Systems & Control Letters 13 (1989) 413–420
North-Holland

A simple procedure for the exact stability robustness computation of polynomials with affine coefficient perturbations *

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Received 26 May 1989
Revised 7 August 1989

Abstract: This paper considers the problem of the stability robustness computation of polynomials with coefficients which are affine functions of the parameter perturbations. A polynomial is said to be stable if its roots are contained in an arbitrarily pre-specified open set in the complex plane, and its stability robustness is then measured by the norm of the smallest parameter perturbation which destabilizes the polynomial. A simple and numerically effective procedure, which is based on the Hahn–Banach theorem of convex analysis and which is applicable for any arbitrary norm, is obtained to compute the stability robustness. The computation is then further simplified for the case when the norm used is the Hölder ∞-norm, 2-norm or 1-norm.

Keywords: Stability robustness; parameter perturbations; polynomials; dual norms; Hahn–Banach theorem.

1. Introduction

Consider a real monic polynomial in a complex variable s. Assume that its coefficients are affine functions of a vector $k \in \mathbb{R}^m$ whose entries represent independent physical parameters. This polynomial can be written as

$$p(s, k) = s^n + a_1(k)s^{n-1} + a_2(k)s^{n-2} + \cdots + a_n(k), \tag{1}$$

where $a_i(k)$, $i = 1, 2, \ldots, n$, are affine functions of k, i.e. there exist a matrix $F \in \mathbb{R}^{n\times m}$ and a vector $g \in \mathbb{R}^n$ such that

$$a(k) := \begin{bmatrix} a_1(k) & a_2(k) & \cdots & a_n(k) \end{bmatrix}' = Fk + g. \tag{2}$$

In many applications, the parameter k is somewhat uncertain but its nominal value is known. With no loss of generality, the nominal value of k is assumed to be zero.

Polynomial models of the form (1)–(2) are encountered in many circumstances in control problems. For example, if a matrix $A \in \mathbb{R}^{n\times n}$ is subject to a unity-rank perturbation of the form

$$A + kb',$$

where $b \in \mathbb{R}^n$ is a known vector and $k \in \mathbb{R}^n$ is the uncertain perturbation, then the characteristic polynomial of the perturbed matrix has the form of (1)–(2). Another example is given by a closed loop MISO (similarly SIMO) system [5]; assume that a MISO system is described by the transfer function

$$\begin{bmatrix} \dfrac{n_1(s)}{d(s)} & \dfrac{n_2(s)}{d(s)} & \cdots & \dfrac{n_r(s)}{d(s)} \end{bmatrix},$$

* This work has been supported by the Natural Sciences and Engineering Research Council of Canada under grant no. A4396.

where

$$d(s) := s^p + (d_1 + \delta d_1)s^{p-1} + \cdots + (d_p + \delta d_p),$$

$$n_i(s) := (n_{i1} + \delta n_{i1})s^{p-1} + (n_{i2} + \delta n_{i2})s^{p-2} + \cdots + (n_{ip} + \delta n_{ip}), \quad i = 1, 2, \ldots, r,$$

and the entries of the parameter vector

$$k = [\delta d_1 \;\; \delta d_2 \; \cdots \; \delta d_p]' \times [\delta n_{11} \;\; \delta n_{12} \; \cdots \; \delta n_{1p}]' \times \cdots \times [\delta n_{r1} \;\; \delta n_{r2} \; \cdots \; \delta n_{rp}]'$$

(where '$\times$' means the Cartesian product) are uncertain and independent. If the system is controlled by a fixed linear time-invariant proper controller, the closed loop characteristic polynomial will be of the form (1)–(2).

In many control problems, a desired property of a polynomial is that all of its roots are located in a pre-specified area in the complex plane. This pre-specified area will be called the stability region in the complex plane and a polynomial is said to be stable if all of its roots are located in the stability region. A natural stability robustness problem then arises for the polynomial of the form (1)–(2): if $p(s, 0)$, the polynomial corresponding to the nominal parameter, is stable, how large can the perturbation k be in order for $p(s, k)$ to maintain stability. More precisely, let us define a norm $\|\cdot\|$ on $\mathbb{R}^m$; we want to find the maximal positive number ρ such that for any parameter perturbation k with $\|k\| < \rho$, the polynomial $p(s, k)$ is stable. Here, the number ρ provides a stability robustness measure for polynomial (1)–(2). The purpose of this paper is to find a procedure to compute ρ when F, g and the stability region are given. It is apparent that ρ depends on the choice of the norm $\|\cdot\|$. In this paper, a simple and numerically effective procedure, which is applicable for any arbitrary norm, is obtained to compute ρ; it is then shown that the computation can be additionally simplified for some commonly used Hölder p-norms. (The Hölder p-norm, $1 \le p \le \infty$, in the vector space $\mathbb{R}^m$ is denoted by $\|\cdot\|_p$ and is defined by $\|k\|_p = [\sum_{i=1}^m |k_i|^p]^{1/p}$ for $k = [k_1 \;\; k_2 \; \cdots \; k_m]' \in \mathbb{R}^m$). The problem of finding such ρ obviously includes the following preliminary problem: given a norm δ of the maximum possible perturbation allowed, determine if $p(s, k)$ is stable for all k with $\|k\| \le \delta$. To distinguish our problem from the preliminary problem, we will call the former problem, the maximal robustness problem and the later problem, the robustness checking problem. It is clear that an answer to the maximal robustness problem automatically provides an answer to the robustness checking problem, but to do the converse requires an extra one-dimensional search.

A special case of the above problems occurs when each column and row of F have at most one nonzero element, the stable region is the open left half part of the complex plane, and the norm in $\mathbb{R}^m$ is the Hölder ∞-norm $\|\cdot\|_\infty$. In this case, the polynomial $p(s, k)$ is called an interval polynomial. The remarkable theorem of Kharitonov [9] gives an elegant solution to the robustness checking problem of an interval matrix. It says that for any $\delta > 0$, all polynomials belonging to an interval polynomial $p(s, k)$ with $\|k\|_\infty \le \delta$ are stable if and only if four specially constructed polynomials are stable. The maximal robustness problem for an interval polynomial can be easily solved in terms of Kharitonov's four polynomials using the Hurwitz stability criteria [7].

The stability of interval polynomials with respect to the open left half part of the complex plane is a very restricted special case of the general problem, in which the matrix F is arbitrary and the stability region is arbitrary. Although some attempts have been made to generalize the result of Kharitonov to the general problem (see [1,2,5] and the references in [3]), no results obtained have the same level of simplicity as Kharitonov's theorem. Recently, a considerable amount of research has lead to the development of feasible numerical methods to solve the maximal robustness problem and the robustness checking problem for polynomials with general affine perturbations. References [1,2] consider the robustness checking problem for the case when the uncertain parameters are known to be contained in a convex polytope in the parameter space, which includes the ball $\|k\| < \delta$ as a special case if $\|\cdot\|$ is the Hölder ∞-norm or 1-norm. The methods used in [1] and [2] are based on the concept of the value set, which is the set of complex numbers $p(s, k)$ with s fixed and k varying in a known set. The computation required in the methods of [1] and [2] are combinatorially explosive with respect to the dimension of k, i.e. the number of independent uncertain parameters, so that they are only realistic to apply when the dimension of k is small. The maximal robustness problem is solved in [4,8] for the Hölder 2-norm case by using the Euclidean space projection theory. The Hölder ∞-norm case is solved in [6] by using the geometry of the value sets and in [13] by using a linear programming method. In this paper, we solve the maximal robustness problem using a unified method for different norms. The method is based on the Hahn–Banach theorem in convex analysis, and is purely algebraic. It turns out to be very simple both conceptually and computationally. For the Hölder 2-norm case, the new method is basically the same as the one used in [4,8]; for the Hölder ∞-norm case, the new method seems simpler than the methods used in [6] and [13] and its computational complexity is only proportional to the dimension of k.

2. Problem formulation

Let $\mathbb{C}$ be the set of all complex numbers. Partition $\mathbb{C}$ into two disjoint subsets $\mathbb{C}_g$ and $\mathbb{C}_b$, i.e. $\mathbb{C} = \mathbb{C}_g \dot{\cup} \mathbb{C}_b$. We assume for technical reasons that $\mathbb{C}_g$ is open. Let $\mathscr{X}$ be a normed linear space defined to be $\mathbb{R}^m$ with norm $\|\cdot\|$. Let $\mathscr{P}$ be the set of all n-th degree real monic polynomials. Then a map from $\mathscr{X}$ to $\mathscr{P}$ is defined by

$$p(s, k) = s^n + [s^{n-1} \quad s^{n-2} \quad \cdots \quad 1](Fk + g). \tag{3}$$

A polynomial in $\mathscr{P}$ is said to be stable if all its roots are in $\mathbb{C}_g$. Define [1]

$$\rho := \inf\{ \|k\|: \ k \in \mathscr{X} \text{ and } p(s, k) \text{ is unstable}\}. \tag{4}$$

The purpose of this paper is to find a procedure to compute ρ when F, g and $\mathbb{C}_g$ are given. If $p(s, 0)$ is unstable, we must have $\rho = 0$, so it is always assumed in the following that $p(s, 0)$ is stable. Alternatively we can write ρ as

$$\rho = \inf\{ \|k\|: \ k \in \mathscr{X} \text{ and } \exists s \in \mathbb{C}_b \text{ such that } p(s, k) = 0\}. \tag{5}$$

Denote the boundary of $\mathbb{C}_g$ by $\partial\mathbb{C}_g$. Then simple continuity arguments show that

$$\rho = \inf\{ \|k\|: \ k \in \mathscr{X} \text{ and } \exists s \in \partial\mathbb{C}_g \text{ such that } p(s, k) = 0\}, \tag{6}$$

which can be rewritten as

$$\rho = \inf_{s \in \partial\mathbb{C}_g} \{\inf\{ \|k\|: \ k \in \mathscr{X} \text{ and } p(s, k) = 0\}\}. \tag{7}$$

Define a function $\tau(s)$: $\partial\mathbb{C}_g \to \mathbb{R}^+ \cup \{\infty\}$ by $\tau(s) = \inf\{ \|k\|: \ k \in \mathscr{X} \text{ and } p(s, k) = 0\}$. It is seen from (7) that the computation of ρ can be accomplished in two phases. The first phase is to find $\tau(s)$ for any fixed $s \in \partial\mathbb{C}_g$. The second phase is to carry out a search over all points in $\partial\mathbb{C}_g$ to find $\inf_{s \in \partial\mathbb{C}_g} \tau(s)$.

Now let $s \in \partial\mathbb{C}_g$ be fixed. The equation $p(s, k) = 0$ becomes

$$[s^{n-1} \quad s^{n-2} \quad \cdots \quad 1]Fk = -s^n - [s^{n-1} \quad s^{n-2} \quad \cdots \quad 1]g. \tag{8}$$

The assumption on the stability of $p(s, 0)$ implies that the right hand side of (8) is nonzero. Let

$$w := \frac{F'[s^{n-1} \quad s^{n-2} \quad \cdots \quad 1]'}{-s^n - [s^{n-1} \quad s^{n-2} \quad \cdots \quad 1]g},$$

and let $u := \Re(w)$, $v := \Im(w)$, the real and the imaginary part of w respectively. Then $u, v \in \mathbb{R}^m$ and (8) is equivalent to

$$u'k = 1 \quad \text{and} \quad v'k = 0. \tag{9}$$

[1] Throughout this paper, we assume $\inf \emptyset = \infty$.

The first phase of the problem to compute ρ then becomes to find

$$\inf\{ \|k\|: k \in \mathscr{K} \text{ and } u'k = 1, v'k = 0\} \tag{10}$$

for any given u, $v \in \mathbb{R}^m$.

The second phase of the problem to compute ρ is usually carried out by a 'brute force search' over $\partial\mathbb{C}_g$. In most applications, $\partial\mathbb{C}_g$ is symmetric to the real axis. Since $p(s, k) = 0$ if and only if $p(\bar{s}, k) = 0$, where $\bar{s}$ is the conjugate of s, it is sufficient to carry out the search over the intersection of $\partial\mathbb{C}_g$ with the closed upper half of the complex plane.

3. Development

In this section a closed form expression for $\inf\{ \|k\|: k \in \mathscr{K}$ and $u'k = 1, v'k = 0\}$ is obtained by using the Hahn–Banach theorem [11].

From the theory of linear equations it is known that if $\text{rank}[u\ v] \neq \text{rank}[\begin{smallmatrix} u & v \\ 1 & 0 \end{smallmatrix}]$, then there exist no $k \in \mathscr{K}$ such that the equations $u'k = 1$ and $v'k = 0$ are satisfied. Thus in this case

$$\inf\{ \|k\|: k \in \mathscr{K} \text{ and } u'k = 1, v'k = 0\} = \infty.$$

Now assume that $\text{rank}[u\ v] = \text{rank}[\begin{smallmatrix} u & v \\ 1 & 0 \end{smallmatrix}] = 1$ or 2. Consider the normed vector space $\mathscr{K}$. Let $\mathscr{K}^*$ be its dual space. Then $\mathscr{K}^*$ is also a normed vector space in which the norm is the dual norm of $\|\cdot\|$. In our problem, $\mathscr{K}^*$ is isometrically isomorphic to the space $\mathbb{R}^m$ with norm $\|\cdot\|^*$ and the norm $\|\cdot\|^*$ is induced from $\|\cdot\|$ by

$$\|x\|^* = \sup_{\substack{k \in \mathscr{K} \\ k \neq 0}} \frac{|x'k|}{\|k\|}$$

for $x \in \mathscr{K}^*$. It is known that if $\|\cdot\|$ is the Hölder p-norm $\|\cdot\|_p$, then $\|\cdot\|^*$ is the Hölder q-norm $\|\cdot\|_q$ with $1/p + 1/q = 1$. Since $\mathscr{K}$ is finite dimensional, $\mathscr{K}$ is isometrically isomorphic to $\mathscr{K}^{**}$, the dual of $\mathscr{K}^*$. This implies that each $k \in \mathscr{K}$ can be considered as a linear functional on $\mathscr{K}^*$. For given u, $v \in \mathbb{R}^m$, consider u, v as vectors in $\mathscr{K}^*$. Let $\tilde{\mathscr{K}} := \{k: k \in \mathscr{K}$ and $u'k = 1, v'k = 0\}$. Then each $k \in \tilde{\mathscr{K}}$ is a linear functional on $\mathscr{K}^*$ and its values on the space spanned by $\{u, v\}$ are determined by the two equations $u'k = 1$ and $v'k = 0$. It is clear that for each $k \in \tilde{\mathscr{K}}$, $\|k\| \geq \|k\,|\text{span}\{u, v\}\|$, where $k\,|\text{span}\{u, v\}$ is the restriction of k to the subspace span $\{u, v\} \subset \mathscr{K}^*$, and its norm is induced by $\|\cdot\|^*$ in span $\{u, v\}$, i.e.

$$\|k\,|\text{span}\{u, v\}\| = \sup_{\substack{x \in \text{span}\{u,v\} \\ x \neq 0}} \frac{|x'k|}{\|x\|^*} = \sup_{\substack{\alpha,\beta \in \mathbb{R} \\ \alpha u + \beta v \neq 0}} \frac{|\alpha|}{\|\alpha u + \beta v\|^*} = \sup_{\substack{\alpha \in \mathbb{R} \\ u + \alpha v \neq 0}} \frac{1}{\|u + \alpha v\|^*}.$$

A critical step in our development is achieved by using the Hahn–Banach theorem [11, p. 104]. The Hahn–Banach theorem simply says that a $k \in \tilde{\mathscr{K}}$ can be found such that $\|k\| = \|k\,|\text{span}\{u, v\}\|$. Therefore the following equality is obtained:

$$\inf\{ \|k\|: k \in \mathscr{K} \text{ and } u'k = 1, v'k = 0\} = \inf\{ \|k\|: k \in \tilde{\mathscr{K}}\} = \sup_{\substack{\alpha \in \mathbb{R} \\ u + \alpha v \neq 0}} \frac{1}{\|u + \alpha v\|^*}. \tag{11}$$

A special case happens when u, v are linearly dependent. Since we have already assumed that $\text{rank}[u\ v] = \text{rank}[\begin{smallmatrix} u & v \\ 1 & 0 \end{smallmatrix}]$, vectors u, v are linearly dependent if and only if $u \neq 0$ and $v = 0$. In this case,

$$\sup_{\substack{\alpha \in \mathbb{R} \\ u + \alpha v \neq 0}} \frac{1}{\|u + \alpha v\|^*} = \frac{1}{\|u\|^*}.$$

The following theorem summarizes the above development.

Theorem 1. *For any given $u, v \in \mathbb{R}^m$,*

$$\inf\{\|k\|: k \in \mathscr{K} \text{ and } u'k=1,\ v'k=0\} = \begin{cases} \infty & \text{if } \operatorname{rank}[u\ v] \neq \operatorname{rank}\begin{bmatrix} u & v \\ 1 & 0 \end{bmatrix}, \\ \dfrac{1}{\|u\|^*} & \text{if } u \neq 0 \text{ and } v = 0, \\ \sup\limits_{\alpha\in\mathbb{R}} \dfrac{1}{\|u+\alpha v\|^*} & \text{if } \operatorname{rank}[u\ v] = 2. \end{cases} \tag{12}$$

From a computational point of view, the only nontrivial case occurring in (12) is the case when $\operatorname{rank}[u\ v]=2$. In this case, $\sup_{\alpha\in\mathbb{R}} 1/\|u+\alpha v\|^*$, or equivalently $\inf_{\alpha\in\mathbb{R}}\|u+\alpha v\|^*$, has to be computed. This computation however is straightforward to do even if $\|\cdot\|^*$ is an arbitrary norm. Simple analysis shows that $\|u+\alpha v\|^*$, to be considered as a function of α, is a continuous convex function on $\mathbb{R}$. As α goes to ∞ or $-\infty$, $\|u+\alpha v\|^*$ goes to ∞. Consequently, $\inf_{\alpha\in\mathbb{R}}\|u+\alpha v\|^*$ is achieved at a finite point and any technique for a convex one-dimensional optimization, such as the Fibonacci search or the golden section search, can be used to find $\inf\|u+\alpha v\|^*$. It will be shown in the next section that if $\|\cdot\|$ is one of the commonly used Hölder p-norms, the computation of $\inf\|u+\alpha v\|^*$ becomes very simple and no one-dimensional optimization is actually required.

4. Computational aspects

This section deals with the computational problem of (12) when the norm $\|\cdot\|$ is $\|\cdot\|_\infty$, $\|\cdot\|_2$ or $\|\cdot\|_1$. The computation is nontrivial only if $\operatorname{rank}[u\ v]=2$. Thus it is assumed in this section that $\operatorname{rank}[u\ v]=2$.

Case I. $\|\cdot\| = \|\cdot\|_\infty$. In this case, $\|\cdot\|^* = \|\cdot\|_1$. Let $u=[u_1\ u_2\ \cdots\ u_m]'$ and $v=[v_1\ v_2\ \cdots\ v_m]'$; then

$$\|u+\alpha v\|_1 = |u_1+\alpha v_1| + |u_2+\alpha v_2| + \cdots + |u_m+\alpha v_m|. \tag{13}$$

A continuous function on $\mathbb{R}$ is called polygonal (or piecewise linear) if there exist finite points $x_1, x_2,\ldots,x_l \in \mathbb{R}$ with $x_1<x_2<\cdots<x_l$ such that the function is linear on $(-\infty, x_1]$, $[x_l, \infty)$ and $[x_i, x_{i+1}]$, $i=1,2,\ldots,l-1$. In this case the points $x_1, x_2,\ldots,x_l$ are called division points. If (13) is considered to be a function of α, then it is a polygonal function with at most m division points. The set of division points is just $\{-u_i/v_i: i=1,2,\ldots,m \text{ and } v_i\neq 0\}$. The supremum and infimum of a polygonal function on $\mathbb{R}$ can only happen at ∞, $-\infty$ or one of its division points. Since $\|u+\alpha v\|_1$ goes to infinity as α goes to ∞ or $-\infty$, its infimum can only be achieved at one of its division points. This proves the following equality:

$$\begin{aligned}\inf\{\|k\|_\infty: k\in\mathscr{K} \text{ and } u'k=1,\ v'k=0\} &= \sup_{\alpha\in\mathbb{R}} \frac{1}{\|u+\alpha v\|_1} \\ &= \max\left\{\frac{1}{\|u+\alpha v\|_1}: \alpha\in\left\{-\frac{u_i}{v_i}: i=1,2,\ldots,m \text{ and } v_i\neq 0\right\}\right\}.\end{aligned} \tag{14}$$

To compute (14), we only need basically to compute the 1-norm of at most m vectors in $\mathbb{R}^m$.

Case II. $\|\cdot\| = \|\cdot\|_2$. In this case, $\|\cdot\|^* = \|\cdot\|_2$. The infimum of $\|u+\alpha v\|_2$ is achieved at the least-squares solution of the linear equation $\alpha v = -u$, which is given by $\alpha = -\|v\|_2^{-2} v'u$. Thus

$$\inf\{\|k\|_2: k\in\mathscr{K} \text{ and } u'k=1,\ v'k=0\} = \sup_{\alpha\in\mathbb{R}}\frac{1}{\|u+\alpha v\|_2} = \frac{\|v\|_2}{\left[\|u\|_2^2\|v\|_2^2-(u'v)^2\right]^{1/2}}. \tag{15}$$

This case is also considered in [8]; the result obtained here is basically the same as in [8].

Case III. $\|\cdot\| = \|\cdot\|_1$. In this case, $\|\cdot\|^* = \|\cdot\|_\infty$. Let $u = [u_1 \; u_2 \; \cdots \; u_m]'$ and $v = [v_1 \; v_2 \; \cdots \; v_m]'$; then

$$\|u + \alpha v\|_\infty = \max\{ |u_1 + \alpha v_1|, \; |u_2 + \alpha v_2|, \ldots, |u_m + \alpha v_m| \}. \tag{16}$$

It is easy to see that $\|u + \alpha v\|_\infty$ is also a polygonal function of α which goes to ∞ as α goes to ∞ or $-\infty$. So its infimum is achieved at one of its division points. However it appears that the division points of $\|u + \alpha v\|_\infty$ can not be obtained as easily as those of $\|u + \alpha v\|_1$. Note that at any division point of $\|u + \alpha v\|_\infty$, we must have $|u_i + \alpha v_i| = |u_j + \alpha v_j|$ for some $i, \; j = 1, 2, \ldots, m$ and $i \neq j$. So the set of division points is contained in the following set

$$\begin{aligned} \Lambda &= \{\alpha\colon \; u_i + \alpha v_i = u_j + \alpha v_j, \; 1 \le i < j \le m\} \cup \{\alpha\colon \; u_i + \alpha v_i = -u_j - \alpha v_j, \; 1 \le i < j \le m\} \\ &= \left\{ -\frac{u_i - u_j}{v_i - v_j}\colon 1 \le i < j \le m \text{ and } v_i - v_j \neq 0 \right\} \cup \left\{ -\frac{u_i + u_j}{v_i + v_j}\colon 1 \le i < j \le m \text{ and } v_i + v_j \neq 0 \right\}. \end{aligned}$$

This proves the following equality:

$$\inf\{ \|k\|_1\colon \; k \in \mathscr{K} \text{ and } u'k = 1, \; v'k = 0\} = \sup_{\alpha \in \mathbf{R}} \frac{1}{\|u + \alpha v\|_\infty} = \max\left\{ \frac{1}{\|u + \alpha v\|}\colon \; \alpha \in \Lambda \right\}. \tag{17}$$

In the worst case, Λ has $m(m-1)$ elements, while the number of the division points of $\|u + \alpha v\|_\infty$ may be much less than $m(m-1)$. It is possible to have a search scheme to find the division points, but this requires extra computational effort. Thus formula (17) should be used at least in the case when m is not too large.

We would like to remark that although we have only considered the computational problem for three different p-norms, the results obtained can be applied to the case when the norm is a weighted version of any of these p-norms, i.e. when $\|k\|$ is defined to be equal to $\|Tk\|_p$, $p = 1$, 2, or ∞, for some nonsingular matrix T. In such a case, we can convert the original problem (3)–(4) to a problem with a standard p-norm by substituting k with $T^{-1}\hat{k}$.

5. An example

The following polynomial is considered in [6]:

$$p(s, k) = s^4 + a_1(k)s^3 + a_2(k)s^2 + a_3(k)s + a_4(k),$$

where $k \in \mathscr{K} = \mathbb{R}^4$ and

$$\begin{bmatrix} a_1(k) \\ a_2(k) \\ a_3(k) \\ a_4(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 10.75 & 0.75 & 7 & 0.25 \\ 32.5 & 7.5 & 12 & 0.5 \\ 18.75 & 18.75 & 10 & 0.5 \end{bmatrix} k + \begin{bmatrix} 12 \\ 47 \\ 70 \\ 50 \end{bmatrix}.$$

The roots of $p(s, 0)$ are -5, -5, $-1 \pm \mathrm{j}$. If the desired stability region is assumed to be the open left part of the complex plane, then the polynomial is nominally stable and the stability robustness measure is given by

$$\rho = \inf_{\omega \in \mathbf{R}} \tau(\mathrm{j}\omega),$$

where $\tau(\mathrm{j}\omega) = \inf\{ \|k\|\colon \; k \in \mathscr{K} \text{ and } p(\mathrm{j}\omega, k) = 0\}$. By using the procedure given in Sections 2–4, we obtain

$$\rho = \begin{cases} \tau(\mathrm{j}\omega)\,|_{\omega=0} = 1.04 & \text{if } \|\cdot\| = \|\cdot\|_\infty, \\ \tau(\mathrm{j}\omega)\,|_{\omega=0} = 1.76 & \text{if } \|\cdot\| = \|\cdot\|_2, \\ \tau(\mathrm{j}\omega)\,|_{\omega=0.71} = 2.00 & \text{if } \|\cdot\| = \|\cdot\|_1. \end{cases}$$

Now assume that the desired stability region is the set $\mathbb{C}_g = \mathbb{C}_{g1} \cup \mathbb{C}_{g2} \cup \mathbb{C}_{g3}$, where

$$\mathbb{C}_{g1} := \{s\colon\ |s-(-1+\mathrm{j})| < 0.25\},\quad \mathbb{C}_{g2} := \{s\colon\ |s-(-1-\mathrm{j})| < 0.25\},$$
$$\mathbb{C}_{g3} := \{s\colon\ |s-(-5)| < 1\}.$$

With respect to the stability region $\mathbb{C}_g$, the nominal polynomial $p(s, 0)$ is stable. The boundary of $\mathbb{C}_g$ is given by $\partial\mathbb{C}_g = \partial\mathbb{C}_{g1} \cup \partial\mathbb{C}_{g2} \cup \partial\mathbb{C}_{g3}$, where

$$\partial\mathbb{C}_{g1} = \{s\colon\ |s-(-1+\mathrm{j})| = 0.25\},\quad \partial\mathbb{C}_{g2} = \{s\colon\ |s-(-1-j)| = 0.25\},$$
$$\partial\mathbb{C}_{g3} = \{s\colon\ |s-(-5)| = 1\}.$$

The stability robustness measure of $p(s, k)$ is then calculated to be as follows on using the proposed procedure:

$$\rho = \inf_{s\in\partial\mathbb{C}_g} \tau(s) = \begin{cases} \tau(s)\,|_{s=-1.17+\mathrm{j}0.81} = 0.30 & \text{if } \|\cdot\| = \|\cdot\|_\infty, \\ \tau(s)\,|_{s=-1.20+\mathrm{j}0.85} = 0.44 & \text{if } \|\cdot\| = \|\cdot\|_2, \\ \tau(s)\,|_{s=-1.23+\mathrm{j}0.91} = 0.47 & \text{if } \|\cdot\| = \|\cdot\|_1. \end{cases}$$

The case when $\|\cdot\| = \|\cdot\|_\infty$ is considered in [6]; the results obtained here and in [6] are consistent.

6. Conclusions

A unified method is obtained to analyze the stability robustness of a polynomial whose coefficients are affine functions of parameter perturbations. The stability robustness is measured by the norm of the smallest parameter perturbation which destabilizes the polynomial. The method is quite general in two senses: (i) the stability region in the complex plane can be an arbitrary open set, which includes the open left half plane and the open unit disc as special cases; (ii) the norm used to measure the stability robustness can be any norm defined on a linear space as long as its dual norm can be numerically evaluated. The computational procedure obtained from this method is conceptually simple and numerically effective, and it can be further simplified if the norm used to measure the stability robustness is among the commonly used Hölder p-norms. If the norm is the Hölder 2-norm, the procedure coincides with some recent results obtained in the literature [4,8]; if the norm is the Hölder ∞-norm, the procedure is an improvement over the existing results.

In most of the applications, using a norm to measure the size of the perturbation is satisfactory. However, there are cases when a norm is not a proper measurement of the size of the perturbation. For example, if it is known that the perturbation is contained in a convex body (containing the origin as an interior point) in the parameter space, it is more natural to use the gauge (or Minkowski functional) [12] of the convex body to measure the size of the perturbation. If the convex body is balanced, i.e. it contains k if and only if it contains $-k$, then the gauge is actually a norm; otherwise it is not a norm. If we use the gauge to measure the size of the perturbation and define the stability robustness in terms of this gauge, then the polytope problem considered by many researchers [1,2,6, etc.] can be covered; in this case it can be shown that the method given in this paper can be generalized to deal with the stability robustness problem for any given convex body. This will appear in a future paper [10].

References

[1] A.C. Barlett, C.V. Hollot and H. Lin, Root locations for an entire polytope of polynomials: it suffices to check the edges, *Math. Control Signals and Systems* **1** (1988) 61–71.

[2] B.R. Barmish, A generalization of Kharitonov's four-polynomial concept for the robust stability problems with linearly dependent coefficient perturbations, *IEEE Trans. Automat. Control* **34** (1989) 157–165.

[3] B.R. Barmish, New tools for robustness analysis, *Proc. IEEE CDC* (1988) 1–6.

[4] R.M. Biernacki, H. Hwang and S.P. Bhattacharyya, Robust stability with structured real parameter perturbations, *IEEE Trans. Automat. Control* **32** (1987) 495–505.

[5] H. Chapellat and S.P. Bhattacharyya, A generalization of Kharitonov's theorem: robust stability of interval plants, *IEEE Trans. Automat. Control* **34** (1989) 306–311.

[6] M. Fu, Polytopes of polynomials with zeros in a prescribed region: new criteria and algorithms, in: M. Milanese, R. Tempo and A. Pecile, Eds., *Robustness in Identification and Control* (1988).

[7] M. Fu and B.R. Barmish, Maximal unidirectional perturbation bounds for stability of polynomials and matrices, *Systems Control Lett.* **11** (1988) 173–179.

[8] D. Hinrichsen and A.J. Prichard, New robustness results for linear systems under real perturbations, *Proc. IEEE CDC* (1988) 1375–1379.

[9] V.L. Kharitonov, Asymptotic stability of an equilibrium position of a family of systems of linear differential equations, *Diffenncjalnyje Uravnenija* **14** (1978) 2086–2088.

[10] L. Qiu and E.J. Davison, A unified approach for the stability robustness of polynomials in a convex set, *Proc. IEEE CDC* (1989), to appear.

[11] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1987).

[12] W. Rudin, *Functional Analysis* (McGraw-Hill, New York, 1973).

[13] A. Tesi and A. Vicino, Robustness analysis for uncertain dynamical systems with structured perturbations, *Proc. IEEE CDC* (1988) 519–525.