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A Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem with Controller Degree Constraint

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***Abstract*— In this paper, we consider a special mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem which is to obtain the optimal $\mathcal{H}_2$ controller under $\mathcal{H}_\infty$ norm and controller degree constraints for an SISO plant. The $\mathcal{H}_\infty$ norm constraint on the controller is to achieve a certain level of closed-loop stability robustness against the gap metric, ν-gap metric, or normalized coprime factor uncertainties. The degree constraint requires the degree of the controller to be bounded by that of the plant. We first characterize the set of all feasible stabilizing controllers that meet both the degree and $\mathcal{H}_\infty$ norm constraints in terms of a finite dimensional convex set. We then carry out an optimization over the convex parameter set to find the controller that gives the optimal $\mathcal{H}_2$ transient performance. Examples and observations are presented, and comparisons with the results obtained by other $\mathcal{H}_2/\mathcal{H}_\infty$ mixed optimization methods are made.**

I. INTRODUCTION

Consider the feedback system in Fig. 1. Let $T_{wz_i}, i = 1, 2$, denote the transfer functions from the exogenous input w to the controlled output z_i. One formulation of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem is to find the stabilizing controller $C(s)$ such that $\|T_{wz_1}\|_\infty \leqslant \alpha$ and $\|T_{wz_2}\|_2$ is minimized. This problem was posed over a decade ago and have remained mostly unsolved to this date.

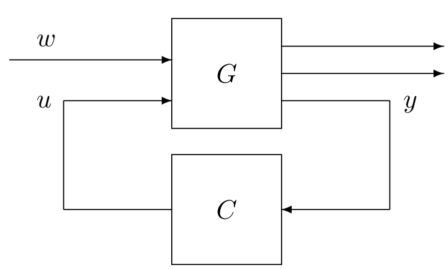

Fig. 1. Feedback system for general mixed control

In such a mixed optimal control problem, the $\mathcal{H}_2$ norm is usually used to measure the transient performance with respect to a known exogenous signal, whereas the $\mathcal{H}_\infty$ norm is usually used to measure the attenuation of unknown disturbance or robustness against dynamic uncertainty. The pure $\mathcal{H}_2$ optimal control problem and the pure $\mathcal{H}_\infty$ problem, corresponding to the cases when one of z_1 and z_2 is absent, have well-known nice solutions. The optimal $\mathcal{H}_2$ controller has the same degree as the given plant, so does at least one of the controllers satisfying the $\mathcal{H}_\infty$ constraint.

There have been quite many attempts to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem, see for example [1], [2], [3], [4], [5], [6], [7], [8], [9]. Bernstein and Haddad [1] gave necessary conditions for optimality of the controllers of a pre-specified order. Doyle, Zhou et al. [4], [5] presented necessary and sufficient conditions for the existence of an optimal controller. However, all these conditions are given in terms of coupled Riccati equations and there are no effective procedures to solve them. Mustafa has shown that the maximum entropy $\mathcal{H}_\infty$ control problem [10] approximates a special mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem in which z_1 and z_2 are the same. In this case the so-called "central controller" minimizes an upper bound of the $\mathcal{H}_2$ norm [11]. Khargonekar and Rotea [2] used another upper bound of the $\mathcal{H}_2$ norm for the case when z_1 is not necessarily the same as z_2 and showed that minimizing this upper bound instead lead to a convex optimization problem over a finite dimensional space. Scherer et al. [6] presented a general framework for multi-objective optimal control based on the linear matrix inequality (LMI) technique and an approximation to the mixed performance specifications. While this framework is able to deal with diverse mixed control problems, numerical experiences show that the "optimal" controller obtained in this way is quite conservative. In all the examples we have tested, the LMI results are not as good as the central controllers. Moreover, the LMI formulation introduces a large number of auxiliary variables which makes the computation rather complicated. Method based on the finite dimensional approximation of the Youla parameter was proposed in [9]. The mixed control problem then reduces to a semi-definite programming (SDP) problem by using the LMI formulation of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms [6]. The drawbacks of this method are that the resulting controller is still conservative and that auxiliary variables are brought in by the LMI formulation. Also the obtained controller may have too high an order compared to the order of the given plant. Sznaier et al. [7] used the same idea and showed that when the Youla parameter is approximated by a higher and higher dimensional space, the controller obtained converges to the true optimal controller.

In [12], it has been shown that the solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem is in general given by an infinite order controller and hence not practical for implementation. Therefore, it is reasonable to modify the problem by putting an explicit bound on the controller order. A reasonable such bound is the order of the given plant since the optimal $\mathcal{H}_2$ controller, the

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optimal $\mathcal{H}_\infty$ controller and the central $\mathcal{H}_\infty$ controllers for all pre-specified performance levels all meet this degree bound. None of the attempts mentioned above explicitly impose such a degree constraint in the controller, but some of them do result in controllers satisfying this constraint, either as the consequence of the approximation or due to the restricted class of controllers considered.

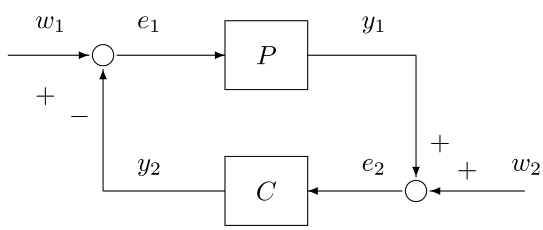

Fig. 2. Feedback system concerned in this paper

In this paper, we will study a special and yet important mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem with the controller degree constraint explicitly spelled out. In particular, we will focus on the feedback system in Fig. 2. The problem to be dealt with in this paper is as follows:

A mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem with controller degree constraint: Given an n-th order plant $P(s)$, find the stabilizing controller $C(s)$ such that

- $\deg C(s) \leqslant n$,
- $b_{P,C} \geqslant \beta$,
- $\|T_{wy}\|_2$ is minimized,

where

$$b_{P,C} = \left\| \begin{bmatrix} 1 \\ C(s) \end{bmatrix} [1+P(s)C(s)]^{-1} [\ 1 \quad P(s)\ ] \right\|_\infty^{-1} \quad (1)$$

$$T_{wy} = \begin{bmatrix} P(s) & P(s)C(s) \\ C(s)P(s) & C(s) \end{bmatrix} [1+C(s)P(s)]^{-1}. \quad (2)$$

Here, T_{wy} is the the closed-loop transfer function from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and its $\mathcal{H}_2$ norm $\|T_{wy}\|_2$ is the transient cost function. The quantity $b_{P,C}$, inverse of an $\mathcal{H}_\infty$ norm, is called *robust stability margin*. It has been demonstrated that this $b_{P,C}$ is a measure of the system robust stability against gap metric [13], ν-gap metric [14], and normalized coprime factor [15] uncertainties. Thus, the requirement $b_{P,C} \geqslant \beta$ ensures a certain level of closed-loop robustness.

The problem will be solved in several steps. We firstly show that the set of controllers that meet $\deg C(s) \leqslant n$, the controller degree constraint, and $b_{P,C} \geqslant \beta$, the robust stability constraint, can be parameterized by a set of $\mathcal{H}_\infty$ functions satisfying an interpolation condition and with a bounded norm and a bounded degree. We then borrow the ideas in [16], [17] to parameterize this set further by a finite dimensional convex set. Finally $\mathcal{H}_2$ optimization over this convex set will be performed to find the controller that minimizes $\|T_{wy}\|_2$.

II. CONTROLLER PARAMETERIZATION

In this section, we will parameterize the required stabilizing controller set $\mathcal{C}_{n,\beta}$ for a given plant $P(s)$:

$$\mathcal{C}_{n,\beta} \triangleq \{C(s) : \deg C(s) \leqslant n, \text{ and } b_{P,C} \geqslant \beta.\} \quad (3)$$

A. Controller degree and closed-loop robustness constraints

By Equation (1),

$$\frac{1}{b_{P,C}} = \left\| \begin{bmatrix} 1 \\ C(s) \end{bmatrix} [1+P(s)C(s)]^{-1} [\ 1 \quad P(s)\ ] \right\|_\infty = \max_{\omega\in\mathbb{R}} \sqrt{\frac{[1+|P(j\omega)|^2][1+|C(j\omega)|^2]}{|1+P(j\omega)C(j\omega)|^2}}.$$

Since $|P(j\omega)|^2 = P(j\omega)P(-j\omega)$, it follows that

$$\frac{[1+|P(j\omega)|^2][1+|C(j\omega)|^2]}{|1+P(j\omega)C(j\omega)|^2} = 1 + \left| \frac{P(-j\omega)-C(j\omega)}{1+P(j\omega)C(j\omega)} \right|^2.$$

So we have

$$b_{P,C} = \frac{1}{\sqrt{1+\max_{\omega\in\mathbb{R}} \left| \frac{P(-j\omega)-C(j\omega)}{1+P(j\omega)C(j\omega)} \right|^2}} = \left\{ 1 + \left\| \frac{P(-s)-C(s)}{1+P(s)C(s)} \right\|_\infty^2 \right\}^{-\frac{1}{2}}. \quad (4)$$

Suppose the plant and the controller have strictly proper transfer functions:

$$P(s) = \frac{b(s)}{a(s)}, \quad C(s) = \frac{h(s)}{g(s)}, \quad (5)$$

where $a(s)$ and $p(s)$ are both monic polynomials. So,

$$\frac{P(-s)-C(s)}{1+P(s)C(s)} = \frac{b(-s)g(s)-a(-s)h(s)}{a(s)g(s)+b(s)h(s)} \frac{a(s)}{a(-s)}.$$

Since $\frac{a(s)}{a(-s)}$ is all-pass, the robust stability margin $b_{P,C}$ can be written as

$$b_{P,C} = \left\{ 1 + \left\| \frac{b(-s)g(s)-a(-s)h(s)}{a(s)g(s)+b(s)h(s)} \right\|_\infty^2 \right\}^{-\frac{1}{2}}. \quad (6)$$

Denote

$$F(s) = \frac{q(s)}{p(s)} \triangleq \frac{b(-s)g(s)-a(-s)h(s)}{a(s)g(s)+b(s)h(s)}, \quad (7)$$

and then

$$b_{P,C} \geqslant \beta \iff \|F(s)\|_\infty \leqslant \gamma \quad (8)$$

where

$$\gamma = \sqrt{\frac{1}{\beta^2} - 1}. \quad (9)$$

By the degree constraint

$$\deg b(s) < \deg a(s) = n, \quad \deg h(s) < \deg g(s) \leqslant n,$$

we get that $F(s)$ is also strictly proper and

$$\deg F(s) \leqslant 2n.$$

Further, $p(s)$ is a monic polynomial.

B. Analytic interpolation

For any controller $C(s) \in \mathcal{C}_{n,\beta}$, its degree is bounded by n, and thus the degree of the function $F(s)$ in Equation (7) is bounded by $2n$. In this subsection, we will develop an analytic interpolation condition for $F(s)$. Denote:

$$a = \begin{bmatrix} 1 \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}, b = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}, g = \begin{bmatrix} 1 \\ g_{n-1} \\ \vdots \\ g_1 \\ g_0 \end{bmatrix}, h = \begin{bmatrix} h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \\ h_0 \end{bmatrix},$$

$$p = \begin{bmatrix} 1 \\ p_{2n-1} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ q_{2n-1} \\ \vdots \\ q_1 \\ q_0 \end{bmatrix}.$$

Since $p(s) = a(s)g(s) + b(s)h(s)$, we have:

$$p(s) = \begin{bmatrix} s^{2n} & s^{2n-1} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} 1 \\ p_{2n-1} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix}$$

$$= \begin{bmatrix} s^{2n} & s^{2n-1} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & 1 \\ 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ g_{n-1} \\ \vdots \\ g_1 \\ g_0 \end{bmatrix}$$

$$+ \begin{bmatrix} s^{2n-2} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} b_{n-1} & 0 & \cdots & 0 \\ b_{n-2} & b_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_0 & b_1 & \cdots & b_{n-1} \\ 0 & b_0 & \cdots & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_0 \end{bmatrix} \begin{bmatrix} h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \\ h_0 \end{bmatrix}.$$

Comparing the coefficients on both sides of the equation,

$$\begin{bmatrix} 1 \\ p_{2n-1} \\ p_{2n-2} \\ \vdots \\ p_n \\ p_{n-1} \\ p_{n-2} \\ \vdots \\ p_0 \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{n-1} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{n-2} & a_{n-1} & \cdots & 0 & b_{n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0 & a_1 & \cdots & 1 & b_1 & b_2 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{n-1} & b_0 & b_1 & \cdots & b_{n-1} \\ 0 & 0 & \cdots & a_{n-2} & 0 & b_0 & \cdots & b_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{array}\right] \begin{bmatrix} 1 \\ g_{n-1} \\ \vdots \\ g_0 \\ -- \\ h_{n-1} \\ h_{n-2} \\ \vdots \\ h_0 \end{bmatrix}.$$

The above equation can be written as:

$$p = M \begin{bmatrix} g \\ h \end{bmatrix}. \tag{10}$$

The $(2n+1) \times (2n+1)$ matrix M, called a Sylvester's resultant matrix, is nonsingular if $a(s)$ and $b(s)$ are coprime polynomials [18]. Thus,

$$\begin{bmatrix} g \\ h \end{bmatrix} = M^{-1} p. \tag{11}$$

In a similar way, $q(s) = b(-s)g(s) - a(-s)h(s)$ yields

$$q = N \begin{bmatrix} g \\ h \end{bmatrix},$$

where

$$N = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ (-1)^{n-1} b_{n-1} & 0 & \cdots & 0 & -(-1)^n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_0 & -b_1 & \cdots & 0 & a_1 & -a_2 & \cdots & -(-1)^n \\ 0 & b_0 & \cdots & (-1)^{n-1} b_{n-1} & -a_0 & a_1 & \cdots & -(-1)^{n-1} a_{n-1} \\ 0 & 0 & \cdots & (-1)^{n-2} b_{n-2} & 0 & a_0 & \cdots & -(-1)^{n-2} a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_0 & 0 & 0 & \cdots & -a_0 \end{array}\right] \tag{12}$$

is a $(2n+1) \times (2n+1)$ matrix. The coefficients of $q(s)$ can be completely determined by those of $p(s)$:

$$q = \Gamma p, \tag{13}$$

where

$$\Gamma \triangleq N M^{-1}. \tag{14}$$

Define $\mathcal{F}$ as the set of functions $F(s) = \dfrac{q(s)}{p(s)}$ such that:

1) $q = \Gamma p$,
2) $\deg F(s) \leqslant 2n$,
3) $F(s) \in \mathcal{H}_\infty$ and $\|F(s)\|_\infty \leqslant \gamma$.

Thus, $\mathcal{F}$ is a set of $\mathcal{H}_\infty$ functions with a bounded norm, a bounded degree and satisfying the interpolation condition. Obviously, for any controller $C(s) = \dfrac{h(s)}{g(s)} \in \mathcal{C}_{n,\beta}$, the corresponding $F(s)$ by Equation (7) is in $\mathcal{F}$. Conversely, for any function $F(s) = \dfrac{q(s)}{p(s)} \in \mathcal{F}$, we can get the unique pair $(g(s), h(s))$ from Equation (11), and $\deg g(s) = n$, $\deg h(s) = n-1$. Consequently, the order of the controller $C(s) = \dfrac{h(s)}{g(s)}$, is bounded by n. Now the following theorem is a direct result of the derivations.

Theorem II.1 *$C(s) \in \mathcal{C}_{n,\beta}$ if and only if $F(s) \in \mathcal{F}$, where $C(s)$ and $F(s)$ are related by Equations (5) and (7).*

C. A quadratic map

The robustness constraint gives:

$$\begin{aligned} b_{P,C} \geqslant \beta &\Leftrightarrow \|F(s)\|_\infty \leqslant \gamma \\ &\Leftrightarrow \quad p(j\omega)p(-j\omega) - \frac{1}{\gamma^2} q(j\omega)q(-j\omega) \geqslant 0, \ \forall\, \omega \in \mathbb{R}. \end{aligned}$$

Hence, there exists a $2n$-th order polynomial $\sigma(s)$, which has no zero in the open right half plane and is called semistable polynomial, such that

$$p(s)p(-s) - \frac{1}{\gamma^2} q(s)q(-s) = \sigma(s)\sigma(-s). \qquad (15)$$

Consider the polynomial multiplication $p(s)p(-s)$, whose coefficients of $s, s^3, \cdots, s^{4n-1}$ are all zeros,

$$\begin{aligned} p(s)p(-s) &= p' \begin{bmatrix} s^{2n} \\ \vdots \\ s^2 \\ s \\ 1 \end{bmatrix} \begin{bmatrix} s^{2n} & \cdots & s^2 & -s & 1 \end{bmatrix} p \\ &= \begin{bmatrix} s^{4n} & s^{4n-2} \cdots & s^2 & 1 \end{bmatrix} \begin{bmatrix} p'A_{2n}p \\ p'A_{2n-1}p \\ \vdots \\ p'A_1 p \\ p'A_0 p \end{bmatrix}, \qquad (16) \end{aligned}$$

where for $k = 0, 1, \cdots, 2n$,

$$A_k(i,j) \triangleq \begin{cases} (-1)^{i-1}, & i+j = 2(2n+1-k) \\ 0, & \text{else} \end{cases}.$$

We write $q(s)q(-s)$ and $\sigma(s)\sigma(-s)$ in similar forms, and then Equation (15) becomes

$$\begin{bmatrix} p'A_{2n}p \\ p'(A_{2n-1} - \frac{1}{\gamma^2}\Gamma' A_{2n-1}\Gamma)p \\ \vdots \\ p'(A_1 - \frac{1}{\gamma^2}\Gamma' A_1\Gamma)p \\ p'(A_0 - \frac{1}{\gamma^2}\Gamma' A_0\Gamma)p \end{bmatrix} = \begin{bmatrix} \sigma' A_{2n}\sigma \\ \sigma' A_{2n-1}\sigma \\ \vdots \\ \sigma' A_1 \sigma \\ \sigma' A_0 \sigma \end{bmatrix} \triangleq \begin{bmatrix} \psi_{2n} \\ \psi_{2n-1} \\ \vdots \\ \psi_1 \\ \psi_0 \end{bmatrix}. \qquad (17)$$

The fact that $p(s)$ is a monic polynomial and $F(s)$ is strictly proper results in that $\sigma(s)$ is also a monic polynomial, $p'A_{2n}p = 1$ and then $\psi_{2n} = 1$. Denote two sets $\bar{\mathcal{P}}$ and $\bar{\Psi}$ as:

$$\bar{\mathcal{P}} \triangleq \left\{ p : \begin{array}{l} p(s) \text{ is a stable and monic polynomial,} \\ \deg p(s) = 2n, \\ \left\| \dfrac{(\Gamma p)(s)}{\gamma p(s)} \right\|_\infty \leqslant 1. \end{array} \right\}$$

$$\bar{\Psi} \triangleq \left\{ \psi : \begin{array}{l} \psi(s^2) = \sigma(s)\sigma(-s), \\ \deg \sigma(s) = 2n, \\ \sigma(s) \text{ is a monic semistable polynomial.} \end{array} \right\}.$$

Therefore,

$$p \in \bar{\mathcal{P}} \iff F(s) = \frac{(\Gamma p)(s)}{p(s)} \in \mathcal{F}. \qquad (18)$$

Equation (17) defines a quadratic map from $\bar{\mathcal{P}}$ to $\bar{\Psi}$:

$$\begin{aligned} G(p) &\triangleq \begin{bmatrix} p'(A_{2n} - \frac{1}{\gamma^2}\Gamma' A_{2n}\Gamma)p \\ p'(A_{2n-1} - \frac{1}{\gamma^2}\Gamma' A_{2n-1}\Gamma)p \\ \vdots \\ p'(A_1 - \frac{1}{\gamma^2}\Gamma' A_1\Gamma)p \\ p'(A_0 - \frac{1}{\gamma^2}\Gamma' A_0\Gamma)p \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ p'(A_{2n-1} - \frac{1}{\gamma^2}\Gamma' A_{2n-1}\Gamma)p \\ \vdots \\ p'(A_1 - \frac{1}{\gamma^2}\Gamma' A_1\Gamma)p \\ p'(A_0 - \frac{1}{\gamma^2}\Gamma' A_0\Gamma)p \end{bmatrix}. \qquad (19) \end{aligned}$$

Directly, the Jacobian matrix of the map $G(p)$ is

$$J_{G(p)} = 2 \times \begin{bmatrix} p'(A_{2n} - \frac{1}{\gamma^2}\Gamma' A_{2n}\Gamma) \\ p'(A_{2n-1} - \frac{1}{\gamma^2}\Gamma' A_{2n-1}\Gamma) \\ \vdots \\ p'(A_1 - \frac{1}{\gamma^2}\Gamma' A_1\Gamma) \\ p'(A_0 - \frac{1}{\gamma^2}\Gamma' A_0\Gamma) \end{bmatrix}. \qquad (20)$$

The map G and its Jacobian matrix $J_{G(p)}$ have the nice properties as stated in the following two theorems, which were proven in [19] based on previous work of Byrnes and Linquist [20].

Theorem II.2 *The map $G : \bar{\mathcal{P}} \to \bar{\Psi}$, is a bijection.*

Theorem II.3 *$J_{G(p)}$, the Jacobian matrix of the map G from $\bar{\mathcal{P}}$ to $\bar{\Psi}$, is invertible in the interior of $\bar{\mathcal{P}}$.*

Given an n-th order plant $P(s)$ and the robust stability margin requirement $b_{P,C} \geqslant \beta$, the interpolation matrix Γ can be obtained by simple computation. Theorem II.1, Formula (18) and Theorem II.2 have established the equivalence of $\mathcal{C}_{n,\beta}$ and $\mathcal{F}$, $\mathcal{F}$ and $\bar{\mathcal{P}}$, $\bar{\mathcal{P}}$ and $\bar{\Psi}$, respectively. i.e.,

$$\mathcal{C}_{n,\beta} \Longleftrightarrow \mathcal{F} \overset{\frac{(\Gamma p)(s)}{p(s)}}{\Longleftrightarrow} \bar{\mathcal{P}} \overset{G(p)}{\rightleftarrows} \bar{\Psi}.$$

Now the controller set $\mathcal{C}_{n,\beta}$ has been characterized by the parameter set $\bar{\Psi}$.

However, for a given $\psi \in \bar{\Psi}$, there is no analytic way to get the $p \in \bar{\mathcal{P}}$ such that $\psi = G(p)$. In next section a numerical method will be introduced to solve Equation (17) utilizing $J_{G(p)}^{-1}$, the inverse of the Jacobian matrix. This method is only feasible in the interior of $\bar{\Psi}$ and $\bar{\mathcal{P}}$ when $J_{G(p)}$ is nonsingular, $b_{P,C} > \beta$, $\|F(s)\|_\infty < \gamma$, and $\sigma(s)$ is strictly stable. In this article we only deal with this case, and leave the boundary case for future work.

D. Parameterization over a convex set

Consider an $2n$-th degree monic and stable polynomial

$$\sigma(s) = s^{2n} + \sigma_{2n-1}s^{2n-1} + \cdots + \sigma_1 s + \sigma_0.$$

For stable $\sigma(s)$ and $\psi(s^2) = \sigma(s)\sigma(-s)$, $\psi \in \Psi$. The Routh criteria tells that $\sigma(s)$ is stable if and only if all elements in the first column of its Routh table are positive. We construct the Routh table as the following tabular.

Construction of the Routh Table

r_{2n}	$r_{(2n)0} = 1$	$r_{(2n)1} = \sigma_{2n-2}$	$\cdots$
r_{2n-1}	$r_{(2n-1)0} = \sigma_{2n-1}$	$r_{(2n-1)1} = \sigma_{2n-3}$	$\cdots$
r_{2n-2}	$r_{(2n-2)0}$	$r_{(2n-2)1}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	
r_2	r_{20}	r_{21}	
r_1	r_{10}		
r_0	r_{00}		

Each row from the third one is computed as

$$r_{ij} = \frac{1}{r_{(i+1)0}} \begin{vmatrix} r_{(i+2)0} & r_{(i+2)(j+1)} \\ r_{i(i+1)0} & r_{(i+1)(j+1)} \end{vmatrix},$$

where $i = 2n-2, \ldots, 0,\ j = 0, \cdots, \left[\frac{i}{2}\right]$.

Denote the first column of the Routh table for $\sigma(s)$ by r. To get r from ψ, we can first do spectral factorization to get $\sigma(s)$, and then r is given by the Routh table of $\sigma(s)$. In the other direction, $\sigma(s)$ is obtained by reconstructing the Routh table from its first column r, and $\psi(s)$ follows by polynomials multiplication. Since $\sigma(s)$ is stable if and only if $r_k \in \mathbb{R}_+$ for $k = 0, 1, \cdots, 2n-1$, we have parameterized the interior of $\bar{\Psi}$ over the open convex set $\mathbb{R}_+^{2n}$.

III. QUADRATIC EQUATIONS AND $\mathcal{H}_2$ OPTIMIZATION

The task of this section is to solve the following problem: Given a $\psi \in \bar{\Psi}$, find the corresponding $p \in \bar{\mathcal{P}}$ such that $\psi = G(p)$. This is to solve the following $2n$ quadratic equations

$$p'\Lambda_k p = \psi_k, \quad k = 0, 1, \cdots, 2n-1. \tag{21}$$

where $\Lambda_k = A_k - \frac{1}{\gamma^2}\Gamma' A_k \Gamma$ are real symmetric, $\psi_{2n} = 1$ by definition and $p \in \bar{\mathcal{P}}$. We have $2n$ equations with $2n$ variables: $p_{2n-1}, \cdots, p_1, p_0$.

It turns out very difficult to solve the quadratic equations and there is no analytic solution available. In [16], a generalized entropy criterion has been developed to compute bounded degree Nevannlina-Pick interpolants. Here, we take a numerical continuation approach based on Theorem II.3 that $J_{G(p)}$, the Jacobian matrix of $G(p)$, is invertible for p in the interior of $\bar{\mathcal{P}}$. Let $H : \mathbb{R}^{2n+1} \times [0,1] \to \mathbb{R}^{2n+1}$ be the convex homotopy defined as:

$$\begin{aligned} H(p,\nu) &\triangleq (1-\nu)(G(p) - \bar{\psi}) + \nu(G(p) - \psi), \\ &= G(p) - \bar{\psi} + \nu(\bar{\psi} - \psi),\ \nu \in [0,1] \end{aligned} \tag{22}$$

where $(\bar{p}, \bar{\psi} = G(\bar{p}))$ is a pair of known initial point. Theorem II.2 guarantees that for each $\nu \in [0,1]$, the equation

$$H(p,\nu) = 0, \quad \text{i.e.,} \quad G(p) = (1-\nu)\bar{\psi} + \nu\psi$$

has a unique solution in the interior of $\bar{\mathcal{P}}$, denoted as $\hat{p}(\nu)$. We call the set $\{\hat{p}(\nu)\}_{\nu=0}^1$ the *trajectory*. The initial point of the trajectory, $\hat{p}(0)$, is the solution to $G(p) = \bar{\psi}$ and the end point of the trajectory, $\hat{p}(1)$, is the solution to $G(p) = \psi$. We trace the trajectory from $\nu = 0$ to $\nu = 1$, and then eventually obtain $\hat{p}(1)$ which is the preimage of ψ. In this process, $J_{G(p)}^{-1}$ is used to predict the position of $\hat{p}(\nu + \delta\nu)$ from $\hat{p}(\nu)$, where $\delta\nu$ is the step length. Details are available in [19] or [21].

Now, we can carry out optimization in the convex parameter set to search the solution that minimizes the $\mathcal{H}_2$ norm $\|T_{wy}\|_2$, and try to answer the following questions: which parameter r corresponds to the optimal $C(s)$ that minimizes $\|T_{wy}\|_2$? Is the optimal controller $C(s)$ in the interior of $\mathcal{C}_{n,\beta}$, or on the boundary?

IV. EXAMPLES AND OBSERVATIONS

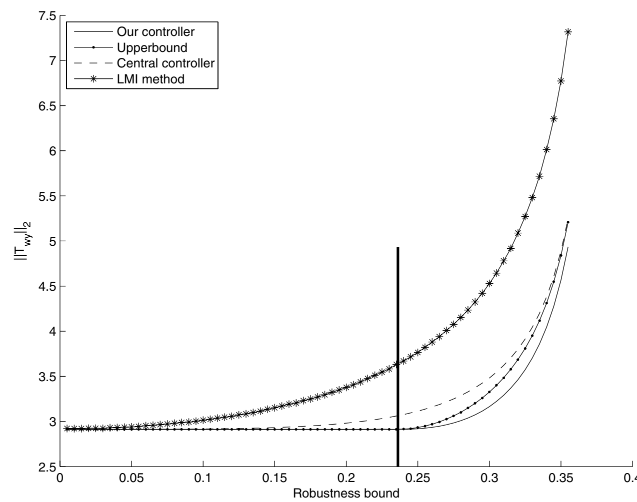

Fig. 3. Comparison of controllers for $P(s) = \frac{1}{s^2}$

The double integrator, $P(s) = \frac{1}{s^2}$, is taken as the benchmark plant to test the performances of each methods. The optimal $b_{P,C}$ given by the optimal $\mathcal{H}_\infty$ controller is 0.3827. Fig. 3 shows the performances of our controllers, the central controllers for various γ [11], controllers obtained by the LMI method [6], and also the upper bounds [22], etc. For a specific robustness bound $\beta = 0.35$, we have:

1) The central controller:

$$C(s) = \frac{17.72s + 7.14}{s^2 + 9.308s + 18.92},$$

$$b_{P,C} = 0.35264, \|T_{yw}\|_2 = 4.8873.$$

2) The LMI controller:

$$C(s) = \frac{17.8s + 7.172}{s^2 + 9.068s + 18.99},$$

$$b_{P,C} = 0.3500, \|T_{yw}\|_2 = 6.7732.$$

3) The upper bound of $\|T_{yw}\|_2$ is: 4.8383.

4) Our controller:

$$C(s) = \frac{14.64s + 5.926}{s^2 + 7.519s + 15.8},$$
$$b_{P,C} = 0.3500, \|T_{yw}\|_2 = 4.5652.$$

Here, the robust stability margins corresponding to the LMI controller and our controller are still greater than the bound β, but they are very close. The central controller is less conservative than the LMI controller, and our controller gives the best transient performance.

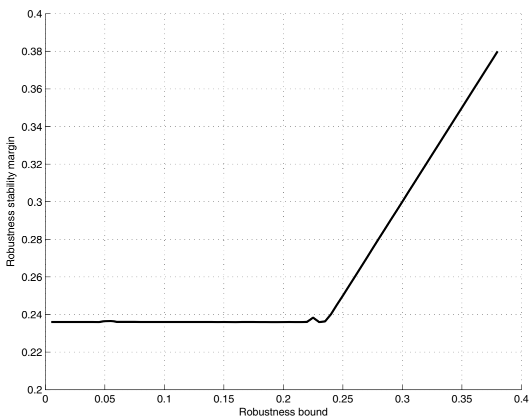

Fig. 4. $b_{P,C}$ bound and $b_{P,C}$ for $P(s) = \frac{1}{s^2}$

The following observations are obtained based on numerical examples:

1) The $\mathcal{H}_2$ norm $\|T_{wy}\|_2$, is unimodal under our parameterization, which will ensure that the minimum point found by optimization is indeed the global solution.
2) See Fig. 3. In case that the optimal $\mathcal{H}_2$ controller is a member of $\mathcal{C}_{n,\beta}$, obviously it is the optimal controller we are looking for. Our algorithm indeed gives it as the solution, while neither the central controller nor the LMI controller does.
3) See Fig. 4. In the non-trivial case that the optimal $\mathcal{H}_2$ controller is not in $\mathcal{C}_{n,\beta}$, the solution provided by our algorithm to the presented mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem with controller degree constraint is on the boundary of $\mathcal{C}_{n,\beta}$. i.e., it brings
$$b_{P,C} = \beta.$$
This drops a hint that we should explore further the boundary case $b_{P,C} = \beta$ in future.

Although supporting theoretical justifications are not available at this moment, the observations hold for the numerical examples we ran and promisingly lead the way for our further exploration of the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem.

V. CONCLUSIONS

In this paper, a special mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem is considered. We aim to find the optimal $\mathcal{H}_2$ controller under controller degree and $\mathcal{H}_\infty$ norm constraints. The set of stabilizing controllers that satisfy both the degree and robustness requirements is characterized over a finite dimensional convex set, in which $\mathcal{H}_2$ optimization is performed to obtain the desired controller that provides optimal nominal transient performance. Examples and observations that may offer insights for future research have been presented. Comparisons show that our method does work better than existing approaches.

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