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A NEW METHOD FOR THE STABILITY ROBUSTNESS DETERMINATION OF STATE SPACE MODELS WITH REAL PERTURBATIONS*

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Abstract

This paper considers the robust stability of a linear time-invariant state space model subject to real plant data perturbations. The problem is to find the distance of a given stable matrix from the set of unstable matrices. A new method, based on the properties of Kronecker product and two other composite matrices, is developed to achieve this aim; this new method makes it possible to distinguish real perturbations from complex ones. Explicit bounds on the distance of a stable matrix to the set of unstable matrices are obtained for both the continuous time and discrete time case. The bounds are applicable only for the case of real plant perturbations; hence they are less conservative to apply than for the case when complex perturbations are allowed. Several examples are given to demonstrate the new bounds; in general, the bounds obtained are "tighter" than results previously reported.

1. Introduction

In the past decade, a great deal of research has been done on the robust stability problem. However, most of the results obtained are based on the transfer function representation of a system, and use frequency domain arguments. Some attention, however, has been paid to the time domain approach of the robust stability problem, e.g. [1]-[5]. Two major methods are used in these papers. One is based on Lyapunov's stability theory [1], [5]; the other is based on frequency domain stability criterion [2]-[4].

This paper develops a new method for the stability robustness analysis of a state space model subject to real perturbations. Specifically, it is desired to determine the distance of a given stable matrix $A\in\mathbb{R}^{n\times n}$ from the set of all unstable matrices in $\mathbb{R}^{n\times n}$, where the distance in $\mathbb{R}^{n\times n}$ is defined by the spectral norm. This problem has been previously considered, e.g. [1]-[3], and some lower bounds of the distance have been obtained. These bounds are derived without assuming that the matrix space is real; therefore they are applicable for both real and complex perturbations. If only real perturbations are present however, the bounds obtained are conservative. In this paper, bounds are obtained assuming that only real perturbations are present. The approach used is based on some properties of the Kronecker product and two other composite matrices, and examples show that the new bounds obtained are less conservative than previous reported ones. The new bounds are easy to compute numerically if A is modest in size.

The structure of this paper is as follows. Section 2 describes the problem to be studied and reviews some existing results on this problem. Section 3 contains some preliminary results on properties of the Kronecker product. The new perturbation bounds are given in Section 4 in terms of the singular values of matrices formed by the Kronecker product and sum. Section 5 discusses various special cases, where it is shown that the new bounds become exact bounds in certain special cases. Section 6 provides an alternative way to view the matrices described in Section 4, which leads to some useful properties of these matrices. Two new composite matrices are defined in Section 7, and their properties are described. In Section 8 several new perturbation bounds are obtained in terms of the composite matrices defined in Section 7. Some numerical examples are given in Section 9. Due to the space limit, some of the proofs are omitted. Readers are referred to [13] for a complete treatment.

The following notation will be used throughout this paper. For an $m\times n$ matrix A, A' is the transpose of A and A^* the conjugate transpose of A. $\sigma_i(A)$, $i=1,2,\ldots,\min(m,n)$, denotes the i-th singular value of A with order $\sigma_1(A)\geq\sigma_2(A)\geq\cdots\geq\sigma_{\min(m,n)}(A)$; in particular, $\sigma_1(A)$ and $\sigma_{\min(m,n)}(A)$ are denoted by $\bar\sigma(A)$ and $\underline\sigma(A)$ respectively. $\|A\|_s$ denotes the spectral norm of A and $\|A\|_F$ denotes the Frobenius norm of A, which have the property that $\|A\|_s=\bar\sigma(A)$, $\|A\|_F=\left[\sum_{i=1}^{\min(m,n)}\sigma_i^2(A)\right]^{1/2}$. If A is square, the trace, determinant and spectrum of A are denoted by $\mathrm{tr}(A)$, $\det(A)$ and $\mathrm{sp}(A)$ respectively, and the i-th eigenvalue of A is denoted by $\lambda_i(A)$ with no specific order imposed.

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2. Development

Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{C}_c$ and $\mathbb{C}_d$ be the stable regions in the complex plane for continuous and discrete time systems, respectively, i.e. $\mathbb{C}_c=\{s\in\mathbb{C},\ \mathrm{Re}(s)<0\}$, $\mathbb{C}_d=\{s\in\mathbb{C},\ |s|<1\}$. A matrix $A\in\mathbb{F}^{n\times n}$ is said to be stable in the continuous (or discrete) time case if $\mathrm{sp}(A)\subset\mathbb{C}_c$ (or $\mathbb{C}_d$); if this is not the case A is said to be unstable. It is desired to find the distance of a given stable matrix $A\in\mathbb{F}^{n\times n}$ from the set of all unstable matrices in $\mathbb{F}^{n\times n}$, which is defined for the continuous and discrete time cases by

$$\mu_{\mathbb{F}}(A) = \inf\{\|\Delta A\|_s : \Delta A\in\mathbb{F}^{n\times n} \text{ and } \mathrm{sp}(A+\Delta A)\not\subset\mathbb{C}_c\} \tag{2.1}$$

and

$$\nu_{\mathbb{F}}(A) = \inf\{\|\Delta A\|_s : \Delta A\in\mathbb{F}^{n\times n} \text{ and } \mathrm{sp}(A+\Delta A)\not\subset\mathbb{C}_d\} \tag{2.2}$$

respectively.

Let the boundary of $\mathbb{C}_c$ and $\mathbb{C}_d$ be denoted by Γ_c and Γ_d respectively. Then it is not hard to see that

$$\mu_{\mathbb{F}}(A) = \inf\{\|\Delta A\|_s : \Delta A\in\mathbb{F}^{n\times n} \text{ and } \mathrm{sp}(A+\Delta A)\cap\Gamma_c\neq\varnothing\} \tag{2.3}$$

$$\nu_{\mathbb{F}}(A) = \inf\{\|\Delta A\|_s : \Delta A\in\mathbb{F}^{n\times n} \text{ and } \mathrm{sp}(A+\Delta A)\cap\Gamma_d\neq\varnothing\}. \tag{2.4}$$

An immediate consequence of (2.1)-(2.4) is that $\mu_{\mathbb{R}}(A)\geq\mu_{\mathbb{C}}(A)$ and $\nu_{\mathbb{R}}(A)\geq\nu_{\mathbb{C}}(A)$ if $A\in\mathbb{R}^{n\times n}$.

The problem of computing $\mu_{\mathbb{F}}(A)$ given A was first considered in [1]. A lower bound of $\mu_{\mathbb{R}}(A)$ was obtained as

$$\mu_{\mathbb{R}}(A)\geq\frac{1}{\bar\sigma(P)}, \tag{2.5}$$

where P satisfies the Lyapunov equation

$$A'P+PA=-2I. \tag{2.6}$$

It is observed that if A' is replaced by A^* in (2.6), $\frac{1}{\bar\sigma(P)}$ becomes a lower bound of $\mu_{\mathbb{C}}(A)$, i.e. if $A\in\mathbb{R}^{n\times n}$, the bounds obtained for $\mu_{\mathbb{C}}(A)$ and $\mu_{\mathbb{R}}(A)$ are the same.

References [2], [3] studied this problem using a frequency domain approach and found that

$$\mu_{\mathbb{C}}(A) = \inf_{\omega\in\mathbb{R}}\underline\sigma(j\omega I-A). \tag{2.7}$$

Clearly, (2.7) gives a lower bound to $\mu_{\mathbb{R}}(A)$, i.e.

$$\mu_{\mathbb{R}}(A)\geq\inf_{\omega\in\mathbb{R}}\underline\sigma(j\omega I-A). \tag{2.8}$$

Since (2.7) gives the exact expression for $\mu_{\mathbb{C}}(A)$ but (2.5) gives only a lower bound for $\mu_{\mathbb{C}}(A)$ (when A' is replaced by A^*), bound (2.8) is tighter than bound (2.5). This is proved in [2] using another approach. If $A\in\mathbb{R}^{n\times n}$ is a normal matrix, bounds (2.5) and (2.8) give the exact value of $\mu_{\mathbb{R}}(A)$ which is equal to the distance between $\mathrm{sp}(A)$ and the imaginary axis [2].

The discrete time version of bounds (2.5)-(2.8) can be obtained using exactly the same approach.

The exact expression of $\mu_{\mathbb{R}}(A)$ for general real matrices has not yet been obtained. Bounds (2.5) and (2.8) share a disadvantage in that they cannot distinguish between real and complex perturbations; this is because the methods used to derive them are not able to make the distinction. In order to reduce the conservatism, a new method which can make the distinction has to be developed. In this paper, such a method is established using the properties of the Kronecker product and other matrix compositions. Lower bounds of $\mu_{\mathbb{R}}(A)$ are found. The new bounds are applicable only to the real matrix space and in general are tighter than (2.5) and (2.8). The new bounds are easy to compute if A is modest in size, do not require a one-dimensional search as required by (2.8), and the computations required are numerically well-defined.

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3. Preliminaries

Let $A=[a_{ij}]\in \mathbb{F}^{m\times n}$, $B=[b_{ij}]\in \mathbb{F}^{p\times q}$. Then the Kronecker product of A and B, denoted by $A\otimes B$, is defined as follows:

$$A\otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{F}^{mp\times nq}. \tag{3.1}$$

If $m=n$ and $p=q$, the Kronecker sum of A and B, denoted by $A\oplus B$, is defined by

$$A\oplus B = A\otimes I_p + I_m\otimes B \in \mathbb{F}^{mp\times mp}. \tag{3.2}$$

The following theorem gives a list of properties of Kronecker product and sum, which will be used in the development.

Theorem 3.1 [6]

(a) If $\alpha,\beta\in \mathbb{F}$, then

$$A\otimes(\alpha B+\beta C) = \alpha(A\otimes B)+\beta(A\otimes C)$$

$$(\alpha A+\beta B)\otimes C = \alpha(A\otimes C)+\beta(B\otimes C).$$

(b) $(A\otimes B)^* = A^*\otimes B^*$.

(c) $(A\otimes B)' = A'\otimes B'$.

(d) $(A\otimes B)(D\otimes C) = AD\otimes BC$.

(e) $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$, if A,B are nonsingular.

(f) If $A\in \mathbb{F}^{m\times m}, B\in \mathbb{F}^{n\times n}$, $\mathrm{sp}(A\otimes B) = \{\lambda_i(A)\lambda_j(B),\ i=1,...,n,\ j=1,...,m\}$.

(g) If $A\in \mathbb{F}^{m\times m}, B\in \mathbb{F}^{n\times n}$, $\mathrm{sp}(A\oplus B) = \{\lambda_i(A)+\lambda_j(B),\ i=1,...,n,\ j=1,...,m\}$.

The following theorem can be easily developed from Theorem 3.1.

Theorem 3.2

(a) If $U,V\in \mathbb{F}^{n\times n}$ are unitary matrices then so is $U\otimes V$.

(b) If $A,B\in \mathbb{F}^{n\times n}$ have singular value decompositions $A=U_1S_1V_1^*$ and $B=U_2S_2V_2^*$, then $A\otimes B$ has a singular value decomposition

$$A\otimes B = (U_1\otimes U_2)(S_1\otimes S_2)(V_1\otimes V_2)^*. \tag{3.3}$$

(c) $\|A\otimes B\|_s = \|A\|_s\cdot\|B\|_s$.

Proof

(a) Using Theorem 3.1(b) and (d), if $U,V\in \mathbb{F}^{n\times n}$ are unitary, then

$$(U\otimes V)^*(U\otimes V) = (U^*\otimes V^*)(U\otimes V) = U^*U\otimes V^*V = I_n\otimes I_n = I_{n^2}.$$

This implies that $U\otimes V$ is unitary.

(b) Using Theorem 3.1(b) and (d), it is easy to show that equality (3.3) follows. Since $U_1\otimes U_2$ and $V_1\otimes V_2$ are unitary and $S_1\otimes S_2$ is positive diagonal, it follows that (3.3) is a singular value decomposition of $A\otimes B$.

(c) This result is an immediate consequence of (b).

□

Remark: The norm equality in Theorem 3.2(c) is actually a special case of the general theory of norms of tensor products [8]. The proof given above, however, is more direct.

4. Robustness Bounds

In what follows, it is always assumed that $A\in \mathbb{R}^{n\times n}$ and that A is stable, i.e. $\mathrm{sp}(A)\subset \mathbb{C}_c$ in the continuous time case and $\mathrm{sp}(A)\subset \mathbb{C}_d$ in the discrete time case. Since only real matrix spaces are considered, we write $\mu(A)$ for $\mu_{\mathbb{R}}(A)$ and $\nu(A)$ for $\nu_{\mathbb{R}}(A)$. To rule out trivial situations, it is assumed that $n\geq 2$.

I. Continuous time case

It is desired to find

$$\mu(A) = \inf\{\|\Delta A\|_s : \Delta A\in \mathbb{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap\Gamma_c\neq\emptyset\}, \tag{4.1}$$

where $\Gamma_c = \{j\omega : \omega\in \mathbb{R}\}$.

Let

$$\mu_1(A) = \inf\{\|\Delta A\|_s : \Delta A\in \mathbb{R}^{n\times n},\ 0\in \mathrm{sp}(A+\Delta A)\} \tag{4.2}$$

$$\mu_2(A) = \inf\{\|\Delta A\|_s : \Delta A\in \mathbb{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap(\Gamma_c\sim\{0\})\neq\emptyset\}, \tag{4.3}$$

where "~" means the difference of two sets.

Then it is clear that

$$\mu_{\mathbb{R}}(A) = \min\{\mu_1(A),\mu_2(A)\}. \tag{4.4}$$

$\mu_1(A)$ can be easily obtained as

$$\mu_1(A) = \underline{\sigma}(A). \tag{4.5}$$

The following analysis will therefore focus on $\mu_2(A)$. Two lemmas are required.

Lemma 4.1 [13]: Given a matrix $B\in \mathbb{R}^{n\times n}$, assume $\mathrm{sp}(B)\cap\{j\omega:\omega\in \mathbb{R}\sim\{0\}\}\neq\emptyset$, then $\mathrm{rank}(B\oplus B)\leq n^2-2$.

Lemma 4.2 [11]: If $B\in \mathbb{R}^{n\times n}$, then for any nonnegative integer $r\leq n$,

$$\min\{\|\Delta B\| : \Delta B\in \mathbb{R}^{n\times n}, \mathrm{rank}(B+\Delta B)\leq r\} = \sigma_{r+1}(B).$$

The following main result on the robust stability of continuous time systems is then obtained.

Theorem 4.1: Given a stable matrix $A\in \mathbb{R}^{n\times n}$, then

$$\mu_2(A)\geq \frac{1}{2}\sigma_{n^2-1}(A\oplus A). \tag{4.6}$$

Proof: If $\|\Delta A\|_s < \frac{1}{2}\sigma_{n^2-1}(A\oplus A)$, then

$$\|\Delta A\oplus\Delta A\|_s = \|\Delta A\otimes I+I\otimes\Delta A\|_s \leq \|\Delta A\otimes I\|_s + \|I\otimes\Delta A\|_s$$

$$= 2\|\Delta A\|_s < \sigma_{n^2-1}(A\oplus A).$$

From Lemma 4.2, we know that

$$\mathrm{rank}[(A+\Delta A)\oplus(A+\Delta A)] = \mathrm{rank}[(A\oplus A)+(\Delta A\oplus\Delta A)] > n^2-2.$$

It follows from Lemma 4.1, therefore, that $A+\Delta A$ has no imaginary eigenvalues.

Therefore, if $\mathrm{sp}(A+\Delta A)\cap(\Gamma_c\sim\{0\})\neq\emptyset$, $\|\Delta A\|_s$ has to be greater than or equal to $\frac{1}{2}\sigma_{n^2-1}(A\oplus A)$.

□

A lower bound of $\mu(A)$ can then be obtained as a consequence of Theorem 4.1 and (4.4)-(4.6).

Corollary 4.1: Given a stable matrix $A\in \mathbb{R}^{n\times n}$, then

$$\mu(A)\geq \min\{\underline{\sigma}(A),\ \frac{1}{2}\sigma_{n^2-1}(A\oplus A)\}. \tag{4.7}$$

The bound (4.7) is in such a simple form that it can be easily computed using standard software. Experience shows that for a matrix A of moderate size, computing (4.7) is in fact faster than computing bounds (2.5) and (2.8), and a large number of examples show that (4.7) is tighter than (2.5) and (2.8). In the next section we will show that the bound (4.7) is exact in some special cases, in particular, for the case when $A\in \mathbb{R}^{2\times 2}$; it is to be noted that bounds (2.5) and (2.8) are in general not exact for arbitrary 2×2 real matrices.

II. Discrete time case

In this case, it is desired to find

$$\nu(A) = \inf\{\|\Delta A\|_s : \Delta A\in \mathbb{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap\Gamma_d\neq\emptyset\}, \tag{4.8}$$

where $\Gamma_d = \{e^{j\omega}:\omega\in \mathbb{R}\}$.

Let

$$\nu_1(A) = \inf\{\|\Delta A\|_s : \Delta A\in \mathbb{R}^{n\times n},\ 1\in \mathrm{sp}(A+\Delta A)\} \tag{4.9}$$

$$\nu_2(A) = \inf\{\|\Delta A\|_s : \Delta A\in \mathbb{R}^{n\times n},\ -1\in \mathrm{sp}(A+\Delta A)\} \tag{4.10}$$

$$\nu_3(A) = \inf\{\|\Delta A\|_s : \Delta A\in \mathbb{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap(\Gamma_d\sim\{1,-1\})\neq\emptyset\}. \tag{4.11}$$

Then

$$\nu_{\mathbb{R}}(A) = \min\{\nu_1(A),\nu_2(A),\nu_3(A)\}. \tag{4.12}$$

$\nu_1(A)$ and $\nu_2(A)$ can be easily obtained as

$$\nu_1(A) = \underline{\sigma}(A-I), \quad \nu_2(A) = \underline{\sigma}(A+I). \tag{4.13}$$

A bound for $\nu_3(A)$ will now be developed. The following lemma is required:

Lemma 4.3 [13]: Given a matrix $B\in \mathbb{R}^{n\times n}$, assume $\mathrm{sp}(B)\cap(\Gamma_d\sim\{1,-1\})\neq\emptyset$, then $\mathrm{rank}(B\otimes B-I)\leq n^2-2$.

Theorem 4.2: Given a stable matrix $A\in \mathbb{R}^{n\times n}$, then

$$\nu_3(A)\geq [\sigma_{n^2-1}(A\otimes A-I)+\bar{\sigma}^2(A)]^{1/2}-\bar{\sigma}(A). \tag{4.14}$$

Proof: If $\|\Delta A\|_s < [\sigma_{n^2-1}(A\otimes A-I)+\bar{\sigma}^2(A)]^{1/2}-\bar{\sigma}(A)$, then

$$\|\Delta A\otimes A+A\otimes\Delta A+\Delta A\otimes\Delta A\|_s \le \|\Delta A\|_s^2+2\|A\|_s\|\Delta A\|_s$$
$$=(\|\Delta A\|_s+\|A\|_s)^2-\|A\|_s^2<\sigma_{n^2-1}(A\otimes A-I).$$

This implies that

$$\text{rank}[(A+\Delta A)\otimes(A+\Delta A)-I]=\text{rank}[(A\otimes A-I)+(\Delta A\otimes A+A\otimes\Delta A+\Delta A\otimes\Delta A)]>n^2-2.$$

It follows from Lemma 4.3, that $\text{sp}(A+\Delta A)\cap(\Gamma_d\sim\{-1,1\})=\varnothing$.

Therefore, if $\text{sp}(A+\Delta A)\cap(\Gamma_d\sim\{1,-1\})\neq\varnothing$, $\|\Delta A\|_s$ has to be greater than or equal to $[\sigma_{n^2-1}(A\otimes A-I)+\bar{\sigma}^2(A)]^{1/2}-\bar{\sigma}(A)$.

□

A lower bound of $\nu(A)$ is then obtained.

Corollary 4.2: Given a stable matrix $A\in\mathbb{R}^{n\times n}$, then

$$\nu(A)\ge\min\{\underline{\sigma}(A-I),\ \underline{\sigma}(A+I),\ [\sigma_{n^2-1}(A\otimes A-I)+\bar{\sigma}^2(A)]^{\frac{1}{2}}-\bar{\sigma}(A)\}(4.15)$$

The bound (4.15) is also very simple to compute, and is exact in some special cases.

5. Discussion of Special Cases

A question which naturally arises is whether or not the bounds given by Corollaries 4.1 and 4.2 are exact, i.e. whether or not the inequalities in (4.7), (4.15) are actually equalities. The answer to this question is not known for arbitrary stable matrices; however, the exact bounds can be established for some special classes of matrices.

I. Continuous time case

From the development of the last section, it is observed that if $\underline{\sigma}(A)\le\frac{1}{2}\sigma_{n^2-1}(A\oplus A)$, then $\mu(A)=\underline{\sigma}(A)$; in this case, $\mu(A)$ is exactly obtained. The exact $\mu_{\mathbb{R}}(A)$ can also be obtained in some other cases.

Theorem 5.1 [13]: If $A\in\mathbb{R}^{n\times n}$ is a stable normal matrix, then

$$\mu(A)=\min\{\underline{\sigma}(A),\ \frac{1}{2}\sigma_{n^2-1}(A\oplus A)\}$$
$$=\min\{-\text{Re}\,\lambda_i(A),\ i=1,2,...,n\}. \qquad (5.1)$$

This theorem implies that the new bound (4.7) is exact if A is normal. It has been shown [1], [2] that the previous bounds have the same property.

Theorem 5.2 [13]: If A is a 2×2 real stable matrix, then

$$\sigma_3(A\oplus A)=-\text{tr}(A) \qquad (5.2)$$

and

$$\mu(A)=\min\{\underline{\sigma}(A),-\frac{1}{2}\text{tr}(A)\}. \qquad (5.3)$$

The 2×2 case has also been studied in [4] where it has been shown that $\mu_{\mathbb{R}}(A)=\min\{\underline{\sigma}(A),-\frac{1}{2}\text{tr}(A)\}$, but no previous general bounds, when applied to the 2×2 case, give the exact answer. This result gives support to the claim that the new bound (4.7) is tighter than previous ones.

II. Discrete time case

Similar to the continuous time case, $\nu(A)$ can be exactly obtained if $\min\{\underline{\sigma}(A-I),\underline{\sigma}(A+I)\}\le[\sigma_{n^2-1}(A\otimes A-I)+\bar{\sigma}^2(A)]^{1/2}-\bar{\sigma}(A)$. For the case when A is normal or A is 2×2, similar results can be proven.

Theorem 5.3 [13]: If $A\in\mathbb{R}^{n\times n}$ is a normal matrix, then

$$\nu(A)=\min\{\underline{\sigma}(A-I),\underline{\sigma}(A+I),[\sigma_{n^2-1}(A\otimes A-I)+\bar{\sigma}^2(A)]^{1/2}-\bar{\sigma}(A)\}$$
$$=\min\{1-|\lambda_i(A)|,\ i=1,2,...,n\}. \qquad (5.4)$$

If $A\in\mathbb{R}^{2\times2}$, it can be shown that $\sigma_3(A)=1-\det(A)$(see [13]). Then bound (4.15) becomes

$$\nu(A)\ge\min\{\underline{\sigma}(A-I),\underline{\sigma}(A+I),[1-\det(A)+\bar{\sigma}^2(A)]^{1/2}-\bar{\sigma}(A)\}. \qquad (5.5)$$

However, it is not clear if this actually gives the exact value of $\nu(A)$.

6. Some Properties of Matrices $A\oplus A$ and $A\otimes A-I$

The bounds developed in section 4 require the singular values of the matrices $A\oplus A$, $A\otimes A-I$ to be determined. Such calculations may be difficult to obtain if n is large. In the following sections, a method to reduce the computation complexity will be developed by introducing two other matrix compositions. These matrix compositions have some similar properties to the Kronecker product, but have smaller dimension. They can also be used to obtain robustness bounds similar to those in section 4.

Let $A\in\mathbb{F}^{n\times n}$. Consider $A\oplus A$ and $A\otimes A-I$ as linear operators on the Hilbert space $\mathbb{F}^{n^2}$, mapping $x\in\mathbb{F}^{n^2}$ to $(A\oplus A)x$ and $(A\otimes A-I)x\in\mathbb{F}^{n^2}$ respectively. The inner product on $\mathbb{F}^{n^2}$ is defined in the usual way, i.e. $(x,y)=x^*y$, $\forall x,y\in\mathbb{F}^{n^2}$. The norm induced by this inner product is the Holder 2-norm $\|\cdot\|_2$.

The $n\times n$ matrix space $\mathbb{F}^{n\times n}$ is also an n^2-dimensional vector space over $\mathbb{F}$. It becomes a Hilbert space, if we define an inner product on it by $(X,Y)=\text{tr}(X^*Y)$, $\forall X,Y\in\mathbb{F}^{n\times n}$. The norm in $\mathbb{F}^{n\times n}$ induced by this inner product is the Frobenius norm $\|\cdot\|_F$. Now define a linear operator Vec: $\mathbb{F}^{n\times n}\to\mathbb{F}^{n^2}$ by

$$\text{Vec}\begin{bmatrix}x_{11}&x_{12}&\cdots&x_{1n}\\x_{21}&x_{22}&\cdots&x_{2n}\\\vdots&\vdots&&\vdots\\x_{n1}&x_{n2}&\cdots&x_{nn}\end{bmatrix}=[x_{11}\cdots x_{n1}\ x_{12}\cdots x_{n2}\cdots x_{nn}]'. \quad (6.1)$$

We need two properties of Vec to proceed.

Lemma 6.1 [6]: Let $X,Y,Z\in\mathbb{F}^{n\times n}$, then

(a) $\text{tr}(X^*Y)=[\text{Vec}(X)]^*\text{Vec}(Y)$,

(b) $\text{Vec}(XYZ)=(Z'\otimes X)\text{Vec}(Y)$.

Lemma 6.1(a) implies Vec is an isomorphism. Under this isomorphism, operators $A\oplus A$ and $A\otimes A-I$ on $\mathbb{F}^{n^2}$ become operators L_c and L_d mapping $X\in\mathbb{F}^{n\times n}$ to $L_c(X)=AX+XA'$ and $L_d(X)=AXA'-X$ respectively. L_c is usually called the Lyapunov transformation. L_d is the discrete time version of the Lyapunov transformation. We will next explore some properties of operators L_c and L_d.

Let $S_1\subset\mathbb{F}^{n\times n}$ be the subspace of all symmetric matrices, and let $S_2\subset\mathbb{F}^{n\times n}$ be the subspace of all skew-symmetric matrices. Formally, $S_1=\{X\in\mathbb{F}^{n\times n}:X'=X\}$ and $S_2=\{X\in\mathbb{F}^{n\times n}:X'=-X\}$. The following two propositions are required in the later development.

Proposition 6.1

$$S_1\perp S_2 \quad\text{and}\quad S_1\dot{+}S_2=\mathbb{F}^{n\times n}.$$

Proof: $\forall X_1\in S_1$ and $X_2\in S_2$,

$$(X_1,X_2)=\text{tr}(X_1^*X_2)=\text{tr}(X_1^*X_2)'=\text{tr}(X_2'X_1'^*)$$
$$=-\text{tr}(X_2X_1^*)=-\text{tr}(X_1^*X_2)=-(X_1,X_2).$$

Thus $(X_1,X_2)=0$, i.e. $X_1\perp X_2$.

This proves $S_1\perp S_2$

$\forall X\in\mathbb{F}^{n\times n}$, let $X_1=\frac{1}{2}(X+X')$, $X_2=\frac{1}{2}(X-X')$.

Then $X_1'=X_1$, $X_2'=-X_2$, i.e. $X_1\in S_1$, $X_2\in S_2$.

So $S_1+S_2=\mathbb{F}^{n\times n}$.

Since $S_1\perp S_2$, the sum is actually a direct sum, which proves that $S_1\dot{+}S_2=\mathbb{F}^{n\times n}$.

□

Proposition 6.1 states that S_1 and S_2 are orthogonal complements to each other.

Proposition 6.2

$$L_c(S_1)\subset S_1,\ L_c(S_2)\subset S_2 \text{ and}$$
$$L_d(S_1)\subset S_1,\ L_d(S_2)\subset S_2.$$

Proof: $\forall X_1\in S_1$, we have $X_1'=X_1$. Thus

$$[L_c(X_1)]'=(AX_1+X_1A')'=AX_1'+X_1'A'=AX_1+X_1A'=L_c(X_2),$$

so that $L_c(S_1)\subset S_1$.

$\forall X_2\in S_2$, we have $X_2'=X_2$. Thus

$$[L_c(X_2)]'=(AX_2+X_2A')'=AX_2'+X_2'A'=-AX_2-X_2A'=-L_c(X_2),$$

so that $L_c(S_2)\subset S_2$.

This proves the result for L_c.

The proof for L_d is almost the same and is omitted.

□

Proposition 6.2 states that $\mathbf{S}_1$ and $\mathbf{S}_2$ are reducing subspaces of $\mathbb{F}^{n\times n}$ for operators L_c and L_d.

Since the operator Vec is an isomorphism from $\mathbb{F}^{n\times n}$ to $\mathbb{F}^{n^2}$, and since $A\oplus A$ and $A\otimes A-I$ are the induced operators of L_c and L_d under Vec, the following two corollaries can be easily obtained.

Corollary 6.1

$$\mathrm{Vec}(\mathbf{S}_1)\perp\mathrm{Vec}(\mathbf{S}_2) \quad\text{and}\quad \mathrm{Vec}(\mathbf{S}_1)\dot{+}\mathrm{Vec}(\mathbf{S}_2)=\mathbb{F}^{n^2}.$$

Corollary 6.2

$$(A\oplus A)\mathrm{Vec}(\mathbf{S}_1)\subset\mathrm{Vec}(\mathbf{S}_1),\ (A\oplus A)\mathrm{Vec}(\mathbf{S}_2)\subset\mathrm{Vec}(\mathbf{S}_2) \text{ and}$$
$$(A\otimes A)\mathrm{Vec}(\mathbf{S}_1)\subset\mathrm{Vec}(\mathbf{S}_1),\ (A\otimes A)\mathrm{Vec}(\mathbf{S}_2)\subset\mathrm{Vec}(\mathbf{S}_2).$$

7. Two Other Composite Matrices

The Kronecker product can be considered as a composition of two matrices. Two other compositions of matrices will now be introduced. These compositions have some similar properties as the Kronecker product, but have smaller dimension, and stability robustness bounds can be obtained in terms of these compositions.

Let $A=[a_{ij}]\in\mathbb{F}^{n\times n}$, $B=[b_{ij}]\in\mathbb{F}^{n\times n}$, $n\geq 2$. Let (i_1,i_2) be the i-th pair of integers in the sequence

$$(1,1),(1,2),\ldots,(1,n),(2,2),(2,3),\ldots,(2,n),(3,3),\ldots,(n-1,n),(n,n). \tag{7.1}$$

Definition 7.1

$$A\overline{\otimes}B=[c_{ij}]\in\mathbb{F}^{\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)},$$

where

$$c_{ij}=\begin{cases} a_{i_1j_1}b_{i_1j_1} & i_1=i_2,\ j_1=j_2,\\ \frac{1}{2}(a_{i_1j_1}b_{i_2j_2}+a_{i_1j_2}b_{i_2j_1}+a_{i_2j_1}b_{i_1j_2}+a_{i_2j_2}b_{i_1j_1}) & i_1\neq i_2,\ j_1\neq j_2,\\ \frac{\sqrt{2}}{2}(a_{i_1j_1}b_{i_2j_2}+a_{i_2j_2}b_{i_1j_1}) & \text{otherwise.}\end{cases} \tag{7.2}$$

Let (r_1,r_2) be the r-th pair of integers in the sequence

$$(1,2),(1,3),\ldots,(1,n),(2,3),\ldots,(2,n),(3,4),\ldots,(n-1,n). \tag{7.3}$$

Definition 7.2

$$A\overline{\overline{\otimes}}B=[d_{rs}]\in\mathbb{F}^{\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)},$$

where

$$d_{rs}=\frac{1}{2}(a_{r_1s_1}b_{r_2s_2}-a_{r_1s_2}b_{r_2s_1}-a_{r_2s_1}b_{r_1s_2}+a_{r_2s_2}b_{r_1s_1}). \tag{7.4}$$

A different version of $\overline{\otimes}$ can be found in [9], but its origin is not given. $\overline{\overline{\otimes}}$ was introduced in [10]. Since no name has been given to $\overline{\otimes}$ and $\overline{\overline{\otimes}}$, they will be called $\overline{\otimes}$-product and $\overline{\overline{\otimes}}$-product respectively. The corresponding sum operations of $\overline{\otimes}$ and $\overline{\overline{\otimes}}$ can be defined as follows:

Definition 7.3

$$A\overline{\oplus}B=A\overline{\otimes}I_n+I_n\overline{\otimes}B\in\mathbb{F}^{\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)}. \tag{7.5}$$

$$A\overline{\overline{\oplus}}B=A\overline{\overline{\otimes}}I_n+I_n\overline{\overline{\otimes}}B\in\mathbb{F}^{\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)}. \tag{7.6}$$

These sum operations will be called $\overline{\oplus}$-sum and $\overline{\overline{\oplus}}$-sum respectively.

Unlike the Kronecker product and sum, $\overline{\otimes}$, $\overline{\overline{\otimes}}$, $\overline{\oplus}$ and $\overline{\overline{\oplus}}$ are defined only for square matrices with the same size.

The operations $\overline{\otimes}$ and $\overline{\overline{\otimes}}$ are closely related to the Kronecker product operation. Recall from the last section that the space $\mathbb{F}^{n^2}$ with inner product $(x,y)=x^*y$, and the space $\mathbb{F}^{n\times n}$ with inner product $(X,Y)=\mathrm{tr}(X^*Y)$, are isomorphic to each other with the isomorphism $\mathrm{Vec}:\mathbb{F}^{n\times n}\to\mathbb{F}^{n^2}$ defined as in (6.1). Subspaces $\mathbf{S}_1$ and $\mathbf{S}_2$ are defined as $\mathbf{S}_1=\{X\in\mathbb{F}^{n\times n}:X'=X\}$, $\mathbf{S}_2=\{X\in\mathbb{F}^{n\times n}:X'=-X\}$.

Define $E_{ij}\in\mathbb{F}^{n\times n}$ to be matrix with 1 in the i,j-th entry and 0 elsewhere. Let (i_1,i_2) be the i-th pair of integers in the sequence (7.1) and let

$$U_i=\begin{cases} E_{i_1i_1} & \text{if } i_1=i_2,\\ \frac{\sqrt{2}}{2}(E_{i_1i_2}+E_{i_2i_1}) & \text{otherwise.}\end{cases} \tag{7.7}$$

Then $\{U_1,U_2,\ldots,U_{\frac{1}{2}n(n+1)}\}$ is an orthonormal basis of $\mathbf{S}_1$.

Let (r_1,r_2) be the r-th pair of integers in the sequence (7.3) and let

$$V_r=\frac{\sqrt{2}}{2}(E_{r_1r_2}-E_{r_2r_1}). \tag{7.8}$$

Then $\{V_1,V_2,\ldots,V_{\frac{1}{2}n(n-1)}\}$ is an orthonormal basis of $\mathbf{S}_2$.

Let $u_i=\mathrm{Vec}(U_i)$, $i=1,2,\ldots,\frac{1}{2}n(n+1)$, and $v_i=\mathrm{Vec}(V_i)$, $i=1,2,\ldots,\frac{1}{2}n(n-1)$. Then $\{u_1,u_2,\ldots,u_{\frac{1}{2}n(n+1)}\}$ is an orthonormal basis of $\mathrm{Vec}(\mathbf{S}_1)$, and $\{v_1,v_2,\ldots,v_{\frac{1}{2}n(n-1)}\}$ is an orthonormal basis of $\mathrm{Vec}(\mathbf{S}_2)$.

Define

$$T_1=[u_1,u_2,\ldots,u_{\frac{1}{2}n(n+1)}]\in\mathbb{F}^{n^2\times\frac{1}{2}n(n+1)} \tag{7.9}$$

$$T_2=[v_1,v_2,\ldots,v_{\frac{1}{2}n(n-1)}]\in\mathbb{F}^{n^2\times\frac{1}{2}n(n-1)}. \tag{7.10}$$

It can be verified that $[T_1\ \ T_2]$ is a real orthogonal matrix.

Proposition 7.1 [13]: Let $A,B\in\mathbb{F}^{n\times n}$. Then

$$A\overline{\otimes}B=T_1'(A\otimes B)T_1 \tag{7.11}$$

$$A\overline{\overline{\otimes}}B=T_2'(A\otimes B)T_2. \tag{7.12}$$

From Corollaries 6.1 and 6.2, the following proposition easily follows:

Proposition 7.2: Let $A\in\mathbb{F}^{n\times n}$. Then

$$T_1'(A\otimes A)T_2=0$$
$$T_2'(A\otimes A)T_1=0$$
$$T_1'(A\oplus A)T_2=0$$
$$T_2'(A\oplus A)T_1=0.$$

Let $T=[T_1\ \ T_2]$. Propositions 7.1 and 7.2 imply that

$$T'(A\otimes A)T=\begin{bmatrix} A\overline{\otimes}A & 0\\ 0 & A\overline{\overline{\otimes}}A\end{bmatrix} \tag{7.13}$$

$$T'(A\oplus A)T=\begin{bmatrix} A\overline{\oplus}A & 0\\ 0 & A\overline{\overline{\oplus}}A\end{bmatrix}. \tag{7.14}$$

Various properties of the $\overline{\otimes},\overline{\overline{\otimes}}$-product and $\overline{\oplus},\overline{\overline{\oplus}}$-sum will now be obtained. Although they can be proved directly from Definitions 7.1 and 7.2, the proof is easier by using Propositions 7.1 and 7.2.

Theorem 7.1 [13]: Let $A,B,C,D\in\mathbb{F}^{n\times n}$; $\alpha,\beta\in\mathbb{F}$. Then

(a) $A\overline{\otimes}(\alpha B+\beta C)=\alpha(A\overline{\otimes}B)+\beta(A\overline{\otimes}C)$
$(\alpha A+\beta B)\overline{\otimes}C=\alpha(A\overline{\otimes}C)+\beta(B\overline{\otimes}C)$.

(b) $(A\overline{\otimes}B)'=A'\overline{\otimes}B'$.

(c) $(A\overline{\otimes}A)(B\overline{\otimes}B)=AB\overline{\otimes}AB$.

(d) $(A\overline{\otimes}A)^{-1}=A^{-1}\overline{\otimes}A^{-1}$ if A is nonsingular.

(e) $(C\overline{\otimes}C)(A\overline{\oplus}A)(C\overline{\otimes}C)^{-1}=CAC^{-1}\overline{\oplus}CAC^{-1}$ if C is nonsingular.
(a)-(e) also hold if $\overline{\otimes}$ and $\overline{\oplus}$ are replaced by $\overline{\overline{\otimes}}$ and $\overline{\overline{\oplus}}$ respectively.

(f) $\|A\overline{\otimes}B\|_s\leq\|A\|_s\cdot\|B\|_s$, $\|A\overline{\overline{\otimes}}B\|_s\leq\|A\|_s\cdot\|B\|_s$.

(g) $\mathrm{sp}(A\overline{\otimes}A)=\{\lambda_i(A)\lambda_j(A),\ l=1,2,\ldots,n,\ j\geq i\}$
$\mathrm{sp}(A\overline{\overline{\otimes}}A)=\{\lambda_i(A)\lambda_j(A),\ l=1,2,\ldots,n-1,\ j>i\}$.

(h) $\mathrm{sp}(A\overline{\oplus}A)=\{\lambda_i(A)+\lambda_j(A),\ l=1,2,\ldots,n,\ j\geq i\}$
$\mathrm{sp}(A\overline{\overline{\oplus}}A)=\{\lambda_i(A)+\lambda_j(A),\ l=1,2,\ldots,n-1,\ j>i\}$.

8. More Robustness Bounds

In this section, the composite matrices introduced in the last section are used to obtain some robustness bounds for stable matrices. We again assume throughout the section that $A \in \boldsymbol{R}^{n\times n}$, $n \geq 2$ and A is stable.

I. Continuous time case

Let $\mu(A)$ be defined as in (4.1), i.e.

$$\mu(A) = \inf\{\|\Delta A\|_s : \Delta A \in \boldsymbol{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap\Gamma_c \neq \varnothing\},$$

where $\Gamma_c = \{j\omega : \omega \in \boldsymbol{R}\}$.

If $\mathrm{sp}(A+\Delta A)\cap\Gamma_c \neq \varnothing$, $(A+\Delta A)\overline{\oplus}(A+\Delta A)$ is singular by Theorem 7.1(h); this leads to the following result.

Theorem 8.1: Given a stable matrix $A \in \boldsymbol{R}^{n\times n}$, then

$$\mu(A) \geq \frac{1}{2}\,\underline{\sigma}(A\overline{\oplus}A). \tag{8.1}$$

Proof: If $\|\Delta A\|_s < \frac{1}{2}\underline{\sigma}(A\overline{\oplus}A)$,

$$\|\Delta A\overline{\oplus}\Delta A\|_s = \|\Delta A\overline{\otimes}I + I\overline{\otimes}\Delta A\|_s \leq \|\Delta A\overline{\otimes}I\|_s + \|I\overline{\otimes}\Delta A\|_s \leq 2\|\Delta A\|_s < \underline{\sigma}(A\overline{\oplus}A).$$

Thus $A\overline{\oplus}A + \Delta A\overline{\oplus}\Delta A = (A+\Delta A)\overline{\oplus}(A+\Delta A)$ is nonsingular, which implies that

$$\mathrm{sp}(A+\Delta A)\cap\Gamma_c = \varnothing.$$

□

Now let $\mu_1(A)$ and $\mu_2(A)$ be defined as in (4.2) and (4.3), i.e.

$$\mu_1(A) = \inf\{\|\Delta A\|_s : \Delta A \in \boldsymbol{R}^{n\times n},\ 0 \in \mathrm{sp}(A+\Delta A)\}$$

$$\mu_2(A) = \inf\{\|\Delta A\|_s : \Delta A \in \boldsymbol{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap(\Gamma_c \sim \{0\}) \neq \varnothing\}.$$

It is known that

$$\mu_1(A) = \underline{\sigma}(A) \quad\text{and}\quad \mu(A) = \min\{\mu_1(A), \mu_2(A)\}.$$

By Theorem 7.1(h), $(A+\Delta A)\overline{\overline{\oplus}}(A+\Delta A)$ is singular if $\mathrm{sp}(A+\Delta A)\cap(\Gamma_c \sim \{0\}) \neq \varnothing$. This leads to the following theorem.

Theorem 8.2: Given a stable matrix $A \in \boldsymbol{R}^{n\times n}$, then

$$\mu_2(A) \geq \frac{1}{2}\,\underline{\sigma}(A\overline{\overline{\oplus}}A). \tag{8.2}$$

Its proof is similar to the proof of Theorem 8.2, so it is omitted.

Corollary 8.1: Given a stable matrix $A \in \boldsymbol{R}^{n\times n}$, then

$$\mu(A) \geq \min\{\underline{\sigma}(A),\ \frac{1}{2}\,\underline{\sigma}(A\overline{\overline{\oplus}}A)\}. \tag{8.3}$$

It is of interest to compare the bounds (8.1), (8.2) and (8.3) with bounds (4.6) and (4.7). Since $A\overline{\oplus}A$ and $A\overline{\overline{\oplus}}A$ have smaller dimensions than $A\oplus A$, it is observed that bounds (8.1), (8.2) and (8.3) are easier to compute. (7.14) shows that the singular values of $A\overline{\oplus}A$ together with those of $A\overline{\overline{\otimes}}A$ are just the singular values of $A\oplus A$; thus either $\underline{\sigma}(A\overline{\oplus}A)$ or $\underline{\sigma}(A\overline{\overline{\oplus}}A)$ must be equal to $\underline{\sigma}(A\oplus A)$. A conjecture, drawn from a large number of examples, is that $\underline{\sigma}(A\overline{\oplus}A) = \underline{\sigma}(A\oplus A)$. If this conjecture is true, bound (8.1) is worse than bound (8.3), since $\underline{\sigma}(A\overline{\oplus}A) \leq \underline{\sigma}(A\overline{\overline{\oplus}}A)$ and $\frac{1}{2}\underline{\sigma}(A\overline{\oplus}A) = \frac{1}{2}\underline{\sigma}(A\oplus A) \leq \underline{\sigma}(A)$. (The last inequality is not directly obvious, but can be proven.) Examples also show that $\underline{\sigma}(A\overline{\overline{\oplus}}A)$ may be greater than $\sigma_{n^2-1}(A\oplus A)$. This implies that bound (8.3) may be less conservative than bound (4.7) (if the conjecture is true).

It can be shown that if A is normal, bounds (8.1) and (8.3) result in the exact value of $\mu(A)$. If $A \in \boldsymbol{R}^{2\times 2}$, the definition of $A\overline{\overline{\oplus}}A$ gives that $A\overline{\overline{\oplus}}A = \mathrm{tr}(A)$; thus bound (8.3) and bound (4.7) both give the exact value of $\mu(A)$ in the 2×2 case. However, bound (8.1) does not give the exact value of $\mu(A)$ for general 2×2 matrices. This result also supports our conjecture.

II. Discrete time case

Recall that $\nu(A)$, $\nu_1(A)$, $\nu_2(A)$ and $\nu_3(A)$ are defined in (4.8)-(4.11) as follows:

$$\nu(A) = \inf\{\|\Delta A\|_s : \Delta A \in \boldsymbol{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap\Gamma_d \neq \varnothing\}$$

$$\nu_1(A) = \inf\{\|\Delta A\|_s : \Delta A \in \boldsymbol{R}^{n\times n},\ 1 \in \mathrm{sp}(A+\Delta A)\}$$

$$\nu_2(A) = \inf\{\|\Delta A\|_s : \Delta A \in \boldsymbol{R}^{n\times n},\ -1 \in \mathrm{sp}(A+\Delta A)\}$$

$$\nu_3(A) = \inf\{\|\Delta A\|_s : \Delta A \in \boldsymbol{R}^{n\times n},\ \mathrm{sp}(A+\Delta A)\cap(\Gamma_d \sim \{1,-1\}) \neq \varnothing\},$$

where $\Gamma_d = \{e^{j\omega} : \omega \in \boldsymbol{R}\}$.

It is clear that

$$\nu_1(A) = \underline{\sigma}(A-I),\ \ \nu_2(A) = \underline{\sigma}(A+I) \quad\text{and}$$

$$\nu(A) = \min\{\nu_1(A), \nu_2(A), \nu_3(A)\}.$$

By Theorem 7.1(g), $(A+\Delta A)\overline{\otimes}(A+\Delta A) - I$ is singular if $\mathrm{sp}(A+\Delta A)\cap\Gamma_d \neq \varnothing$, and $(A+\Delta A)\overline{\overline{\otimes}}(A+\Delta A) - I$ is singular if $\mathrm{sp}(A+\Delta A)\cap(\Gamma_d \sim \{1,-1\}) \neq \varnothing$. The following results are now obtained.

Theorem 8.3: Given a stable matrix $A \in \boldsymbol{R}^{n\times n}$, then

$$\nu(A) \geq [\underline{\sigma}(A\overline{\otimes}A - I) + \bar{\sigma}^2(A)]^{1/2} - \bar{\sigma}(A). \tag{8.5}$$

Proof: If $\|\Delta A\|_s \leq [\underline{\sigma}(A\overline{\otimes}A - I) + \bar{\sigma}^2(A)]^{1/2} - \bar{\sigma}(A)$, then

$$\|\Delta A\overline{\otimes}A + A\overline{\otimes}\Delta A + \Delta A\otimes\Delta A\|_s \leq \|\Delta A\|_s^2 + 2\|A\|_s\|\Delta A\|_s$$

$$= (\|\Delta A\|_s + \|A\|_s)^2 - \|A\|_s^2 < \underline{\sigma}(A\overline{\otimes}A - I),$$

which implies that $A\overline{\otimes}A - I + \Delta A\overline{\otimes}A + A\overline{\otimes}\Delta A + \Delta A\overline{\otimes}\Delta A = (A+\Delta A)\overline{\otimes}(A+\Delta A) - I$ is nonsingular.

Thus $\mathrm{sp}(A+\Delta A)\cap\Gamma_c = \varnothing$.

□

Theorem 8.4: Given a stable matrix $A \in \boldsymbol{R}^{n\times n}$, then

$$\nu_3(A) \geq [\underline{\sigma}(A\overline{\overline{\otimes}}A - I) + \bar{\sigma}^2(A)]^{1/2} - \bar{\sigma}(A). \tag{8.6}$$

The proof of Theorem 8.4 is similar to that of Theorem 8.3 and so is omitted.

Corollary 8.2: Given a stable matrix $A \in \boldsymbol{R}^{n\times n}$, then

$$\nu(A) \geq \min\{\underline{\sigma}(A+I), \underline{\sigma}(A-I), [\underline{\sigma}(A\overline{\overline{\otimes}}A - I) + \bar{\sigma}^2(A)]^{1/2} - \bar{\sigma}(A)\}. \tag{8.7}$$

For the discrete time case, we also can note that bounds (8.5)-(8.7) are easier to compute than bounds (4.14) and (4.15). Since (7.13) implies that

$$T'(A\otimes A - I)T = \begin{bmatrix} A\overline{\otimes}A - I & 0 \\ 0 & A\overline{\overline{\otimes}}A - I \end{bmatrix}.$$

This implies that either $\underline{\sigma}(A\overline{\otimes}A - I)$ or $\underline{\sigma}(A\overline{\overline{\otimes}}A - I)$ must be equal to $\underline{\sigma}(A\otimes A - I)$. A conjecture similar to the one suggested for the continuous case is made that $\underline{\sigma}(A\overline{\otimes}A - I) = \underline{\sigma}(A\otimes A - I)$. If this is true, then bound (8.7) will give the "best" result. For the case when A is normal, both (8.5) and (8.6) give the exact value of $\nu(A)$ (which is the distance of $\mathrm{sp}(A)$ to Γ_d). If A is a 2×2 matrix, Definition 7.2 gives $A\overline{\overline{\otimes}}A - I = \det(A) - 1$; so bound (5.5) can also be obtained from (8.7).

9. Numerical Examples

Several examples are presented to demonstrate the new bounds obtained, and to compare them with previous bounds. Examples 1-3 are for continuous systems, and Example 4 is for a discrete system.

Example 1

A stable 5×5 matrix obtained from a linear quadratic optimal control design was considered in [1], using the bound (2.5). The matrix is

$$A = \begin{bmatrix} -0.201 & 0.755 & 0.351 & -0.075 & 0.033 \\ -0.149 & -0.696 & -0.160 & 0.110 & -0.048 \\ 0.081 & 0.004 & -0.189 & -0.003 & 0.001 \\ -0.173 & 0.802 & 0.251 & -0.804 & 0.056 \\ 0.092 & -0.467 & -0.127 & 0.075 & -1.162 \end{bmatrix}.$$

Previous bounds give

bound (2.5): $\mu_{\boldsymbol{R}}(A) \geq \mu_{\boldsymbol{C}}(A) \geq 0.077$

bounds (2.7), (2.8): $\mu_{\boldsymbol{R}}(A) \geq \mu_{\boldsymbol{C}}(A) = 0.1116$.

On applying the new bounds to A, we obtain the following results:

$$\underline{\sigma}(A) = 0.1116$$

$$\sigma_{n^2}(A\oplus A)=0.1716,\quad \sigma_{n^2-1}(A\oplus A)=0.3480,\quad \sigma_{n^2-2}(A\oplus A)=0.3604$$

$$\underline{\sigma}(A\overline{\oplus}A)=0.1716,\quad \underline{\sigma}(A\overline{\overline{\oplus}}A)=0.3604,$$

which implies that

bound (4.7): $\mu_{\mathbb{R}}(A)\geq\min\{\underline{\sigma}(A),\frac{1}{2}\sigma_{n^2-1}(A\oplus A)\}=0.1116$

bound (8.1): $\mu_{\mathbb{R}}(A)\geq\frac{1}{2}\underline{\sigma}(A\overline{\oplus}A)=0.0858$

bound (8.3): $\mu_{\mathbb{R}}(A)\geq\min\{\underline{\sigma}(A),\frac{1}{2}\underline{\sigma}(A\overline{\overline{\oplus}}A)\}=0.1116.$

Since $\min\{\underline{\sigma}(A),\frac{1}{2}\underline{\sigma}(A\overline{\overline{\oplus}}A)\}=\underline{\sigma}(A)$, we actually have that $\mu_{\mathbb{R}}(A)=0.1116$. The conjecture that $\underline{\sigma}(A\overline{\oplus}A)=\underline{\sigma}(A\oplus A)$ holds in this case. We also have that $\underline{\sigma}(A\overline{\overline{\oplus}}A)=\sigma_{n^2-2}(A\oplus A)>\sigma_{n^2-1}(A\oplus A)$. This shows that it may be possible to obtain a tighter bound using (8.3) instead of (4.7).

Example 2

This is an example for which a perturbation more easily moves two poles to the imaginary axis rather than move one pole to the origin. In this case, the new bounds give a great improvement compared to previous ones. The matrix considered is as follows:

$$A=\begin{bmatrix}0 & 1 & 100\\ -10 & -1 & 2\\ -1 & 1 & -110\end{bmatrix}.$$

The robustness bounds are as follows:

bound (2.5): $\mu_{\mathbb{R}}(A)\geq 0.1626$

bound (2.8): $\mu_{\mathbb{R}}(A)\geq 0.5093$

bound (4.7): $\mu_{\mathbb{R}}(A)\geq\min\{\underline{\sigma}(A),\frac{1}{2}\sigma_{n^2-1}(A\oplus A)\}$

$=\min\{1.4704,\frac{1}{2}\times 1.3342\}=0.6671$

bound (8.1): $\mu_{\mathbb{R}}(A)\geq\frac{1}{2}\underline{\sigma}(A\overline{\oplus}A)=0.1894$

bound (8.3): $\mu_{\mathbb{R}}(A)\geq\min\{\underline{\sigma},\frac{1}{2}\underline{\sigma}(A\overline{\overline{\oplus}}A)\}$

$=\{1.4704,\frac{1}{2}\times 1.3342\}=0.6671.$

The best result is given by (4.7) and (8.3). Again the conjecture is true, and bound (8.1) is worse than bound (8.3).

Example 3

In this example, we will determine how conservative the bound $\mu_{\mathbb{C}}(A)$ can be when used as a lower bound for $\mu_{\mathbb{R}}(A)$. The matrix to be considered is:

$$A=\begin{bmatrix}-1 & k\\ -1 & -1\end{bmatrix},\qquad \text{where } k\geq 1.$$

The eigenvalues of A are $-1\pm j\sqrt{k}$ and

$$\sigma_3(A\oplus A)=\underline{\sigma}(A\overline{\overline{\oplus}}A)=-\frac{1}{2}\mathrm{tr}(A)=1$$

$$\underline{\sigma}(A)=\frac{3+k^2-\sqrt{(k^2+1)^2-8k+4}}{2}>1.$$

From Theorem 5.2 or Theorem 8.3, we obtain $\mu_{\mathbb{R}}(A)=1$. However, the result of (2.7) is

$$\mu_{\mathbb{C}}(A)=\inf_{\omega\in\mathbb{R}}\underline{\sigma}(j\omega I-A)\leq\underline{\sigma}(j\omega I-A)\,|_{\omega=\sqrt{k-1}}=\frac{(1+k)^2-\sqrt{(1+k)^4-16k}}{2},$$

which goes to zero as k goes to infinity. This shows that the conservatism of $\mu_{\mathbb{C}}(A)$ as a bound of $\mu_{\mathbb{R}}(A)$ can become arbitrarily large. Since bounds (4.7) and (8.3) give the exact value of $\mu_{\mathbb{R}}(A)$ when A is 2×2, there can be an arbitrary degree of improvement over the previous bounds when they are applied to matrices with any size.

Example 4

This example is for the discrete time case. The following 7×7 matrix is the closed loop state matrix of a heated rod controlled by a dead-beat controller [12]:

$$A=\begin{bmatrix}-0.1373 & 0.2139 & -0.2831 & 0.2792 & -0.2177 & -0.1298 & 0.0666\\ 0.0002 & -0.0163 & -0.0438 & -0.0657 & -0.0669 & -0.0473 & -0.0275\\ 0.0469 & 0.0718 & 0.0896 & 0.0782 & 0.0493 & 0.0224 & 0.0074\\ 0.0373 & 0.0712 & 0.1124 & 0.1292 & 0.1124 & 0.0712 & 0.0373\\ 0.0074 & 0.0224 & 0.0498 & 0.0782 & 0.0896 & 0.0718 & 0.0469\\ -0.0275 & -0.0473 & -0.0669 & 0.0657 & -0.0438 & -0.0163 & 0.0002\\ -0.0666 & -0.1298 & -0.2177 & -0.2792 & -0.2831 & -0.2139 & -0.1373\end{bmatrix}.$$

The following data are computed

$$\underline{\sigma}(A-I)=0.6705,\quad \underline{\sigma}(A+I)=0.6642,\quad \bar{\sigma}(A)=0.8306$$

$$\sigma_{n^2-1}(A\otimes A-I)=0.9375,\quad \underline{\sigma}(A\overline{\otimes}A-I)=0.7129,\quad \underline{\sigma}(A\overline{\overline{\otimes}}A-I)=0.9375.$$

The stability robustness bounds are as follows:

bound (4.15): $\nu_{\mathbb{R}}(A)\geq 0.4451$

bound (8.5): $\nu_{\mathbb{R}}(A)\geq 0.3538$

bound (8.7): $\nu_{\mathbb{R}}(A)\geq 0.4451.$

As anticipated, bound (8.7) is a better result than bound (8.5).

10. Conclusions

A new method for the robust stability problem of linear time-invariant state space models with real perturbations is considered in this paper. The method is based on the properties of the Kronecker product and two other composite matrices. Explicit bounds on the magnitude of unstructured real perturbations which do not destabilize a linear time-invariant stable system are obtained. The bounds obtained are shown to be tighter than previously reported ones, and are exact for a wider class of matrices. The bounds are easy to compute, and although the dimensions of the composite matrices are in the order of n^2, recent studies have shown that it is possible to compute the required singular values of these composite matrices without actually constructing them [14]. The new method can also be used to deal with the robust stability problem when the perturbations are structured.

Some questions in this framework remain to be answered: (i)What relationships exist between the bounds obtained in this paper? Which bound is the tightest? (ii)What relationships exist between the new bounds and bounds previously reported? (iii)How far are the new bounds to the exact value of $\mu(A)$ or $\nu(A)$?

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