Since $\frac{a(s)}{a(-s)}$ is all-pass, $b_{P,C}$ can be written as

$$b_{P,C} = \left\{1+\left\|\frac{b(-s)g(s)-a(-s)h(s)}{a(s)g(s)+b(s)h(s)}\right\|_\infty^2\right\}^{-\frac{1}{2}}. \quad (6)$$

Denote

$$F(s) = \frac{q(s)}{p(s)} \triangleq \frac{b(-s)g(s)-a(-s)h(s)}{a(s)g(s)+b(s)h(s)}, \quad (7)$$

and then

$$b_{P,C} \geqslant \beta \iff \|F(s)\|_\infty \leqslant \gamma \tag{8}$$

where

$$\gamma = \sqrt{\frac{1}{\beta^2} - 1}. \tag{9}$$

By the degree constraint

$$\deg b(s) < \deg a(s) = n, \quad \deg h(s) < \deg g(s) \leqslant n,$$

we get that $F(s)$ is also strictly proper and

$$\deg F(s) \leqslant 2n.$$

Further, $p(s)$ is monic.

2.2 Analytic interpolation

For any controller $C(s) \in \mathcal{C}_{n,\beta}$, its degree is bounded by n, and thus the degree of the function $F(s)$ in Equation (7) is bounded by $2n$. In this subsection, we will develop an analytic interpolation condition for $F(s)$. Denote:

$$a = \begin{bmatrix} 1 \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}, b = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}, g = \begin{bmatrix} 1 \\ g_{n-1} \\ \vdots \\ g_1 \\ g_0 \end{bmatrix}, h = \begin{bmatrix} h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \\ h_0 \end{bmatrix},$$

$$p = \begin{bmatrix} 1 \\ p_{2n-1} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ q_{2n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix}.$$

Since $p(s) = a(s)g(s) + b(s)h(s)$, we have:

$$p(s) = \begin{bmatrix} s^{2n} & s^{2n-1} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} 1 \\ p_{2n-1} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix}$$

$$= \begin{bmatrix} s^{2n} & s^{2n-1} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & 1 \\ 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ g_{n-1} \\ \vdots \\ g_1 \\ g_0 \end{bmatrix}$$

$$+ \begin{bmatrix} s^{2n-2} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} b_{n-1} & 0 & \cdots & 0 \\ b_{n-2} & b_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_0 & b_1 & \cdots & b_{n-1} \\ 0 & b_0 & \cdots & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_0 \end{bmatrix} \begin{bmatrix} h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \\ h_0 \end{bmatrix}.$$

Comparing the coefficients on both sides of the equation,

$$\begin{bmatrix} 1 \\ p_{2n-1} \\ \vdots \\ p_n \\ p_{n-1} \\ p_{n-2} \\ \vdots \\ p_0 \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{n-2} & a_{n-1} & \cdots & 0 & b_{n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & \cdots & 1 & b_1 & b_2 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{n-1} & b_0 & b_1 & \cdots & b_{n-1} \\ 0 & 0 & \cdots & a_{n-2} & 0 & b_0 & \cdots & b_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{array}\right] \begin{bmatrix} 1 \\ g_{n-1} \\ \vdots \\ g_0 \\ \hline h_{n-1} \\ h_{n-2} \\ \vdots \\ h_0 \end{bmatrix}.$$

We denote the above equation as:

$$p = Mv. \tag{10}$$

The $(2n+1) \times (2n+1)$ matrix M, called a Sylvester's resultant matrix, is nonsingular if $a(s)$ and $b(s)$ are coprime polynomials [18]. Thus,

$$\begin{bmatrix} g \\ h \end{bmatrix} = v = M^{-1}p. \tag{11}$$

In a similar way, $q(s) = b(-s)g(s) - a(-s)h(s)$ yields $q = Nv$, where

$$N = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ (-1)^{n-1}b_{n-1} & 0 & \cdots & 0 & -(-1)^n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_0 & -b_1 & \cdots & 0 & a_1 & -a_2 & \cdots & -(-1)^n \\ 0 & b_0 & \cdots & (-1)^{n-1}b_{n-1} & -a_0 & a_1 & \cdots & -(-1)^{n-1}a_{n-1} \\ 0 & 0 & \cdots & (-1)^{n-2}b_{n-2} & 0 & a_0 & \cdots & -(-1)^{n-2}a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_0 & 0 & 0 & \cdots & -a_0 \end{array}\right] \tag{12}$$

is a $(2n+1) \times (2n+2)$ matrix. The coefficients of $q(s)$ can be completely determined by those of $p(s)$:

$$q = \Gamma_0 p, \quad \Gamma_0 \triangleq NM^{-1}. \tag{13}$$

Define $\mathcal{F}$ as the set of functions $F(s) = \dfrac{q(s)}{p(s)}$ such that:

1. $q = \Gamma_0 p$,
2. $\deg F(s) \leqslant 2n$,
3. $F(s) \in \mathcal{H}_\infty$ and $\|F(s)\|_\infty \leqslant \gamma$.

Thus, $\mathcal{F}$ is a set of $\mathcal{H}_\infty$ functions with a bounded norm, a bounded degree and satisfying the interpolation condition. Obviously, for any controller $C(s) = \dfrac{h(s)}{g(s)} \in \mathcal{C}_{n,\beta}$, the corresponding $F(s)$ by Equation (7) is in $\mathcal{F}$. Conversely, for any function $F(s) = \dfrac{q(s)}{p(s)} \in \mathcal{F}$, we can get the unique pair $(g(s), h(s))$ from Equation (11), and $\deg g(s) = n$, $\deg h(s) = n-1$. Consequently, the order of the controller $C(s) = \dfrac{h(s)}{g(s)}$, is bounded by n. Now the following theorem is a direct result of the derivations.

Theorem 2.1 *$C(s) \in \mathcal{C}_{n,\beta}$ if and only if $F(s) \in \mathcal{F}$, where $C(s)$ and $F(s)$ are related by Equation (5) and (7).*

2.3 A quadratic map

The robustness constraint gives:

$$\begin{aligned} b_{P,C} \geqslant \beta &\Leftrightarrow \|F(s)\|_\infty \leqslant \gamma \\ &\Leftrightarrow \quad p(j\omega)p(-j\omega) - \frac{1}{\gamma^2} q(j\omega)q(-j\omega) \geqslant 0, \ \forall\, \omega \in \mathbb{R}. \end{aligned}$$

Hence, there exists an $2n$-th order polynomial $\sigma(s)$, which has no zero in the open right half plane, such that

$$p(s)p(-s) - \frac{1}{\gamma^2} q(s)q(-s) = \sigma(s)\sigma(-s). \tag{14}$$

Consider the polynomial multiplication $p(s)p(-s)$, whose coefficients of $s, s^3, \cdots, s^{4n-1}$ are all zeros,

$$\begin{aligned} p(s)p(-s) &= p' \begin{bmatrix} s^{2n} \\ \vdots \\ s \\ 1 \end{bmatrix} \begin{bmatrix} s^{2n} & \cdots & -s & 1 \end{bmatrix} p \\ &= \begin{bmatrix} s^{4n} & \cdots & s^2 & 1 \end{bmatrix} \begin{bmatrix} p'A_{2n}p \\ \vdots \\ p'A_1 p \\ p'A_0 p \end{bmatrix}, \end{aligned} \tag{15}$$

where for $k = 0, 1, \cdots, 2n$,

$$A_k(i,j) \triangleq \begin{cases} (-1)^{i-1}, & i+j = 2(2n+1-k) \\ 0, & \text{else} \end{cases}.$$

We write $q(s)q(-s)$ and $\sigma(s)\sigma(-s)$ in similar forms, and then Equation (14) becomes

$$\begin{bmatrix} p'A_{2n}p \\ p'(A_{2n-1} - \Gamma' A_{2n-1}\Gamma)p \\ \vdots \\ p'(A_1 - \Gamma' A_1 \Gamma)p \\ p'(A_0 - \Gamma' A_0 \Gamma)p \end{bmatrix} = \begin{bmatrix} \sigma' A_{2n}\sigma \\ \sigma' A_{2n-1}\sigma \\ \vdots \\ \sigma' A_1 \sigma \\ \sigma' A_0 \sigma \end{bmatrix} \triangleq \begin{bmatrix} \psi_{2n} \\ \psi_{2n-1} \\ \vdots \\ \psi_1 \\ \psi_0 \end{bmatrix} \tag{16}$$

where

$$\Gamma \triangleq \frac{1}{\gamma}\Gamma_0.$$

The fact $p(s)$ is monic and $F(s)$ is strictly proper results in that $\sigma(s)$ is also monic, $p'A_{2n}p = 1$ and then $\psi_{2n} = 1$.

Denote two sets $\bar{\mathcal{P}}$ and $\bar{\Psi}$ as:

$$\bar{\mathcal{P}} \triangleq \left\{ p : \begin{array}{l} p(s) \text{ is stable and monic}, \deg p(s) = 2n, \\ \left\| \dfrac{(\Gamma p)(s)}{p(s)} \right\|_\infty \leqslant 1. \end{array} \right\}$$

$$\bar{\Psi} \triangleq \left\{ \psi : \begin{array}{l} \psi(s^2) = \sigma(s)\sigma(-s), \\ \deg \sigma(s) = 2n, \sigma(s) \text{ is monic}, \\ \sigma(s) \text{ has no zero in the open right half plane.} \end{array} \right\}$$

Therefore,

$$p \in \bar{\mathcal{P}} \iff F(s) = \gamma \frac{(\Gamma p)(s)}{p(s)} \in \mathcal{F}. \tag{17}$$

Equation (16) defines a quadratic map from $\bar{\mathcal{P}}$ to $\bar{\Psi}$:

$$G(p) \quad \triangleq \quad \begin{bmatrix} p'(A_{2n} - \Gamma' A_{2n}\Gamma)p \\ p'(A_{2n-1} - \Gamma' A_{2n-1}\Gamma)p \\ \vdots \\ p'(A_0 - \Gamma' A_0 \Gamma)p \end{bmatrix}. \tag{18}$$

Directly, the Jacobian matrix of the map $G(p)$ is

$$J_{G(p)} = 2 \times \begin{bmatrix} p'(A_{2n} - \Gamma' A_{2n}\Gamma) \\ p'(A_{2n-1} - \Gamma' A_{2n-1}\Gamma) \\ \vdots \\ p'(A_0 - \Gamma' A_0 \Gamma) \end{bmatrix}. \tag{19}$$

The map G and its Jacobian $J_{G(p)}$ have the nice properties as stated in the following theorems, which were proven in [19] based on previous work of Byrnes and Linquist [20].

Theorem 2.2 *The map $G : \bar{\mathcal{P}} \to \bar{\Psi}$, is a bijection.*

Theorem 2.3 *$J_{G(p)}$, the Jacobian matrix of the map G from $\bar{\mathcal{P}}$ to $\bar{\Psi}$, is invertible in the interior of $\bar{\mathcal{P}}$.*

Given an n-th order plant $P(s)$ and the robust stability margin requirement $b_{P,C} \geqslant \beta$, the interpolation matrix Γ (or Γ_0) can be obtained by simple computation. Theorem 2.1, Formula (17) and Theorem 2.2 have established the equivalence of $\mathcal{C}_{n,\beta}$ and $\mathcal{F}$, $\mathcal{F}$ and $\bar{\mathcal{P}}$, $\bar{\mathcal{P}}$ and $\bar{\Psi}$, respectively. i.e.,

$$\mathcal{C}_{n,\beta} \Longleftrightarrow \mathcal{F} \overset{\gamma \frac{(\Gamma p)(s)}{p(s)}}{\Longleftrightarrow} \bar{\mathcal{P}} \overset{G(p)}{\rightleftarrows} \bar{\Psi}.$$

Now the controller set $\mathcal{C}_{n,\beta}$ has been characterized by the parameter set $\bar{\Psi}$. However, for a given $\psi \in \bar{\Psi}$, there is no analytic way to get the $p \in \bar{\mathcal{P}}$ such that $\psi = G(p)$. In next section a numerical method will be introduced to solve Equation (16) utilizing $J^{-1}_{G(p)}$, the inverse of the Jacobian matrix.

2.4 Parameterization over a convex set

Consider an $2n$-th degree monic and stable polynomial

$$\sigma(s) = s^{2n} + \sigma_{2n-1}s^{2n-1} + \cdots + \sigma_1 s + \sigma_0.$$

For stable $\sigma(s)$ and $\psi(s^2) = \sigma(s)\sigma(-s)$, $\psi \in \Psi$. The Routh criteria tells that $\sigma(s)$ is stable if and only if all elements in the first column of its Routh table are positive. We construct the Routh table as the following tabular.

Construction of the Routh Table

r_{2n}	$r_{(2n)0} = 1$	$r_{(2n)1} = \sigma_{2n-2}$	$\cdots$
r_{2n-1}	$r_{(2n-1)0} = \sigma_{2n-1}$	$r_{(2n-1)1} = \sigma_{2n-3}$	$\cdots$
r_{2n-2}	$r_{(2n-2)0}$	$r_{(2n-2)1}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	
r_2	r_{20}	r_{21}	
r_1	r_{10}		
r_0	r_{00}		

Each row from the third one is computed as

$$r_{ij} = \frac{1}{r_{(i+1)0}} \begin{vmatrix} r_{(i+2)0} & r_{(i+2)(j+1)} \\ r_{i(i+1)0} & r_{(i+1)(j+1)} \end{vmatrix},$$

where $i = 2n-2, \ldots, 0,\ j = 0, \cdots, \left[\frac{i}{2}\right]$. Denote the first column of the Routh table for $\sigma(s)$ by r, and also denote the map from $\psi \in \Psi$ to r and the inverse map as:

$$r = \varphi(\psi), \quad \psi = \varphi^{-1}(r).$$

It is easy to get r from ψ by the definition. As for the inverse map, we reconstruct the Routh table from r, and then get σ or ψ. Since $\sigma(s)$ is stable if and only if $r_k \in \mathbb{R}_+$ for $k = 0, 1, \cdots, 2n-1$, we have parameterized the interior of $\bar{\Psi}$ over the open convex set $\mathbb{R}_+^{2n}$.

3 $\mathcal{H}_2$ OPTIMIZATION

This section is to solve the following problem: Given a $\psi \in \bar{\Psi}$, find the corresponding $p \in \bar{\mathcal{P}}$ such that $\psi = G(p)$. This is to solve the following $2n$ quadratic equations

$$p'\Lambda_k p = \psi_k, \quad k = 0, 1, \cdots, 2n-1. \qquad (20)$$

where $\Lambda_k = A_k - \Gamma' A_k \Gamma$ are real symmetric, $\psi_{2n} = 1$ by definition and $p \in \bar{\mathcal{P}}$. We have $2n$ equations with $2n$ variables: $p_{2n-1}, \cdots, p_1, p_0$. It turns out difficult to solve the equations and there is no analytic solution available. In [16], a generalized entropy criterion has been developed to compute bounded degree Nevannlina-Pick interpolants. Here, we take a numerical continuation approach based on Theorem 2.3 that $J_{G(p)}$, the Jacobian of $G(p)$, is invertible for p in the interior of $\bar{\mathcal{P}}$. Let $H : \mathbb{R}^{2n+1} \times [0,1] \to \mathbb{R}^{2n+1}$ be the convex homotopy defined as:

$$\begin{aligned} H(p,\nu) &\triangleq (1-\nu)(G(p) - \bar{\psi}) + \nu(G(p) - \psi), \\ &= G(p) - \bar{\psi} + \nu(\bar{\psi} - \psi),\ \nu \in [0,1] \quad (21) \end{aligned}$$

where $(\bar{p}, \bar{\psi} = G(\bar{p}))$ is a pair of known initial point. Theorem 2.2 guarantees that for each $\nu \in [0,1]$, the equation

$$H(p,\nu) = 0, \quad \text{i.e.,} \quad G(p) = (1-\nu)\bar{\psi} + \nu\psi$$

has a unique solution in the interior of $\bar{\mathcal{P}}$, denoted as $\hat{p}(\nu)$. We call the set $\{\hat{p}(\nu)\}_{\nu=0}^{1}$ the *trajectory*. The initial point of the trajectory, $\hat{p}(0)$, is the solution to $G(p) = \bar{\psi}$ and the end point, $\hat{p}(1)$, is the solution to $G(p) = \psi$. We trace the trajectory from $\nu = 0$ to $\nu = 1$, and then eventually obtain $\hat{p}(1)$ which is the preimage of ψ. In this process, $J_{G(p)}^{-1}$ is used to predict the position of $\hat{p}(\nu + \delta\nu)$ from $\hat{p}(\nu)$, where $\delta\nu$ is the step length. Details are available in [19] or [21]. Now, we can carry out optimization in the convex parameter set to search the solution that minimizes the LQG cost function $\|T_{wy}\|_2^2$, and try to answer the following questions: which parameter r corresponds to the optimal $C(s)$ that minimizes $\|T_{wy}\|_2$? Is the optimal controller $C(s)$ in the interior of $\mathcal{C}_{n,\beta}$, or on the boundary?

4 EXAMPLES AND OBSERVATIONS

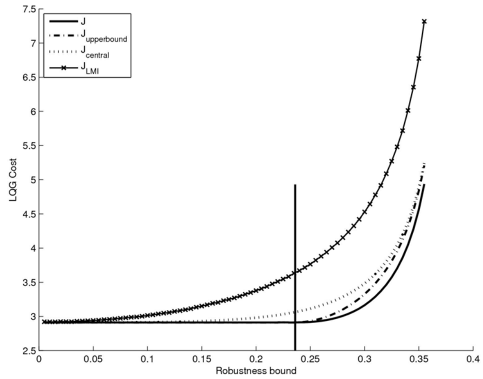

Figure 3: Comparison of controllers for $P(s) = \frac{1}{s^2}$

The double integrator, $P(s) = \frac{1}{s^2}$, is taken as the benchmark plant to test the performances of each methods. The optimal $b_{P,C}$ given by the optimal $\mathcal{H}_\infty$ controller is 0.3827. Fig. 3 shows the performances of our controllers, the central controllers for various γ [11], controllers obtained by the LMI method [6], and also the upper bounds [22], etc. For a specific robustness bound $\beta = 0.35$, we have:

1. The central controller:

$$C(s) = \frac{17.72s + 7.14}{s^2 + 9.308s + 18.92},$$

$$b_{P,C} = 0.35264, \|T_{yw}\|_2 = 4.8873.$$

2. The LMI controller:

$$C(s) = \frac{17.8s + 7.172}{s^2 + 9.068s + 18.99},$$

$$b_{P,C} = 0.3500, \|T_{yw}\|_2 = 6.7732.$$

3. The upper bound of $\|T_{yw}\|_2$ is: 4.8383.

4. Our controller:

$$C(s) = \frac{14.64s + 5.926}{s^2 + 7.519s + 15.8},$$

$$b_{P,C} = 0.3500, \|T_{yw}\|_2 = 4.5652.$$

Here, the robust stability margins corresponding to the LMI controller and our controller are still greater than the bound β, but they are very close. The central controller is less conservative than the LMI controller, and our controller gives the best transient performance.

Observations are obtained based on numerical examples:

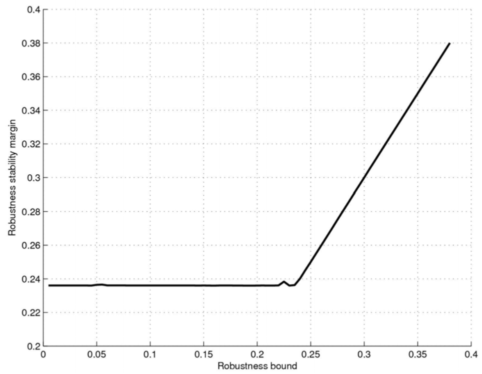

Figure 4: $b_{P,C}$ bound and $b_{P,C}$ for $P(s) = \frac{1}{s^2}$

1. The LQG cost $\|T_{wy}\|_2^2$, is unimodal under our parameterization, which will ensure that the minimum point found by optimization is indeed the global solution.

2. See Fig. 3. In case that the optimal $\mathcal{H}_2$ controller is a member of $\mathcal{C}_{n,\beta}$, obviously it is the optimal controller we are looking for. Our algorithm indeed gives it as the solution, while neither the central controller nor the LMI controller does.

3. See Fig. 4. In the non-trivial case that the optimal $\mathcal{H}_2$ controller is not in $\mathcal{C}_{n,\beta}$, the solution provided by our algorithm to the degree and robustness constrained LQG problem is almost on the boundary of $\mathcal{C}_{n,\beta}$. i.e., $b_{P,C} \approx \beta$. This drops a hint that we should explore the boundary case in future.

Although theoretical justifications are not available at this moment, the observations hold for the numerical examples we ran and promisingly lead the way for our further exploration of the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem.

5 CONCLUSIONS

This paper deals with a special mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem. We aim to find the optimal LQG controller under degree and robustness constraints. The set of stabilizing controllers that satisfy both the degree and robustness requirements is characterized over a finite dimensional convex set, in which $\mathcal{H}_2$ optimization is performed to obtain the desired controller that provides optimal nominal transient performance. Examples and observations that may offer insights for future research have been presented. Comparisons show that our method does work better than existing approaches.

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