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A Summary on the Real Stability Radius and Real Perturbation Values

Bo Bernhardsson, Anders Rantzer, Li Qiu

1. Introduction. The stability radius problem has an interesting history in the mathematical and control theory literature. Stability radii also occur in numerical analysis in relation with the so called pseudospectra introduced by Trefethen and the analysis of stability of numerical solvers for ordinary differential equations.

There are several different definitions of so called "stability radii". The theory for the complex stability radius is equivalent to H_{∞} -theory and can be connected to Riccati equations. Towards the end of 1980's, attention was focused on the real stability radius. Hinrichsen, Pritchard, and associates studied various properties of the real stability radius and surveyed their results in [5]. It was also studied by people in numerical linear algebra, see [9]. Several lower bounds on the real stability radius were obtained in [12]. We will here focus on presenting the recent ideas behind the calculation of the real stability radius and the connected "real perturbation values" of a matrix.

Consider the following problem closely connected with the computation of the real stability radius: Given a complex matrix $M \in \mathbf{C}^{p \times m}$, compute the so called "real perturbation values" of M:

(1)
$$\tau_k(M) := \left[\min\{\|\Delta\| : \Delta \in \mathbf{R}^{m \times p} \text{ and } \operatorname{rank}(I_m - \Delta M) = m - k\}\right]^{-1}$$

Note that Δ is here assumed *real*, while M is a complex matrix. The size of the matrix Δ is measured in induced 2-norm, i.e. as the largest singular value and the inverse is taken for later notational convenience. When M is real, $\tau_k(M) = \sigma_k(M)$, where $\sigma_k(M)$ denote the standard singular values of M ordered nondecreasingly.

We have recently shown the following easily computable formula for the real perturbation values:

(2)
$$\tau_k(M) = \inf_{\gamma \in (0,1]} \sigma_{2k} \left(\begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \right).$$

This result generalizes a formula for τ_1 obtained in [10], [11] to arbitrary k. The new more general proof is quite different and sheds new light on the previous results on the real stability radius. The problem is by (2) reduced to a simple one-parametric minimization, where actually only k local minima can occur. It is also possible to give a constructive method for finding a minimizing Δ . For a discussion of numerical issues connected with the computation of (2) and the real stability radius see the contribution by van Dooren et.al. in this book.

The (structured) real stability radius of a matrix quadruple $(A, B, C, D) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m} \times \mathbf{R}^{p \times n} \times \mathbf{R}^{p \times m}$ satisfies

$$r_R(A, B, C, D) := \inf_{\Delta \in \mathbf{R}^{m \times p}} \{ \|\Delta\| : A + B(I - \Delta D)^{-1} \Delta C \text{ is unstable} \\ \text{or } \det(I - \Delta D) = 0 \}.$$
$$= \left[\sup_{s \in \partial C_g} \tau_1(D + C(sI - A)^{-1}B) \right]^{-1}$$

where ∂C_g denotes the boundary of the stability domain C_g . This should be compared with the corresponding well known result for complex perturbations

$$r_{C}(A, B, C, D) := \inf_{\Delta \in \mathbf{C}^{m \times p}} \{ \|\Delta\| : A + B(I - \Delta D)^{-1} \Delta C \text{ is unstable} \\ \text{or } \det(I - \Delta D) = 0 \}.$$
$$= \left[\sup_{s \in \partial C_{g}} \sigma_{1}(D + C(sI - A)^{-1}B) \right]^{-1}$$

which connects the complex stability radius with H_{∞} -theory.

It is also relatively easy to show that

(3)
$$\tau_k(M) = \max_{\dim(S)=k} \min_{z \in S} \frac{\|\operatorname{Re}(Mz)\|}{\|\operatorname{Re}(z)\|}.$$

Compare this with the classical variational formula for the singular values

(4)
$$\sigma_k(M) = \max_{\dim(S)=k} \min_{z \in S} \frac{\|Mz\|}{\|z\|}.$$

The real perturbation values seem to be new interesting entities connected to a complex matrix . Many results for the singular values have counterparts for the real perturbation values. The analysis is related to other interesting areas in mathematics, such as Hermitian-symmetric inequalities, consimilarity and quaternions.

Due to page limitations it will not be possible to give all details here. An extended version of this paper will appear elsewhere.

2. Calculation of the Real Perturbation Values. Formula (2) follows from: THEOREM 2.1. Given a matrix $C \in \mathbb{C}^{p \times m}$ the following four conditions are equivalent, where $A = M^*M - \tau^2 I$, $B = M^TM - \tau^2 I$, $\Delta \in \mathbb{R}^{m \times p}$ and S_k is a complex matrix of rank k:

(5)
$$\exists S_k, \Delta : \|\Delta\| \le \tau^{-1} \text{ and } (I - \Delta M)S_k = 0$$

(6)
$$\exists S_k: \begin{bmatrix} S_k & 0\\ 0 & \overline{S}_k \end{bmatrix}^* \begin{bmatrix} A & \overline{B}\\ B & \overline{A} \end{bmatrix} \begin{bmatrix} S_k & 0\\ 0 & \overline{S}_k \end{bmatrix} \ge 0$$

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(7)
$$\inf_{|\alpha| \le 1} \lambda_{2k} \left(\left[\begin{array}{cc} A & \alpha \overline{B} \\ \alpha B & \overline{A} \end{array} \right] \right) \ge 0$$

(8)
$$\inf_{\gamma \in (0,1]} \sigma_{2k} \left(\begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \right) \geq \tau$$

Proof That (5) is equivalent to (6) follows from the fact that given two complex matrices U and V there exists a real contraction, i.e. a real matrix Δ with $||\Delta|| \leq 1$, such that $\Delta U = V$ if and only if

$$\begin{bmatrix} U^* \\ U^T \end{bmatrix} \begin{bmatrix} U & \overline{U} \end{bmatrix} \ge \begin{bmatrix} V^* \\ V^T \end{bmatrix} \begin{bmatrix} V & \overline{V} \end{bmatrix}.$$

If (6) is true then it is easy to see that the same statement with B replaced with -B is also true. By convexity one can then replace B by αB for any $\alpha \in [-1, 1]$. Hence (6) implies (7). That (7) is equivalent to (8) can be seen as follows: Since

$$P_{M}(\gamma) := \begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} = D \begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix} D^{-1}$$

where $D = \begin{bmatrix} i\gamma I & i\gamma I \\ I & -I \end{bmatrix}$

we can use the fact that congruence transformations do not change the sign of eigenvalues to show that

$$\lambda_{2k}(P_M(\gamma)^*P_M(\gamma) - \tau^2 I) \ge 0 \iff \lambda_{2k} \left(\begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix}^* D^* D \begin{bmatrix} M & 0 \\ 0 & \overline{M} \end{bmatrix} - \tau^2 D^* D \right) \ge 0$$

However, because $D^*D = \begin{bmatrix} \gamma^2 + 1 & \gamma^2 - 1 \\ \gamma^2 - 1 & \gamma^2 + 1 \end{bmatrix}$ this is equivalent to

$$\lambda_{2k} \left(\left[\begin{array}{cc} A & 0 \\ 0 & \overline{A} \end{array} \right] + \frac{\gamma^2 - 1}{\gamma^2 + 1} \left[\begin{array}{cc} 0 & \overline{B} \\ B & 0 \end{array} \right] \right) \ge 0$$

With $\alpha := (\gamma^2 - 1)/(\gamma^2 + 1)$ this proves that (7) is equivalent to (8) with the interval (0, 1] replaced by $(0, \infty)$. The singular values of $P_M(\gamma)$ and $P_M(\gamma^{-1})$ are however the same. This concludes the proof of the theorem except for the implication (7) \Rightarrow (6). We will not prove that in full generality here. The following lemma however illustrates the main idea.

Remark The implication $(7) \Rightarrow (6)$ would be relatively easy to prove if A and B where diagonal. The existence of a simultaneous *- and T-block diagonalization of A and B. required in the next lemma is actually not a strong restriction. If the eigenvalues to the generalized eigenvalue problem

$$\lambda \left[\begin{array}{cc} 0 & \overline{B} \\ B & 0 \end{array} \right] - \left[\begin{array}{cc} A & 0 \\ 0 & \overline{A} \end{array} \right].$$

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are simple, S can be constructed from the corresponding eigenvectors. In the case of multiple eigenvalues the simultaneous block-diagonalization might however not be possible. One must then replace the block-diagonal matrices with matrices of a form parallel to those in the Jordan canonical form. The proof then becomes slightly more technical. For a more extensive discussion see [3]. See also the related references [1], [6], [14], [15] and [16].

LEMMA 2.2. Let A and B a be defined as in the previous theorem. Assume that there exists a complex non-singular matrix S such that

(9)
$$S^*AS = \Lambda \quad and \quad S^TBS = I$$

where $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_r)$ is a block-diagonal matrix with blocks of the form $\Lambda_j = \lambda_j \in \mathbb{R} \text{ or } \Lambda_j = \begin{bmatrix} 0 & \overline{\lambda}_j \\ \lambda_j & 0 \end{bmatrix}$. Then (6) and (7) are equivalent to the following condition

(*) Let n_1, n_2 denote the number of $\lambda_j \geq 1$ and non-real λ_j respectively and let the real eigenvalues smaller than 1 be ordered so that $1 > \lambda_1 \geq \lambda_2 \geq \dots$. Then either $n_1 + n_2 \geq k$ or else

$$\lambda_j + \lambda_{2k-2n_1-2n_2+1-j} \ge 0, \quad j = 1, \dots, k - n_1 - n_2.$$

Proof After a congruence transformation condition (7) means that $\begin{bmatrix} \Lambda & \alpha I \\ \alpha I & \overline{\Lambda} \end{bmatrix}$ has 2k positive eigenvalues for all $\alpha \in [-1, 1]$. The eigenvalues can be studied block by block. Because of symmetry it is enough to study $\alpha \in [0, 1]$. The eigenvalues are given by $\lambda_j \pm \alpha$ if $\lambda_j \in R$ and $\pm (\alpha^2 + |\lambda_j|^2 \pm 2\alpha \operatorname{Re}(\lambda_j))^{1/2}$ otherwise. If the eigenvalues of the matrix above is plotted as function of α there will be 2n curves. The n_2 non-real blocks of Λ give rise to $4n_2$ curves, half of them above 0, half of them below 0. The real blocks will give $2n - 4n_2$ straight lines, half of them with slope 1, half with slope -1 as a function of α . The eigenvalues with $\lambda_j \geq 1$ give $2n_1$ lines above 0 for all $\alpha \in [-1, 1]$. Drawing a diagram and figuring out the condition for existence of a total of 2k curves above 0 for all $\alpha \in [-1, 1]$ one arrives at the somewhat involved condition (*).

From (*) it is easy to construct the nonsingular matrix S_k needed in (6) by using the blocks corresponding to the 2k curves above 0. In fact we construct S_k in the following way. Let S be given as above and put for each real eigenvalue ≥ 1

$$s_j = S \begin{bmatrix} 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}^T, \quad j = 1, \dots, n_1.$$

so that $s_i^* A s_j \ge 1$ and $s_j^T B s_j = 1$. For each pair of complex eigenvalues put

$$s_j = S \begin{bmatrix} 0 & \dots & 1 & i & \dots & 0 \end{bmatrix}^T, \quad j = n_1 + 1, \dots, n_1 + n_2$$

Assuming $\operatorname{Im}(\lambda_j) < 0$ this gives $s_j^* A s_j = -2 \operatorname{Im}(\lambda_j) > 0$ and $s_j^T B s_j = 0$. Put for each pair with $\lambda_j + \lambda_{2k-2n_1-2n_2+1-j} \ge 0$

$$s_j = S \begin{bmatrix} 0 & \dots & 1 & i & \dots & 0 \end{bmatrix}^T, \quad j = n_1 + n_2 + 1, \dots, k$$

This gives $s_j^* A s_j = \lambda_j + \lambda_{2k-2n_1-2n_2+1-j} \ge 0$ and $s_j^T B s_j = 0$. With $S_k = \begin{bmatrix} s_1 & \dots & s_k \end{bmatrix}^T$ we hence have

$$S_k^*AS_k = \operatorname{diag}(\Lambda_c, \Lambda_d) \quad ext{and} \quad S_k^TBS_k = \operatorname{diag}(I, 0),$$

where $\Lambda_c \geq I$ and $\Lambda_d \geq 0$. From this (6) follows. That (6) implies (7) was proved in the previous theorem.

2.1. The Courant-Fischer Type Formula. Formula (3) follows from

LEMMA 2.3. Let $A = M^*M - \tau^2 I$, $B = M^TM - \tau^2 I$ and assume S_k denotes a matrix of rank k. Then the following conditions are equivalent

 $(6) \exists S_k : \begin{bmatrix} S_k & 0\\ 0 & \overline{S}_k \end{bmatrix}^* \begin{bmatrix} A & \overline{B}\\ B & \overline{A} \end{bmatrix} \begin{bmatrix} S_k & 0\\ 0 & \overline{S}_k \end{bmatrix} \ge 0$ $(**) \exists S_k : z^* Az \ge \operatorname{Re}(z^T Bz), \quad \forall z \in \operatorname{Im}(S_k)$ $(3') \exists S_k : \|\operatorname{Re}(Mz)\|^2 \ge \tau^2 \|\operatorname{Re}(z)\|^2, \quad \forall z \in \operatorname{Im}(S_k)$

Proof That (6) is equivalent to (**) follows from Lemma 3.1 below. Conditions (**) and (3') are equivalent since

$$\begin{aligned} \|\operatorname{Re} \left(MS_{k}w\right)\|^{2} &\geq \tau^{2} \|\operatorname{Re} \left(S_{k}w\right)\|^{2}, \quad \forall w \in \mathbf{C}^{k} \\ \Leftrightarrow \left(MS_{k}w + \overline{M} \, \overline{S_{k}} \, \overline{w}\right)^{*} \left(MS_{k}w + \overline{M} \, \overline{S_{k}} \, \overline{w}\right) &\geq \tau^{2} \left(S_{k}w + \overline{S_{k}} \, \overline{w}\right)^{*} \left(S_{k}w + \overline{S_{k}} \, \overline{w}\right) \\ \Leftrightarrow w^{*}S_{k}^{*} \left(M^{*}M - \tau^{2}I\right)S_{k}w &\geq -\operatorname{Re} \left(w^{T}S_{k}^{T} \left(M^{T}M - \tau^{2}I\right)S_{k}w\right), \quad \forall w \in \mathbf{C}^{k} \\ \Leftrightarrow w^{*}S_{k}^{*} \left(M^{*}M - \tau^{2}I\right)S_{k}w &\geq \operatorname{Re} \left(w^{T}S_{k}^{T} \left(M^{T}M - \tau^{2}I\right)S_{k}w\right), \quad \forall w \in \mathbf{C}^{k} \end{aligned}$$

2.2. Some Properties of the Real Perturbation Values. The following properties follow directly from the definition of $\tau_k(M)$: LEMMA 2.4.

(11)
$$\tau_k(\overline{M}) = \tau_k(M)$$

By using (2) for $\tau_1(M)$ and the fact that $\sigma_2(A) \ge \inf_{z \in S_2} ||Az||/||z||$ for any subspace S_2 of dimension 2 one can also prove that LEMMA 2.5. $\tau_1(M)$ is continuous in M at points M where $\operatorname{Im}(M) \neq 0$.

3. Connections to Other Parts of Mathematics.

3.1. Hermitian-Symmetric Inequalities. Pairs (A, B) where A is Hermitian and B complex symmetric matrix occur occasionally in analysis, for instance in quadratic Hermitian-symmetric inequalities:

(13)
$$z^*Az = \sum_{i,j=1}^n a_{ij}\overline{z}_i z_j \ge \left|\sum_{i,j=1}^n b_{ij}z_i z_j\right| = \left|z^T Bz\right|, \quad \forall z \in \mathbf{C}^n.$$

Such inequalities are surveyed in [4] where the following theorem is proved: LEMMA 3.1. The following six statements are equivalent

(i)
$$z^*Az \ge |z^TBz|$$
, $\forall z \in \mathbf{C}^n$
(ii) $x^*Ax + y^*Ay \ge 2 |x^TBy|$, $\forall x, y \in \mathbf{C}^n$
(iii) $x^*Ax + y^*Ay \ge 2 \operatorname{Re}(x^TBy)$, $\forall x, y \in \mathbf{C}^n$
(iv) the $2n \times 2n$ matrix
 $\mathcal{A} = \begin{bmatrix} A & \overline{B} \\ B & \overline{A} \end{bmatrix}$
is nonnegative definite, that is, $\zeta \mathcal{A}\zeta \ge 0$, $\forall \zeta \in \mathbf{C}^{2n}$
(v) $\zeta^*\mathcal{A}\zeta \ge 0$, for all $\zeta \in \mathbf{C}^{2n}$ of the form
 $\zeta = \begin{bmatrix} z \\ \overline{z} \end{bmatrix}$ where $z \in \mathbf{C}^n$

(vi)
$$z^*Az \ge \operatorname{Re}(z^TBz), \ \forall z \in \mathbf{C}^n$$

There are several interesting instances of such inequalities, for example the Grunsky inequalities in the theory of univalent functions. Hermitian-symmetric inequalities also occur in analytic continuation, harmonic analysis and the moment problem for complex measures. Some of these applications are described below.

3.1.1. Grunsky Inequalities. The most celebrated example of Hermitian-symmetric inequalities is probably the Grunsky inequalities in the classical theory of univalent analytic functions: If f(z) is a normalized (i.e. f(0) = 0, f'(0) = 1) analytic function on the unit disc, then a necessary and sufficient condition that f be univalent, is that

$$\sum_{i,j=1}^n x_i \overline{x}_j \log \frac{1}{1-z_i \overline{z}_j} \ge \left| \sum_{i,j=1}^n x_i x_j \log \left[\frac{z_i z_j}{f(z_i) f(z_j)} \frac{f(z_i) - f(z_j)}{z_i - z_j} \right] \right|$$

for all $x_1, \ldots, x_n \in \mathbb{C}$, all z_1, \ldots, z_n in the unit disc, and all $n = 1, 2, \ldots$. Of course, the difference quotient is interpreted as $f'(z_i)$ if $z_i = z_j$.

3.1.2. The Moment Problem. Consider a finite sequence of complex numbers a_0 , a_1, \ldots, a_{2N} where a_0 is real and N is a positive integer. Define $a_{-n} = \overline{a}_n$, for $n = 1, 2, \ldots, 2N$. In [4] it is shown that the following conditions are equivalent:

(a) There exists an infinite sequence of complex numbers $(a_j)_{j=2N+1}^{\infty}$ such that the function $f(z) = a_0 + 2a_1z + 2a_2z^2 + 2a_3z^3 + \dots$ is analytic in the unit disc of the complex plane and satisfies

$$\operatorname{Re} f(z) \ge 0, \quad |z| \le 1.$$

(b)

$$\sum_{i,j=0}^N a_{i-j}c_i\overline{c}_j \ge \left|\sum_{i,j=0}^N a_{i+j}c_ic_j\right|, \ \forall c_0, c_1, \dots, c_N \in \mathbf{C}.$$

3.1.3. Reproducing Kernel Hilbert Spaces. Let Ω be a finite domain in the complex *z*-plane which is bounded by *n* closed analytic curves $C_{\nu}, \nu = 1, 2, ..., n$. The Green function $g(z, \zeta)$ of Ω is defined by the following properties

(a) $g(z,\zeta)$ is harmonic for $\zeta \in \Omega$ fixed except for $z = \zeta$.

- (b) $g(z,\zeta) + \log |z-\zeta|$ is harmonic in the neighborhood of $z = \zeta$.
- (c) $g(z,\zeta) \equiv 0$ for $z \in \partial \Omega$ and $\zeta \in \Omega$.

The kernel functions are defined by

$$K(z,\overline{\zeta}) = -rac{2}{\pi}rac{\partial^2 g(z,\zeta)}{\partial z\partial\overline{\zeta}}, \quad L(z,\zeta) = -rac{2}{\pi}rac{\partial^2 g(z,\zeta)}{\partial z\partial\zeta}.$$

The following Hermitian/symmetric relations follow from the definitions:

$$K(z,\overline{\zeta}) = K(\zeta,\overline{z}), \quad L(z,\zeta) = L(\zeta,z).$$

One often also introduces the function $l(z,\zeta) = 1/\pi(z-\zeta)^2 - L(z,\zeta)$. One can show that for any Ω :

$$\sum_{i,j=1}^{n} x_i \overline{x}_j K(z_i, \overline{z}_j) \ge \left| \sum_{i,j=1}^{n} x_i x_j l(z_i, z_j) \right|, \quad \forall x_i \in \mathbf{C}, \ \forall z_i \in \Omega,$$

which is yet another example of a Hermitian-symmetric inequality. For a discussion of simultaneous diagonalization of K and l see [2]. See also [13] for an interesting discussion on connections to Hilbert transforms and the Fredholm integral equation.

3.2. Consimilarity. We say that two matrices C, D are consimilar if there is a nonsingular P such that $\overline{P}^{-1}CP = D$. A mapping $T: V \to W$ between complex vector spaces V and W is called an *antilinear transformation* if

$$T(\alpha x + \beta y) = \overline{\alpha}T(x) + \overline{\beta}T(y), \quad \forall \alpha, \beta \in \mathbf{C}, x, y \in V.$$

Just as similar matrices are matrix representations of a linear transformation in different bases, consimilar matrices are matrix representations of an antilinear transformation in different bases. For a collection of results for consimilarity and more references see [8, Chapter 4]. Consimilarity is closely related to the simultaneous * and T-diagonalization in (9), see [8, Chapter 4.6], [7].

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