Proceedings of the 7th Asian Control Conference,
Hong Kong, China, August 27-29, 2009

A Polynomial Solution to an $\mathcal{H}_\infty$ Robust Stabilization Problem

Yu Liang and Li Qiu

***Abstract*— In this paper, we present a polynomial solution to a particular $\mathcal{H}_\infty$ control problem, which arises in the robust stabilization of a SISO plant when the uncertainty is described by the gap metric, ν-gap metric, pointwise gap metric, or normalized corprime factor perturbation. Compared with the existing state space solution, the polynomial solution appears simpler and more elementary. It boils down to a rational function pole-placement problem and hence is equivalent to a polynomial Diophantine equation. The derivation of this polynomial solution is based on the existing results of the Nehari problem. We first reduce the original problem to the Nehari problem, and then find the closed-loop characteristic polynomial from the solution to the Nehari problem. Finally, we obtain the desired controller by rational function pole-placement.**

I. Introduction

The $\mathcal{H}_\infty$ norm can be interpreted as a measure of how large the frequency response of the system gets. We often wish to make the $\mathcal{H}_\infty$ norm of a closed-loop transfer function small for two reasons. Firstly, we wish to make the energy gain of the system small in order to satisfy performance objectives. Secondly, by making certain closed-loop $\mathcal{H}_\infty$ norms sufficiently small, it is possible to guarantee robustness in the face of modeling errors.

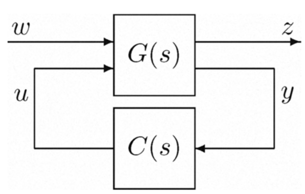

Fig. 1. Feedback system for $\mathcal{H}_\infty$ control problem

Consider the feedback system in Fig. 1. The optimal $\mathcal{H}_\infty$ control problem is to design a controller $C(s)$, so that the overall system is internally stable and the $\mathcal{H}_\infty$ norm of the closed-loop transfer function from w to z,

$$\|T_{wz}(s)\|_\infty = \sup_\omega \bar{\sigma}(T_{wz}(j\omega))$$

is minimized.

Knowing the minimum $\mathcal{H}_\infty$ norm may be useful since it sets a limit on what we can achieve. However, in practice it is often not necessary or sometimes even undesirable to design an optimal controller, by considering the economy and time conditions. And suboptimal controllers, which are very close to the optimal ones in the norm sense, may also have other nice properties. The suboptimal $\mathcal{H}_\infty$ control problem is given $\alpha > 0$, to design a stabilizing controller $C(s)$, so that

$$\|T_{wz}(s)\|_\infty \le \alpha.$$

In $\mathcal{H}_\infty$ control theory, the suboptimal controllers are not unique, and the behavior of each solution varies. For example, one of these, which called the central controller, is suggested to achieve a nominal performance of the closed-loop system. In fact, as α tends to ∞, the central controller actually approaches the $\mathcal{H}_2$ controller.

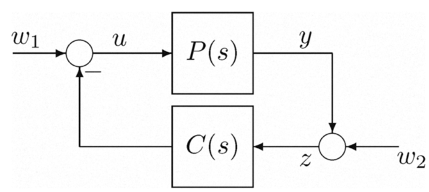

Fig. 2. Feedback system concerned in this paper

In this paper, we will study a special $\mathcal{H}_\infty$ robust stabilization problem when the uncertainty is described by the gap metric [3], [10], ν-gap metric [9], pointwise gap metric [6], or normalized coprime factor perturbation [4]. In particular, we will focus on the feedback system in Fig. 2. Consider the transfer function from w_2 and w_1 to y and u,

$$F(s) = \begin{bmatrix} P(s) \\ 1 \end{bmatrix} [1 + C(s)P(s)]^{-1} \begin{bmatrix} -C(s) & 1 \end{bmatrix}.$$

Define

$$b_{P,C} = \|F(s)\|_\infty^{-1}. \tag{1}$$

The quantity $b_{P,C}$, inverse of an $\mathcal{H}_\infty$ norm, is called robust stability margin, which is a measure of the system robust stability [3], [7], [8], [9]. We shall see that the norm can never be made smaller than 1, which means that, for any $P(s)$ and $C(s)$, $b_{P,C}$ lies in the range $[0, 1]$.

The problem to be dealt with in this paper is described as follows:

Given an n-th order proper plant $P(s) = \dfrac{b(s)}{a(s)}$, find a controller $C(s)$ so that $b_{P,C} \ge \beta$ for a given $\beta \in [0, 1]$.

A solution to this problem in the state space form is given in [4] as follows:

This work was supported by Hong Kong Research Grants Council

Yu LIANG is with the Department of Electronic & Computer Engineering, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong `rina@ust.hk`

Li QIU is with Faculty of the Department of Electronic & Computer Engineering, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong `eeqiu@ust.hk`

978-89-956056-9-1/09/©2009 ACA

Start from a minimal realization of the plant

$$P(s) = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right].$$

1) Solve two particular algebraic Riccati equations:

$$(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{D}^*\boldsymbol{C})^*\boldsymbol{X} + \boldsymbol{X}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{D}^*\boldsymbol{C}) - \boldsymbol{X}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^*\boldsymbol{X} + \boldsymbol{C}^*\boldsymbol{R}^{-1}\boldsymbol{C} = 0,$$

$$(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^*\boldsymbol{R}^{-1}\boldsymbol{C})\boldsymbol{Y} + \boldsymbol{Y}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^*\boldsymbol{R}^{-1}\boldsymbol{C})^* - \boldsymbol{Y}\boldsymbol{C}^*\boldsymbol{R}^{-1}\boldsymbol{C}\boldsymbol{Y} + \boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^* = 0,$$

where $\boldsymbol{R} = \boldsymbol{I} + \boldsymbol{D}\boldsymbol{D}^*$ and $\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{D}^*\boldsymbol{D}$.
Let $\lambda_{\max}$ denotes the eigenvalue with the largest magnitude. If $\beta \geq [1 - \lambda_{\max}(\boldsymbol{Y}\boldsymbol{X}(\boldsymbol{I}+\boldsymbol{Y}\boldsymbol{X})^{-1})]^{1/2}$, there is no solution and exit. Otherwise proceed.

2) A controller for the selected β is given by

$$C(s) = \left[\begin{array}{c|c} \boldsymbol{A}_c & \beta^{-2}\boldsymbol{W}_1^{*-1}\boldsymbol{Y}\boldsymbol{C}^* \\ \hline \boldsymbol{B}^*\boldsymbol{X} & -\boldsymbol{D}^* \end{array}\right]$$

in which

$$\begin{aligned} \boldsymbol{A}_c &= \boldsymbol{A} + \boldsymbol{B}\boldsymbol{F} + \beta^{-2}\boldsymbol{W}_1^{*-1}\boldsymbol{Y}\boldsymbol{C}^*(\boldsymbol{C} + \boldsymbol{D}\boldsymbol{F}), \\ \boldsymbol{F} &= -\boldsymbol{S}^{-1}(\boldsymbol{D}^*\boldsymbol{C} + \boldsymbol{B}^*\boldsymbol{X}), \\ \boldsymbol{W}_1 &= \boldsymbol{I} + (\boldsymbol{X}\boldsymbol{Y} - \beta^{-2}\boldsymbol{I}). \end{aligned}$$

The state space solution involves solving two Riccati equations, which are nonlinear matrix equations, and other matrix computations. To program this solution in Matlab, we need the Control System Toolbox.

In this paper, we will develop a simpler polynomial solution which only involves elementary polynomial and matrix computations, and can be easily programmed using basic Matlab functions.

II. Polynomial Solution

The design procedure of the polynomial solution is given as follows:

Start from the transfer function of the plant $P(s) = \dfrac{b(s)}{a(s)}$.

1) *Spectral factorization:*
Find a stable polynomial

$$d(s) = d_0 s^n + d_1 s^{n-1} + \cdots + d_n$$

such that

$$a(-s)a(s) + b(-s)b(s) = d(-s)d(s).$$

2) *Eigenvalue and eigenvector computation:*
Define the companion matrix of $d(s)$ as

$$\boldsymbol{A}_d = \begin{bmatrix} -\frac{d_1}{d_0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{d_{n-1}}{d_0} & 0 & \cdots & 1 \\ -\frac{d_n}{d_0} & 0 & \cdots & 0 \end{bmatrix}$$

and set

$$\boldsymbol{H} = \left(\begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^* \begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix} \right)^{-1} \times \begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^* \begin{bmatrix} b(\boldsymbol{A}_d) \\ -a(\boldsymbol{A}_d) \end{bmatrix} \boldsymbol{J},$$

where

$$\boldsymbol{J} = \begin{bmatrix} (-1)^{n-1} & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix}.$$

Let γ^* be the eigenvalue of $\boldsymbol{H}$ with the largest magnitude. If $\beta^2 \geq \dfrac{1}{1+(\gamma^*)^2}$, there is no solution and exit. Otherwise proceed.

3) *Rational function pole-placement:*
For $\gamma = \sqrt{\beta^{-2} - 1}$, define a polynomial

$$v(s) = d(s) + 2e(s)\boldsymbol{H}^2(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1} \begin{bmatrix} d_1 \\ 0 \\ d_3 \\ \vdots \end{bmatrix},$$

where $e(s) = [s^{n-1} \quad s^{n-2} \quad \cdots s \quad 1]$. The unique n-th order strictly proper controller $C(s) = \dfrac{q(s)}{p(s)}$ solved from the Diophantine equation

$$a(s)p(s) + b(s)q(s) = d(s)v(s)$$

is a guaranteed robust stabilizing controller.

To illustrate the new polynomial solution and to compare it with the state space solution, let us consider a simple numerical example:

Given a proper plant $P(s) = \dfrac{b(s)}{a(s)} = \dfrac{1}{s(s+1)}$ and $\beta = 0.5$. Find a controller $C(s)$ such that $b_{P,C} \geq \beta$.

A. State space Solution

Obtain a minimal realization of the plant from the transfer function as

$$P(s) = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & -1 & 1 \\ \hline 1 & -1 & 0 \end{array}\right].$$

1) Using Matlab Control System Toolbox to solve the two algebraic Riccati equations, we get

$$\boldsymbol{X} = \begin{bmatrix} 1.7321 & -0.7321 \\ -0.7321 & 0.4641 \end{bmatrix}, \boldsymbol{Y} = \begin{bmatrix} 1.7321 & 0.7321 \\ 0.7321 & 0.4641 \end{bmatrix}.$$

Check $\beta < [1 - \lambda_{\max}(\boldsymbol{Y}\boldsymbol{X}(\boldsymbol{I}+\boldsymbol{Y}\boldsymbol{X})^{-1})]^{1/2} = 0.5671$, solution exists.

2) A desired controller is

$$C(s) = \left[\begin{array}{cc|c} -5.6524 & 4.9204 & -4.6524 \\ -2.6233 & 0.8913 & -1.6233 \\ \hline 1 & -0.2679 & 0 \end{array}\right] = \frac{4.217s + 4.652}{s^2 + 4.761s + 7.87}.$$

B. Polynomial Solution

1) Solving $a(-s)a(s) + b(-s)b(s) = d(s)d(-s) = s^4 - s^2 + 1$, we get $d(s) = s^2 + \sqrt{3}s + 1$.
2) The companion matrix is

$$\boldsymbol{A}_d = \begin{bmatrix} -\sqrt{3} & 1 \\ -1 & 0 \end{bmatrix},$$

so we get

$$\boldsymbol{H} = \begin{bmatrix} 2-\sqrt{3} & \sqrt{3}-1 \\ \sqrt{3}-1 & 1 \end{bmatrix},$$

whose eigenvalue with the largest magnitude is $\gamma^* = 1.4524$. Check $\beta^2 < \dfrac{1}{1+(\gamma^*)^2} = 0.3216$, solution exists.
3) Define

$$v(s) = s^2 + \frac{16+21\sqrt{3}}{11}s + \frac{45+24\sqrt{3}}{11}.$$

Rational function pole-placement gives a desired controller

$$C(s) = \frac{(36+6\sqrt{3})s + 18 + 20\sqrt{3}}{11s^2 + (16+21\sqrt{3})s + 45 + 24\sqrt{3}} = \frac{4.217s + 4.652}{s^2 + 4.761s + 7.87}.$$

For a given plant, the two methods give the same solutions. But the polynomial solution is probably easier to understand and avoids using Matlab Control System Toolbox.

III. Derivation of the Polynomial Solution

In this section, we will explain how to derive the polynomial solution. First, we will reduce the original problem to the Nehari problem, then solve the Nehari problem to find the closed-loop characteristic polynomial corresponding to a desired controller, and finally obtain the controller by rational function pole-placement.

A. The Nehari Problem

By the definition of $b_{P,C}$ in Equation (1),

$$\begin{aligned} \frac{1}{b_{P,C}} &= \left\| \begin{bmatrix} P(s) \\ 1 \end{bmatrix} [1 + C(s)P(s)]^{-1} \begin{bmatrix} C(s) & 1 \end{bmatrix} \right\|_\infty \\ &= \sup_{\omega\in\mathbb{R}} \sqrt{\frac{[1+|P(j\omega)|^2][1+|C(j\omega)|^2]}{1+|P(j\omega)C(j\omega)|^2}} \\ &= \sup_{\omega\in\mathbb{R}} \sqrt{\frac{[1+P(j\omega)P(-j\omega)][1+C(j\omega)C(-j\omega)]}{[1+P(j\omega)C(j\omega)][1+P(-j\omega)C(-j\omega)]}} \\ &= \sup_{\omega\in\mathbb{R}} \sqrt{1 + \left|\frac{P(-j\omega)-C(j\omega)}{1+P(j\omega)C(j\omega)}\right|^2}. \end{aligned}$$

Therefore,

$$b_{P,C} = \left\{ 1 + \left\| \frac{P(-s)-C(s)}{1+P(s)C(s)} \right\|_\infty^2 \right\}^{-\frac{1}{2}},$$

and

$$b_{P,C} \geq \beta \iff \gamma_{P,C} = \left\| \frac{P(-s)-C(s)}{1+P(s)C(s)} \right\|_\infty \leq \gamma$$

where $\gamma = \sqrt{\beta^{-2}-1}$.

Given an n-th order proper plant $P(s) = \dfrac{b(s)}{a(s)}$, let

$$d(s) = d_0 s^n + d_1 s^{n-1} + \cdots + d_n$$

be a stable polynomial satisfying the spectral factorization

$$a(-s)a(s) + b(-s)b(s) = d(s)d(-s).$$

Solve the Diophantine equation

$$a(s)p_0(s) + b(s)q_0(s) = d^2(s),$$

then $C_0(s) = \dfrac{q_0(s)}{p_0(s)}$ is one of the stabilizing controllers.

Define

$$M(s) = \frac{a(s)}{d(s)},\ N(s) = \frac{b(s)}{d(s)}, \tag{2}$$

$$X(s) = \frac{p_0(s)}{d(s)},\ Y(s) = \frac{q_0(s)}{d(s)}, \tag{3}$$

so $M(s)$, $N(s)$, $X(s)$, $Y(s)$ are all stable transfer functions satisfying

$$P(s) = \frac{N(s)}{M(s)},\ C_0(s) = \frac{Y(s)}{X(s)},$$

and

$$M(s)X(s) + N(s)Y(s) = 1.$$

By Youla-Kucera parametrization [11], the set of all stabilizing controllers is given by

$$\mathcal{S}(P) = \left\{ C(s) = \frac{Y(s)+M(s)Q(s)}{X(s)-N(s)Q(s)},\ Q(s) \text{ stable} \right\}. \tag{4}$$

For controllers of such form,

$$\begin{aligned} &\frac{P(-s)-C(s)}{1+P(s)C(s)} \\ =&\frac{\dfrac{N(-s)}{M(-s)} - \dfrac{Y(s)+M(s)Q(s)}{X(s)-N(s)Q(s)}}{1 + \dfrac{N(s)}{M(s)}\dfrac{Y(s)+M(s)Q(s)}{X(s)-N(s)Q(s)}} \\ =&\frac{M(s)}{M(-s)}[N(-s)X(s) - M(-s)Y(s) - Q(s)]. \end{aligned}$$

Since $\dfrac{M(s)}{M(-s)}$ is all pass, it follows that

$$\gamma_{P,C} = \|G(-s) - Q(s)\|_\infty,$$

where

$$G(s) := N(s)X(-s) - M(s)Y(-s). \tag{5}$$

Substituting Equations (2) and (3) into Equation (5),

$$G(s)=\frac{b(s)p_0(-s)-a(s)q_0(-s)}{d(s)d(-s)}.$$

It appears that $G(s)$ has denominator $d(s)d(-s)$ with degree $2n$.

Lemma 1: $G(s)=\frac{r(s)}{d(s)}$ for some polynomial $r(s)$ with $\deg r(s)\le n$.

Proof: Denote $w(s):=b(s)p_0(-s)-a(s)q_0(-s)$.

Carrying polynomial calculations, we get

$$a(-s)w(s)=[b(s)d(-s)-q_0(-s)d(s)]d(-s), \quad (6)$$
$$b(-s)w(s)=[p_0(-s)d(s)-a(s)d(-s)]d(-s). \quad (7)$$

As we all know, $a(s)$ and $b(s)$ are coprime. It follows from Euclid's algorithm that there exist polynomials $x(s)$ and $y(s)$ so that

$$a(s)x(s)+b(s)y(s)=1. \quad \text{(Bezout's identity)} \quad (8)$$

Therefore,

$$\begin{aligned}&w(s)=b(s)p_0(-s)-a(s)q_0(-s)\\ =&[a(-s)x(-s)+b(-s)y(-s)][b(s)p_0(-s)-a(s)q_0(-s)]\\ =&r(s)d(-s)\end{aligned} \quad (9)$$

in which

$$\begin{aligned}r(s):=\ & x(-s)[b(s)d(-s)-q_0(-s)d(s)]\\ &+y(-s)[p_0(-s)d(s)-a(s)d(-s)].\end{aligned} \quad (10)$$

By comparing the degree of polynomials on both sides of Equation (9), we know that $\deg r(s)\le n$. Hence,

$$G(s)=\frac{b(s)p_0(-s)-a(s)q_0(-s)}{d(s)d(-s)}=\frac{r(s)d(-s)}{d(s)d(-s)}=\frac{r(s)}{d(s)},$$

for $r(s)$ defined in Equation (10). ■

The original problem can now be reduced to another problem: For a stable $G(s)$ given in (5) corresponding to the given plant $P(s)$, design $Q(s)\in\mathcal{H}_\infty$ so that

$$\|G(-s)-Q(s)\|_\infty\le\gamma,\ \gamma=\sqrt{\beta^{-2}-1}. \quad (11)$$

Such a $Q(s)$ may not be optimal, but it can be as near optimality as we wish. This problem is called a suboptimal Nehari problem.

At this moment, to design a controller $C(s)$ for a given plant $P(s)$ so that $\gamma_{P,C}\le\gamma$, the direct steps are:

1) Obtain $G(s)$ from the original given plant $P(s)$;
2) Solve the Nehari problem to get $Q(s)$;
3) Substitute $Q(s)$ back into (4) to get $C(s)$.

However, the procedures above involve a lot of complicated calculations. We are thinking to find some simpler way to solve this problem. As we all know, if we could give the closed-loop characteristic polynomial of the system, the controller is easily solved by pole-placement.

B. The Hankel Operator

Consider the set of rational functions

$$\mathcal{S}_{d(s)}=\left\{\frac{k(s)}{d(s)}:k(s)=k_1s^{n-1}+\cdots+k_n,\quad k_i\in\mathbb{R}\right\}.$$

Given a proper $G(s)=\frac{r(s)}{d(s)}$ and take any $\frac{k(s)}{d(s)}\in\mathcal{S}_{d(s)}$, then

$$G(s)\frac{k(-s)}{d(-s)}=\frac{r(s)k(-s)}{d(s)d(-s)}$$

is a strictly proper rational function with poles at the roots of $d(s)$ and their mirror images with respect to the imaginary axis. This function could be uniquely decomposed into

$$\frac{r(s)k(-s)}{d(s)d(-s)}=\frac{m(s)}{d(s)}+\frac{n(s)}{d(-s)}$$

where both terms on the right hand side are strictly proper.

Consider the map $\frac{k(s)}{d(s)}\mapsto\frac{m(s)}{d(s)}$, which is a linear operator from $\mathcal{S}_{d(s)}$ to $\mathcal{S}_{d(s)}$. We call it the Hankel operator with symbol $G(s)$, denoted by $\mathbf{\Gamma}_{G(s)}$.

The largest singular value of $\mathbf{\Gamma}_{G(s)}$ is called the Hankel norm of $G(s)$ and is denoted by $\|G(s)\|_H$. Since there exists an orthonormal basis under which the matrix representation of the Hankel operator $\mathbf{\Gamma}_{G(s)}$ is symmetric [8], so $\mathbf{\Gamma}_{G(s)}$ is a self-adjoint operator, i.e., $\mathbf{\Gamma}_{G(s)}=\mathbf{\Gamma}^*_{G(s)}$. Thus, the singular values of $\mathbf{\Gamma}_{G(s)}$ equal the absolute values of the eigenvalues of $\mathbf{\Gamma}_{G(s)}$. The Hankel operator can be represented by a matrix if a basis in $\mathcal{S}_{d(s)}$ is chosen, and all its matrix representations share the same eigenvalues. Define the matrix representation of $\mathbf{\Gamma}_{G(s)}$ under the standard basis as the standard Hankel matrix, denoted by $\boldsymbol{H}$. So $\|G(s)\|_H=|\lambda_{\max}(\boldsymbol{H})|$, where $\lambda_{\max}$ denotes the eigenvalue with the largest magnitude. Nehari's theorem [1] states that

$$\min_{Q(s)\in\mathcal{H}_\infty}\|G(-s)-Q(s)\|_\infty=\|G(s)\|_H\,,$$

that is to say, for the problem described in (11), we can only find a solution for $\gamma>\|G(s)\|_H$.

Define the companion matrix as

$$\boldsymbol{A}_d=\begin{bmatrix}-\frac{d_1}{d_0} & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{d_{n-1}}{d_0} & 0 & \cdots & 1\\ -\frac{d_n}{d_0} & 0 & \cdots & 0\end{bmatrix}.$$

From [1], [2], [5], we know that

$$\boldsymbol{H}=d^{-1}(-\boldsymbol{A}_d)r(\boldsymbol{A}_d)\boldsymbol{J},$$

where

$$\boldsymbol{J}=\begin{bmatrix}(-1)^{n-1} & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1\end{bmatrix}.$$

To follow this direct way of calculating $\boldsymbol{H}$, we have to calculate the $r(s)$ corresponding to $G(s)$ in Lemma 1, whose steps may be a bit complicated. An alternative method which avoids the computation of $r(s)$ can be obtained as given in the following lemma.

Lemma 2: The standard Hankel matrix of $\boldsymbol{\Gamma}_{G(s)}$ is given by

$$\boldsymbol{H} = \begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^{\dagger} \begin{bmatrix} b(\boldsymbol{A}_d) \\ -a(\boldsymbol{A}_d) \end{bmatrix} \boldsymbol{J},$$

where $\begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^{\dagger}$ is a left inverse of $\begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}$.

Proof: From Equatuions (6), (7), and (9), we get

$$a(-s)r(s) = b(s)d(-s) - q_0(-s)d(s),$$
$$b(-s)r(s) = p_0(-s)d(s) - a(s)d(-s).$$

Substituting $\boldsymbol{A}_d$ into the two equations above, we have

$$a(-\boldsymbol{A}_d)r(\boldsymbol{A}_d) = b(\boldsymbol{A}_d)d(-\boldsymbol{A}_d), \quad (12)$$
$$b(-\boldsymbol{A}_d)r(\boldsymbol{A}_d) = -a(\boldsymbol{A}_d)d(-\boldsymbol{A}_d), \quad (13)$$

since $d(\boldsymbol{A}_d) = 0$ by Cayley-Hamilton theorem.

Combining Equations (12) and (13), we get

$$\begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix} r(\boldsymbol{A}_d) = \begin{bmatrix} b(\boldsymbol{A}_d) \\ -a(\boldsymbol{A}_d) \end{bmatrix} d(-\boldsymbol{A}_d). \quad (14)$$

Since $a(s)$ and $b(s)$ are coprime, the tall matrix $\begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}$ has full column rank. Then

$$\boldsymbol{H} = d(-\boldsymbol{A}_d)^{-1} r(\boldsymbol{A}_d)\boldsymbol{J} = \begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^{\dagger} \begin{bmatrix} b(\boldsymbol{A}_d) \\ -a(\boldsymbol{A}_d) \end{bmatrix} \boldsymbol{J}.$$

For convenience, we can simply choose the Moore-Penrose pseudoinverse as a left inverse, i.e.,

$$\begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^{\dagger} = \left(\begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^{*} \begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix} \right)^{-1} \begin{bmatrix} a(-\boldsymbol{A}_d) \\ b(-\boldsymbol{A}_d) \end{bmatrix}^{*}.$$

■

C. Solution to the Nehari Problem

Denote the standard basis of $\mathcal{S}_{d(s)}$ as

$$E(s) = \frac{e(s)}{d(s)} = \frac{1}{d(s)}[s^{n-1} \quad \cdots s \quad 1].$$

For a proper transfer function $G(s)$ with $\|G(s)\|_H < \gamma$, decompose it as the sum of a constant term and a strictly proper term

$$G(s) = c + \hat{G}(s),$$

where $c = G(\infty)$ and $\hat{G}(s) = \dfrac{\hat{r}(s)}{d(s)}$ is strictly proper. Assume

$$\hat{G}(s) = \frac{\hat{r}(s)}{d(s)} = \frac{r_1 s^{n-1} + \cdots + r_n}{d_0 s^n + \cdots + d_n} \in \mathcal{S}_{d(s)},$$

and we have $\left\|\hat{G}(s)\right\|_H = \|G(s)\|_H < \gamma$.

Denote the vector representation of $\hat{r}(s)$ under the standard basis $e(s)$ by $\hat{\boldsymbol{r}}$. Define

$$U(s) = \gamma E(s)(\gamma^2 \boldsymbol{I} - \boldsymbol{H}^2)^{-1}\hat{\boldsymbol{r}},$$
$$V(s) = 1 + E(s)\boldsymbol{H}(\gamma^2 \boldsymbol{I} - \boldsymbol{H}^2)^{-1}\hat{\boldsymbol{r}}.$$

A solution to this Nehari problem is stated in [2] as:

Lemma 3: Let $\hat{G}(s) \in \mathcal{H}_\infty$ be a rational, strictly proper, stable function with $\left\|\hat{G}(s)\right\|_H < \gamma$. One of the $\hat{Q}(s) \in \mathcal{H}_\infty$ satisfying $\left\|\hat{G}(-s) - \hat{Q}(s)\right\|_\infty \le \gamma$ is given by

$$\hat{Q}(s) = \hat{G}(-s) - \gamma U(-s)V^{-1}(s).$$

From Lemma 3, we know that for $G(s) = c + \hat{G}(s)$, a $Q(s) \in \mathcal{H}_\infty$ satisfying $\|G(-s) - Q(s)\|_\infty \le \gamma$ is

$$Q(s) = c + \hat{Q}(s) = G(-s) - \gamma U(-s)V^{-1}(s). \quad (15)$$

Let $V(s) = \dfrac{v(s)}{d(s)} = 1 + \dfrac{\hat{v}(s)}{d(s)}$. According to the definition of Hankel operator, we have

$$G(s)V(-s) = G(s) + \boldsymbol{\Gamma}_{G(s)}\frac{\hat{v}(s)}{d(s)} + \frac{z_1(s)}{d(-s)}.$$

Since

$$\begin{aligned} V(s) &= 1 + \frac{1}{\gamma}\boldsymbol{\Gamma}_{G(s)}U(s), \\ \boldsymbol{\Gamma}_{G(s)}\frac{\hat{v}(s)}{d(s)} &= E(s)\boldsymbol{H}^2(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1}\hat{\boldsymbol{r}} \\ &= -E(s)(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1}\hat{\boldsymbol{r}} \\ &\quad +\gamma^2 E(s)(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1}\hat{\boldsymbol{r}} \\ &= -\hat{G}(s) + \gamma U(s), \end{aligned}$$

so

$$G(s)V(-s) - \gamma U(s) = c + \frac{z_1(s)}{d(-s)}.$$

Then Equation (15) is equivalent to

$$Q(s) = G(-s) - \gamma U(-s)V^{-1}(s) = \frac{z_2(s)}{v(s)}, \quad (16)$$

where $z_2(s) = c \cdot d(s) + z_1(-s)$.

Theorem 1: For the controller corresponding to such a $Q(s)$ in (15), the closed-loop poles of the system are exactly the poles of $Q(s)$ and the roots of $d(s)$.

Proof: Choose arbitrary $C(s) = \dfrac{q(s)}{p(s)} \in \mathcal{S}(P)$, then

$$\begin{aligned} Q(s) &= \frac{X(s)C(s) - Y(s)}{N(s)C(s) + M(s)} \\ &= \frac{p_0(s)q(s) - q_0(s)p(s)}{a(s)p(s) + b(s)q(s)} := \frac{h(s)}{c(s)}, \end{aligned} \quad (17)$$

which implies that the poles of $Q(s)$ are closed-loop poles.

Combining (16) and (17), we know that $c(s)$ and $h(s)$ have an n-th order common divisor $c_h(s)$, i.e.,

$$c(s) = v(s)c_h(s), \; h(s) = z_2(s)c_h(s).$$

Assume

$$\boldsymbol{\Gamma}_{G(s)}\frac{\hat{v}(s)}{d(s)} = \frac{f(s)}{d(s)}.$$

Carrying polynomial calculations, we get

$$q(s)d^2(s) = [a(s)z_2(s) + q_0(s)v(s)]c_h(s), \quad (18)$$

$$p(s)d^2(s) = [p_0(s)v(s) - b(s)z_2(s)]c_h(s), \quad (19)$$

$$d(s)d(-s)q(s) = [a(s)k(-s) + b(-s)v(s)]c_h(s), \quad (20)$$

$$d(s)d(-s)p(s) = [a(-s)v(s) - b(s)k(-s)]c_h(s). \quad (21)$$

where $k(s) = c \cdot d(s) - f(s) - r(s)$.

From Equations (18) and (19), we could see that $c_h(s)$ divides $q(s)d^2(s)$ and it also divides $p(s)d^2(s)$. $p(s)$ and $q(s)$ are coprime, so $c_h(s)$ divides $d^2(s)$. Similarly, Equations (20) and (21) imply that $c_h(s)$ divides $d(s)d(-s)$. So $c_h(s)$ divides $d(s)$, i.e., $c_h(s) = d(s)$, since they have the same degree. Hence the closed-loop poles of the system are exactly the poles of $Q(s)$ and the roots of $d(s)$.

■

Theorem 1 shows that to find a controller $C(s)$ such that $b_{P,C} \leq \beta$, it is sufficient to obtain $d(s)$ and find the poles of $Q(s)$ in (15), or the zeros of $V(s)$.

Let

$$\bar{d}(s) = d(s) - (-1)^n d(-s) = 2(d_1 s^{n-1} + d_3 s^{n-3} + \cdots),$$

then

$$\hat{G}(s)\frac{\bar{d}(-s)}{d(-s)} = \frac{\hat{r}(s)}{d(s)} - \frac{(-1)^n \hat{r}(s)}{d(-s)},$$

which means $\mathbf{\Gamma}_{\hat{G}(s)}\dfrac{\bar{d}(s)}{d(s)} = \hat{G}(s)$. Since the Hankel operator does not depend on the constant term c in $G(s)$ [8], i.e.,

$$\mathbf{\Gamma}_{G(s)}\frac{\bar{d}(s)}{d(s)} = \mathbf{\Gamma}_{\hat{G}(s)}\frac{\bar{d}(s)}{d(s)},$$

so

$$E(s)\boldsymbol{H}\bar{\boldsymbol{d}} = E(s)\hat{\boldsymbol{r}},$$

where $\bar{\boldsymbol{d}}$ is the vector representation of $\bar{d}(s)$ under the standard basis. This implies that $\boldsymbol{H}\bar{\boldsymbol{d}} = \hat{\boldsymbol{r}}$.

It is easy to verify that

$$\boldsymbol{H}(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1} = (\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1}\boldsymbol{H},$$

then $V(s)$ can be modified as

$$V(s) = 1 + E(s)\boldsymbol{H}^2(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1}\bar{\boldsymbol{d}},$$

which means

$$v(s) = d(s) + [s^{n-1} \quad \cdots s \quad 1]\boldsymbol{H}^2(\gamma^2\boldsymbol{I} - \boldsymbol{H}^2)^{-1}\bar{\boldsymbol{d}}$$

such that $\dfrac{v(s)}{d(s)} = V(s)$. Hence by Theorem 1, a desired n-th order controller $C(s) = \dfrac{q(s)}{p(s)}$ is solved from rational function pole-placement

$$a(s)p(s) + b(s)q(s) = d(s)v(s).$$

For this controller, in general, $b_{P,C}$ is neither of optimal possible value $b^*(P)$ nor the boundary value β. It is rather somewhere in between.

IV. Conclusions

We have presented the polynomial solution to a special $\mathcal{H}_\infty$ control problem for an SISO system. Compared to the standard state space solution in literature, this polynomial solution is arguably easier to understand and requires less mathematical background. To use this solution, people only need knowledge on basic linear algebra and control theory, for example, matrix operations and polynomial operations. Moreover, it is easier for programming. Matlab users do not need to install extra control system toolbox.

Although the polynomial solution is simple and efficient to apply, its derivation procedures are not so simple. The first step is to reduce the original problem to Nehari Problem, then solve the Nehari Problem in polynomial approach to find the closed-loop characteristic polynomial. Finally, apply rational function pole-placement to obtain the desired controller.

References

[1] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, Society for Industrial and Applied Mathematics, 2005.

[2] P. Fu, X. Zhao and L. Qiu, "Solution to the Nehari problem using the routh table", *Proc. 12th Mediterranean Conference on Control and Automation*, 2004.

[3] T. T. Georgiou and M. C. Smith, "Optimal robustness in the gap metric", *IEEE Transactions on Automatic Control*, pp. 673-686, Vol. 35, 1990.

[4] K. Glover and D. C. McFarlane, "Robust stabilization of normalized coprime vector plant descriptions with $\mathcal{H}_\infty$-bounded uncertainty", *IEEE Trans. Automat. Contr.*, Vol. 5, No.8, 1989.

[5] Y. Liang, *$\mathcal{H}_2$ Optimal Control under Robust Stability and Controller Degree Constraint*, Mphil thesis, HKUST, Hong Kong, July 2009.

[6] L. Qiu, and E.J. Davison, "Pointwise gap metrics on transfer matrices", *IEEE Trans. on Automatic Control*, Vol. 37, pp. 741-758, 1992.

[7] L. Qiu and K. Zhou, *Introduction to Feedback Control*, Prentice Hall, Upper Saddle River, New Jersey, 2009.

[8] L. Qiu and K. Zhou, "Pre-classical tools for post-modern control", *Proc. 42nd IEEE Conference on Decision and Control*, Vol. 5, pp. 4945-4950, 2003.

[9] G. Vinnicombe, *Uncertainty and feedback*, Imperial College Press, 2001.

[10] G. Zames. and A. K. El-Sakkary, "Unstable systems and feedback: The gap metric", *Proc. Allerton Conf.*, pp. 380-385, 1980.

[11] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, New Jersey, 1996.