[16] R. I. Leine and N. van de Wouw, *Stability and Convergence of Mechanical Systems With Unilateral Constraints*. Berlin/Heidelberg, Germany: Springer-Verlag, 2008, vol. 36, Lecture Notes in Applied and Computational Mechanics.

[17] M. D. P. Monteiro Marques, *Differential Inclusions in Nonsmooth Mechanical Problems. Shocks and Dry Friction*, ser. Progress in Nonlinear Differential Equations and Their Applications 9. Basel, Switzerland: Birkhauser, 1993.

[18] M. Osorio and J. A. Moreno, "Dissipative design of observers for multivalued nonlinear systems," in *Proc. IEEE Conf. Decision and Control*, 2006, pp. 5400–5405.

[19] R. T. Rockafellar and R. J. B. Wets, *Variational Analysis*. Berlin: Springer, 1998.

[20] A. J. van der Schaft and J. M. Schumacher, "The complementary-slackness class of hybrid systems," *Mathem. Control, Signals and Syst.*, vol. 9, pp. 266–301, 1996.

[21] F. Vasca, L. Iannelli, M. K. Camlibel, and R. Frasca, "A new perspective for modeling power electronics converters: complementarity framework," *IEEE Trans. Power Electron.*, vol. 24, no. 2, pp. 456–468, Feb. 2009.

An Average Performance Limit of MIMO Systems in Tracking Multi-Sinusoids With Partial Signal Information

Weizhou Su, *Senior Member, IEEE*, Li Qiu, *Fellow, IEEE*, and Jie Chen, *Fellow, IEEE*

***Abstract*—This paper studies the best achievable reference tracking performance of MIMO linear time-invariant (LTI) feedback systems with partial reference information. The reference signal to be tracked is a multi-tone sinusoidal signal. It is assumed that, other than the instantaneous values of the reference signal, only the frequencies of the sinusoidal components are known. The tracking performance is measured by the energy of the tracking error. With this partial information of the reference signal, we consider an averaged performance measure and obtain an explicit expression of the best achievable performance. The expression shows how the harmonic frequencies and the zero directions may affect the performance, and further, how a performance degradation may result under the available partial information.**

***Index Terms*—Linear systems, nonminimum phase zeros, optimal control, performance limitation, tracking.**

I. Introduction

This paper studies an optimal tracking problem for MIMO feedback systems with a linear time-invariant (LTI) plant. The reference to be considered consists of multi-tone sinusoidal components, with a step and a finite number of harmonics at given frequencies. The tracking performance of the system is measured by the integral square of the tracking error signal. We desire to obtain an explicit expression of the best achievable tracking performance, and our essential objective is to demonstrate via such a result how in a general MIMO setting, this performance may suffer from incomplete information of the reference input available to the tracking controller. Here by the incomplete information of the reference signal, we mean that while the signal's instantaneous values can be accessed for tracking, its evolution is nevertheless unknown and must be estimated. In other words, the reference information is only given partially, and as such, the tracking objective is to be met in an information-constrained setting.

The work continues the authors' previous studies in [4] and [5], where the full information of the reference signal, i.e., the entire history and evolution of the reference signal, is accessible by the feedback controller. One should note that this assumption is always true when the reference is a step signal. In the case of a multi-tone sinusoidal signal, however, the assumption requires that the magnitudes and phases of the harmonics must all be available to the controller, which can be rather demanding. The problem of tracking under partial information, as alluded to above, presents a more realistic scenario. This case was investigated for SISO systems in [5], with respect to a pure sinusoid, which led to an explicit formula of the best tracking performance and shows that an additional cost must be paid when dealing with partial information. The present work seeks to generalize the result to MIMO systems with a multi-tone signal consisting of arbitrarily many harmonics. This will allow us to see how in a general MIMO system the information contents of the reference signal and the directionality properties of the plant may couple to constrain the tracking objective. Indeed, we show that while in the full information case an averaged performance index is solely determined by the harmonic frequencies of the reference and the zero locations of the plant, under partial information it will also be affected by the zero directions.

Our work is none but a new endeavor in a long series of work devoted to the understanding and characterization of the fundamental performance limitation in reference signal tracking. Notable earlier results in this effort includes [1], which revealed that the performance limit of an MIMO LTI system in tracking a step signal is determined by the nonminimum-phase zeros of the plant. This work has since been extended to various settings, e.g., with respect to sinusoidal signals [4], for discrete time systems [3], [6], and for systems subject to modeling uncertainties or disturbances [2].

The notation used throughout this paper is fairly standard. For any complex number, vector and matrix, denote their conjugate, conjugate transpose, real and imaginary parts by $\bar{(\cdot)}$, $(\cdot)^*$, $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$, respectively. The phase or argument of a nonzero complex number is denoted by $\angle(\cdot)$. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open right and left half planes be denoted by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. $\mathcal{L}_2$ is the standard frequency domain Lebesgue space. $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspaces of $\mathcal{L}_2$ containing functions that are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. It is well-known that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ constitute orthogonal complements in $\mathcal{L}_2$. The Euclidean vector norm and the norm in the space $\mathcal{L}_2$ are both denoted by $\|\cdot\|_2$. The Frobenius norm of a matrix is denoted by $\|\cdot\|_F$. $\mathcal{RH}_\infty$ is the set of all stable, rational transfer matrices. Finally, the angle between two unitary vectors u, v is defined by $\cos\langle u, v\rangle := |u^*v|$, which quantifies the orientation between the two subspaces spanned by u and v, respectively.

II. Problem Statement

The system under consideration in this paper is shown in Fig. 1. Here $P(s)$ is the transfer function of a given plant whose output $z(t)$ and measurement $y(t)$ may not be the same, $K(s)$ is the transfer function of a two degree of freedom (2DOF) controller, S is a signal generator,

Manuscript received October 18, 2008; revised March 09, 2009 and May 14, 2009. Current version published August 05, 2009. This work was supported in part by the Natural Science Foundation of China under the grants NSFC-60774057, NSFC-60834003 and NSFC-60628301, by the Hong Kong Research Grants Council under the grant HKUST618906, and by the NSF/USA under Grant ECCS-0801874. Recommended by Associate Editor D. Arzelier.

W. Su is with School of Automation Science and Engineering, South China University of Technology, Guangzhou 510640, China (e-mail: wzhsu@scut.edu.cn).

L. Qiu is with Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China (e-mail: eeqiu@ust.hk).

J. Chen is with Department of Electrical Engineering, University of California Riverside, Riverside, CA 92521-0425 USA (e-mail: jchen@ee.ucr.edu).

Digital Object Identifier 10.1109/TAC.2009.2023969

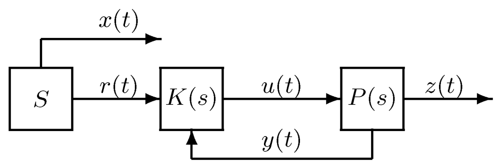

Fig. 1. Two-parameter control structure with partial reference information.

whose output $r(t)$ is a multitone sinusoidal reference signal with a step and l harmonics at the given frequencies ω_k, $k = 1, \ldots, l$, written as

$$r(t) = \sum_{k=0}^{l} \left(\bar{v}_k e^{-j\omega_k t} + v_k e^{j\omega_k t}\right) = 2\sum_{k=0}^{l} [\mathrm{Re}(v_k)\cos\omega_k t + \mathrm{Im}(v_k)\sin\omega_k t] \quad (1)$$

where $\omega_0 = 0$ and $2v_0$ is the magnitude of the step signal. We use the vector

$$v = [\bar{v}_{-l}^* \quad \cdots \quad \bar{v}_{-1}^* \quad v_0^* \quad v_1^* \quad \cdots \quad v_l^*]^*$$

to capture the magnitude and phase information of all frequency components of the reference. The signal $x(t)$ is the internal state of the signal generator S, which includes all information of the parameter vector v and can be selected as

$$x(t) = \begin{bmatrix} \bar{v}_1 e^{-j\omega_1 t} + v_1 e^{j\omega_1 t} \\ \vdots \\ \bar{v}_l e^{-j\omega_l t} + v_l e^{j\omega_l t} \end{bmatrix}. \quad (2)$$

Denote the k-th component of $x(t)$ by $x_k(t)$. Then the reference signal $r(t)$, the output of the signal generator, is the sum of $x_k(t)$, $k = 1, \ldots, l$, i.e.,

$$r(t) = \sum_{k=1}^{l} x_k(t). \quad (3)$$

Write $P(s) = \left[\begin{smallmatrix} G(s) \\ H(s) \end{smallmatrix}\right]$, where $G(s)$ is the transfer function from $u(t)$ to $z(t)$ and $H(s)$ the transfer function from $u(t)$ to $y(t)$. The typical tracking problem is to design a controller $K(s)$ so that the closed-loop system is internally stabilized and the plant output $z(t)$ asymptotically tracks the signal $r(t)$.

In [4], we presented a complete result for the tracking performance limit under the assumption that the controller can access $x(t)$, and hence all the past and future values of the reference signal in advance. This case is referred to as the full reference information case [5]. In contrast, herein we limit the controller access to $r(t)$ only, referred as the partial reference information case [5]. Additionally, we assume that ω_k are known. This reference information entails that $r(t)$ is fully available to $K(s)$, but v_k may not be known; in other words, both the magnitudes and phases of the harmonic components of $r(t)$ are unknown. Intuitively, one expects that due to this information constraint, a performance degradation is likely to result. This constitutes the central issue that we attempt to explore for general MIMO LTI systems in the present paper.

The tracking performance is quantified by the energy measure

$$J(v) = \int_0^\infty \|r(t) - z(t)\|_2^2 \, dt = \int_0^\infty \|e(t)\|_2^2 \, dt. \quad (4)$$

For this measure to be meaningful, we make the following assumptions throughout the paper.

Assumption 1:

1) $P(s), G(s)$ and $H(s)$ have the same unstable poles.
2) $G(s)$ has no zero at $-j\omega_k, j\omega_k, k = 0, 1, \ldots, l$.

The first condition means that the measurement can be used to stabilize the system and at the same time does not introduce any additional unstable modes. This condition is satisfied in the case of output feedback, where $y(t) = z(t)$, and that of state feedback, where $y(t)$ is the state vector of the system $G(s)$. Alternatively, an equivalent condition is that if $P(s) = \left[\begin{smallmatrix} N(s) \\ L(s) \end{smallmatrix}\right] M^{-1}(s)$ is a coprime factorization, then $N(s)M^{-1}(s)$ and $L(s)M^{-1}(s)$ are also coprime factorizations. The second condition is necessary for the tracking problem to be well-posed.

In the full information case, the optimal tracking performance with respect to v is given by

$$J_{\mathrm{opt}}(v) = \inf_K J(v),$$

which, naturally, depends on v. An averaged performance cost independent of v can be obtained by averaging $J_{\mathrm{opt}}(v)$ over a set of random vectors with zero mean, unit covariance, and uncorrelated conjugate, that is

$$J_{\mathrm{opt}} = \boldsymbol{E}\left\{J_{\mathrm{opt}}(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I, \boldsymbol{E}(vv^T) = 0\right\}.$$

Explicit expressions for $J_{\mathrm{opt}}(v)$ and J_{opt} were obtained in [4]. It should be noted that the statistical properties $\boldsymbol{E}(vv^*) = I, \boldsymbol{E}(vv^T) = 0$ dictate that $\boldsymbol{E}[\mathrm{Re}(v)\mathrm{Re}(v)^T] = \boldsymbol{E}[\mathrm{Im}(v)\mathrm{Im}(v)^T] = (1/2)I$, $\boldsymbol{E}(\mathrm{Re}(v)\mathrm{Im}(v)^T) = 0$; i.e., the coefficients $2\mathrm{Re}(v_k)$ and $2\mathrm{Im}(v_k)$ of the reference in real form (1) are independent and the covariances of the coefficients are $2I$.

In the partial information case, since v_k are not available to the controller, it is generally not possible to design a controller to minimize $J(v)$. Instead, it is more plausible to adopt an average performance measure in the following sense. Consider again the set of random vectors v with zero mean, unit covariance, and uncorrelated conjugate. An average tracking performance can be formulated as

$$E = \boldsymbol{E}\left\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I, \boldsymbol{E}(vv^T) = 0\right\}.$$

The best achievable average performance, thus, is given by

$$E_{\mathrm{opt}} = \inf_K E.$$

It is immediately clear that $E_{\mathrm{opt}} \geq J_{\mathrm{opt}}$. Note that in the full information case, $E_{\mathrm{opt}} = J_{\mathrm{opt}}$, a consequence borne out of the fact that the optimal tracking controller is independent of v [4]. This desirable property, however, no longer holds when only partial information is available, which leads to the outcome that $E_{\mathrm{opt}} > J_{\mathrm{opt}}$; this strict inequality was established in [5] for one sinusoid and will be seen true in general in the present work. Indeed, our purpose herein is to find an analogous expression for E_{opt}, with which we may analyze why and how much E_{opt} is in excess of J_{opt}.

We shall need to conduct an allpass factorization on $G(s)$, which we quote from [4]. Suppose that $G(s)$ has nonminimum phase zeros $z_1, \ldots, z_m$. Then $G(s)$ can be factorized in the form of

$$G(s) = G_1(s) \cdots G_m(s) G_0(s)$$

where $G_0(s)$ has only minimum phase zeros, and

$$G_i(s) = \frac{s - z_i}{s + z_i^*}\eta_i\eta_i^* + V_i V_i^* = U_i \begin{bmatrix} \frac{s-z_i}{s+z_i^*} & 0 \\ 0 & I \end{bmatrix} U_i^* \quad (5)$$

with η_i being a unitary vector and $U_i = [\eta_i \quad V_i]$ a unitary matrix. The vector η_i is referred to as a directional vector associated with the zero z_i. We denote

$$G_{in}(s) = G_1(s)\cdots G_m(s).$$

When $G(s)$ contains only one pair of conjugate complex nonminimum phase zeros z and z^*, with the directional vector of z being η, then we may take $z_1 = z$, $\eta_1 = \eta$, $z_2 = z^*$ ([9], [10])

$$\eta_2 = \sqrt{\gamma}\left(\overline{\eta} - \frac{\alpha\beta^*}{z^*}\eta\right) \tag{6}$$

and

$$G_{in}(s) = G_1(s)G_2(s) = I - \frac{4\alpha\gamma}{s^2+2\alpha s+|z|^2} \times \left[s\mathrm{Re}\left(\eta\eta^* - \frac{\alpha\beta}{z}\overline{\eta}\eta^*\right) + \mathrm{Re}(z\eta\eta^* - \alpha\beta\overline{\eta}\eta^*)\right] \tag{7}$$

where

$$\alpha = \mathrm{Re}(z), \qquad \beta = \eta^T\eta, \qquad \gamma = \left(1 - \frac{\alpha^2|\beta|^2}{|z|^2}\right)^{-1}. \tag{8}$$

Since $N(s)M^{-1}(s)$ is a coprime factorization of $G(s)$, $G(s)$ and $N(s)$ share the same nonminimum phase zeros, and likewise, $N(s)$ can be factorized as

$$N(s) = G_{in}(s)N_0(s) \tag{9}$$

where $N_0(s)$ is an outer function (for details, see, e.g., [7]).

III. Main Results

Our main result in this paper is the following explicit formula of the best performance attainable under the partial reference information.

Theorem 1: Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$ with directional vectors $\eta_1, \eta_2, \ldots, \eta_m$. Then

$$E_{\mathrm{opt}} = 2\sum_{k=0}^{l}\sum_{i=1}^{m}\left(\frac{1}{z_i^*+j\omega_k} + \frac{1}{z_i - j\omega_k}\right) + \sum_{k=1}^{l}\frac{\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2}{2\omega_k}. \tag{10}$$

Furthermore, denote

$$Z = \mathrm{diag}\left(\sqrt{\mathrm{Re}(z_1)}, \ldots, \sqrt{\mathrm{Re}(z_m)}\right),$$

$$R = \begin{bmatrix} 1 & \eta_1^*\eta_2 & \cdots & \eta_1^*\eta_m \\ \eta_2^*\eta_1 & 1 & \cdots & \eta_2^*\eta_m \\ \vdots & \vdots & \vdots & \vdots \\ \eta_m^*\eta_1 & \eta_m^*\eta_2 & \cdots & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} -z_1^* & -2\gamma_{12}\eta_1^*\eta_2 & \cdots & -2\gamma_{1m}\eta_1^*\eta_m \\ 0 & -z_2^* & \cdots & -2\gamma_{2m}\eta_2^*\eta_m \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -z_m^* \end{bmatrix}$$

where $\gamma_{ij} = \sqrt{\mathrm{Re}(z_i)\mathrm{Re}(z_j)}$, $i, j = 1, \ldots, m$. Then

$$\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 = 16\omega_k^2\left\|R^{1/2}Z\left(\omega_k^2 I + A^2\right)^{-1}ZR^{1/2}\right\|_F^2. \tag{11}$$

The proof is given in the Appendix.

It can be shown, as in [5], that with full reference information, the optimal tracking performance is given by

$$J_{\mathrm{opt}} = 2\sum_{k=0}^{l}\sum_{i=1}^{m}\left(\frac{1}{z_i^*+j\omega_k} + \frac{1}{z_i - j\omega_k}\right).$$

Hence, Theorem 1 exhibits that a performance degradation will necessarily result when only partial reference information is available. This result also gives the indication that in an MIMO system, the additional cost not only depends on the nonminimum phase zeros, but is likely to depend on the relative orientation of the zero directional vectors as well; the latter is seen from the dependence of E_{opt} on the inner products $\eta_i^*\eta_j$. While this aspect appears difficult to characterize in general, an appeal to special and extreme cases may yield useful insight. In this vein, we note from [5], or by a direct calculation of (11), that if $G(s)$ has two nonminimum phase zeros, then

$$\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 = 4\left[\sin^2\langle\eta_1,\eta_2\rangle\left(\sin^2\theta_1^{(k)} + \sin^2\theta_2^{(k)}\right) + \cos^2\langle\eta_1,\eta_2\rangle\sin^2\left(\theta_1^{(k)}+\theta_2^{(k)}\right)\right] \tag{12}$$

where z_1, z_2 are nonminimum phase zeros with directional vectors η_1, η_2, and $\theta_i^{(k)} = 2\angle(z_i - j\omega_k)$, $i = 1, 2$. This result reveals that indeed, the performance degradation depends on the angle between the zero directional vectors.

For SISO systems, we may also obtain a simple expression when the plant has more than two nonminimum phase zeros. A direct calculation of (11) shows that for a SISO plant $G(s)$ with nonminimum phase zeros $z_1, z_2, \ldots, z_m$

$$E_{\mathrm{opt}} = 2\sum_{k=0}^{l}\sum_{i=1}^{m}\left(\frac{1}{z_i^*+j\omega_k} + \frac{1}{z_i - j\omega_k}\right) + \sum_{k=1}^{l}\frac{2}{\omega_k}\sin^2 2\left[\sum_{i=1}^{m}\angle(z_i - j\omega_k)\right]. \tag{13}$$

In particular, if $G(s)$ has only one real nonminimum phase zero z, then

$$E_{\mathrm{opt}} = 2\sum_{k=0}^{l}\left(\frac{1}{z+j\omega_k} + \frac{1}{z - j\omega_k}\right) + \sum_{k=1}^{l}\frac{2}{\omega_k}\sin^2 2\angle(z - j\omega_k). \tag{14}$$

These formulas extend a result in [5] with respect to one sinusoid.

Note from (12) that when $\cos\langle\eta_1,\eta_2\rangle = 0$, i.e., the two directional vectors are orthogonal, then

$$\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 = 4\left(\sin^2\theta_1^{(k)} + \sin^2\theta_2^{(k)}\right).$$

Thus, in light of (14), the two zeros contribute to E_{opt} independently of one another. On the other hand, when $\cos\langle\eta_1,\eta_2\rangle = 1$, i.e., the two directional vectors are parallel, then

$$\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 = 4\sin^2\left(\theta_1^{(k)}+\theta_2^{(k)}\right).$$

As seen from (13), in this case, the zeros act in the same fashion as they do in a SISO system.

When the two nonminimum phase zeros are further confined to be a pair of conjugate zeros, the zero coupling effect can be quantified more explicitly. The following corollary addresses this special case.

Corollary 1: Suppose that $G(s)$ has a pair of conjugate nonminimum phase zeros z, z^*, and the corresponding zero directional vector for z is η. Then

$$\begin{aligned}\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 &= \frac{4}{1-|\eta^T\eta|^2\cos^2\angle z} \\ &\times \left\{\left(1-|\eta^T\eta|^2\right)\left(\sin^2\theta_1^{(k)}+\sin^2\theta_2^{(k)}\right)\right. \\ &\left.+|\eta^T\eta|^2\sin^2\angle z\sin^2\left(\theta_1^{(k)}+\theta_2^{(k)}\right)\right\} \qquad (15)\end{aligned}$$

where $\theta_1^{(k)} = 2\angle(z-j\omega_k), \theta_2^{(k)} = 2\angle(z^*-j\omega_k)$. Furthermore

$$\begin{aligned}&\left(1-|\eta^T\eta|\right)\frac{32\gamma\alpha^2\omega_k^2}{(|z|^2-\omega_k^2)^2+4\alpha^2\omega_k^2} \\ &\leq \|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 \\ &\leq \left(1+|\eta^T\eta|\right)\frac{32\gamma\alpha^2\omega_k^2}{(|z|^2-\omega_k^2)^2+4\alpha^2\omega_k^2} \qquad (16)\end{aligned}$$

with α, β, and γ given by (8).

Proof: We first note from (6) that

$$\begin{aligned}\cos^2\langle\eta_1,\eta_2\rangle &= \left|\sqrt{\gamma}\left(\eta^*\overline{\eta} - \frac{\alpha\beta^*}{z^*}\right)\right|^2 \\ &= \gamma|\beta^*|^2\left|1-\frac{\alpha}{z^*}\right|^2 = \gamma|\eta^T\eta|^2\sin^2\angle z.\end{aligned}$$

In light of (12), and the fact that $\cos^2\angle z = \alpha^2/|z|^2$, we obtain (15). To establish (16), we use the allpass factor (7). Denote

$$\Gamma = \eta\eta^* - \frac{\alpha\beta}{z}\overline{\eta}\eta^*, \quad \Lambda = z\eta\eta^* - \alpha\beta\overline{\eta}\eta^*.$$

It follows by a direct calculation that

$$\begin{aligned}\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 &= \frac{16\alpha^2\gamma^2\cdot 4\omega_k^2}{\left[(|z|^2-\omega_k^2)^2+4\alpha^2\omega_k^2\right]^2} \\ &\times \left\|2\alpha\mathrm{Re}(\Lambda) - \left(|z|^2-\omega_k^2\right)\mathrm{Re}(\Gamma)\right\|_F^2.\end{aligned}$$

We then evaluate

$$\begin{aligned}&\left\|2\alpha\mathrm{Re}(\Lambda) - \left(|z|^2-\omega_k^2\right)\mathrm{Re}(\Gamma)\right\|_F^2 \\ &= \left\|2\alpha\left(\frac{\overline{\Lambda}+\Lambda}{2}\right) - \left(|z|^2-\omega_k^2\right)\left(\frac{\overline{\Gamma}+\Gamma}{2}\right)\right\|_F^2.\end{aligned}$$

Write

$$g = 2\alpha(z\eta - \alpha\beta\overline{\eta}) - \left(|z|^2-\omega_k^2\right)\left(\eta - \frac{\alpha\beta}{z}\overline{\eta}\right).$$

It can be shown that

$$2\alpha\left(\frac{\overline{\Lambda}+\Lambda}{2}\right) - \left(|z|^2-\omega_k^2\right)\left(\frac{\overline{\Gamma}+\Gamma}{2}\right) = \frac{1}{2}(g\eta^* + \overline{g}\eta^T).$$

As such

$$\begin{aligned}&\left\|2\alpha\mathrm{Re}(\Lambda) - \left(|z|^2-\omega_k^2\right)\mathrm{Re}(\Gamma)\right\|_F^2 \\ &= \frac{1}{4}\|g\eta^* + \overline{g}\eta^T\|_F^2 \\ &= \frac{1}{4}\mathrm{Tr}\left(\begin{bmatrix}1 & \beta^* \\ \beta & 1\end{bmatrix}\begin{bmatrix}g^*g & g^*\overline{g} \\ g^Tg & g^*g\end{bmatrix}\right) \\ &= \frac{1}{2}\left(\|g\|^2 + \mathrm{Re}(\beta g^Tg)\right).\end{aligned}$$

The remaining proof is then completed by evaluating $\|g\|^2$, which yields

$$\|g\|^2 = \frac{1}{\gamma}\left[\left(|z|^2-\omega_k^2\right)^2 + 4\alpha^2\omega_k^2\right],$$

and by noting the inequalities $-|\beta|\|g\|^2 \leq \mathrm{Re}(\beta g^Tg) \leq |\beta|\|g\|^2$. □

IV. Conclusion

In this paper, the fundamental performance limit of general MIMO feedback systems in tracking multitone sinusoidal signals is studied. It is assumed that, in addition to the instantaneous values of the reference signal, only the harmonic frequencies of the signal are known, whilst the information of magnitudes and phases of the harmonics is unavailable to the controller. This formulation presents a more realistic problem of tracking with a partial information structure, in contrast to the previous studies where the controller can access the complete information (past and future values) of the reference signal. A formula for the best achievable average tracking error has been derived, which shows that a performance degradation results due to the lack of full information for tracking. The degradation is found to be dependent on the nonminimum phase characteristics of the plant and the harmonic frequencies of the reference signal, and as shown in special cases, also on the angles between the zero directional vectors.

Appendix

Proof of the Theorem 1:

Proof: Let $G(s) = N(s)M^{-1}(s)$ be a coprime factorization. Then using the parameterization of all stabilizing 2DOF controllers [7], we find the transfer function from $x(t)$ to $z(t)$ to be $N(s)Q(s)$, where $Q(s)$ is an arbitrary $\mathcal{RH}_\infty$ transfer function which can be designed. Denote the Laplace transforms of $r(t)$ and $x(t)$ by $R(s)$ and $X(s)$, respectively. For a fixed v, the tracking performance $J(v)$ defined in (4) can be written as

$$J(v) = \|R(s) - N(s)Q(s)X(s)\|_2^2. \qquad \text{(A-1)}$$

Since $x(t)$ is not available, $X(s)$ is unknown. Decompose $Q(s)$ as

$$Q(s) = [Q_1(s), \ldots, Q_l(s)].$$

It follows that

$$R(s) - N(s)Q(s)X(s) = \sum_{k=1}^{l}[I - N(s)Q_k(s)]X_k(s). \qquad \text{(A-2)}$$

Substituting (A-2) into (A-1) leads to

$$J(v) = \left\|\sum_{k=1}^{l}[I - N(s)Q_k(s)]X_k(s)\right\|_2^2.$$

Similarly, $X_k(s)$ are unknown. However, by averaging $J(v)$ with $\boldsymbol{E}(vv^*) = I$, $\boldsymbol{E}(vv^T) = 0$, we find

$$\boldsymbol{E}\{J(v)\} = \boldsymbol{E}\left\{\sum_{k=1}^{l}\left\|[I - N(s)Q_k(s)] \times \left(\frac{\bar{v}_k}{s+j\omega_k} + \frac{v_k}{s-j\omega_k}\right)\right\|_2^2\right\} = \sum_{k=1}^{l}\left\|[I - N(s)Q_k(s)]\begin{bmatrix}\frac{1}{s+j\omega_k}I \\ \frac{1}{s-j\omega_k}I\end{bmatrix}\right\|_2^2. \quad \text{(A-3)}$$

Thus, in the partial information case while it may not be possible to find a $Q(s) \in \mathcal{RH}_\infty$ such that $J(v)$ is well-defined, one may design a $Q(s) \in \mathcal{RH}_\infty$ for $\boldsymbol{E}\{J(v)\}$ to be. It follows from the inner-outer factorization of $N(s)$ given in (9) that

$$\boldsymbol{E}\{J(v)\} = \sum_{k=1}^{l}\left\|\left[G_{in}^{-1}(s) - N_0(s)Q_k(s)\right]\begin{bmatrix}\frac{1}{s+j\omega_k}I \\ \frac{1}{s-j\omega_k}I\end{bmatrix}\right\|_2^2.$$

Let

$$e_{1k}(s) = G_{in}^{-1}(s)\begin{bmatrix}\frac{1}{s+j\omega_k}I \\ \frac{1}{s-j\omega_k}I\end{bmatrix} - \begin{bmatrix}\frac{G_{in}^{-1}(-j\omega_k)}{s+j\omega_k} \\ \frac{G_{in}^{-1}(j\omega_k)}{s-j\omega_k}\end{bmatrix} \quad \text{(A-4)}$$

and

$$e_{2k}(s) = \begin{bmatrix}\frac{G_{in}^{-1}(-j\omega_k)}{s+j\omega_k} \\ \frac{G_{in}^{-1}(j\omega_k)}{s-j\omega_k}\end{bmatrix} - N_0(s)Q_k(s)\begin{bmatrix}\frac{1}{s+j\omega_k}I \\ \frac{1}{s-j\omega_k}I\end{bmatrix}. \quad \text{(A-5)}$$

The averaged cost function $\boldsymbol{E}\{J(v)\}$ is further written as

$$\boldsymbol{E}\{J(v)\} = \sum_{k=1}^{l}\|e_{1k}(s) + e_{2k}(s)\|_2^2.$$

Notice the facts that $e_{1k}(s)$ belongs $\mathcal{H}_2^\perp$, and $e_{2k}(s)$ to $\mathcal{H}_2$ for some properly selected $Q_k(s) \in \mathcal{RH}_\infty$. This leads to

$$\boldsymbol{E}\{J(v)\} = \sum_{k=1}^{l}\left[\|e_{1k}(s)\|_2^2 + \|e_{2k}(s)\|_2^2\right]. \quad \text{(A-6)}$$

To obtain an explicit formula for $\|e_{1k}(s)\|_2^2$, let us consider

$$\left\|\frac{G_{in}^{-1}(s)}{s+j\omega_k} - \frac{G_{in}^{-1}(-j\omega_k)}{s+j\omega_k}\right\|_2^2 \quad \text{and} \quad \left\|\frac{G_{in}^{-1}(s)}{s-j\omega_k} - \frac{G_{in}^{-1}(j\omega_k)}{s-j\omega_k}\right\|_2^2$$

respectively. A simple manipulation gives rise to

$$\left\|\frac{G_{in}^{-1}(s)}{s+j\omega_k} - \frac{G_{in}^{-1}(-j\omega_k)}{s+j\omega_k}\right\|_2^2 = \left\|\frac{G_1^{-1}(s)}{s+j\omega_k} - \frac{G_1^{-1}(-j\omega_k)}{s+j\omega_k} + \Delta_k(s)\right\|_2^2 \quad \text{(A-7)}$$

where

$$\Delta_k(s) = \left[I - G_2(s), \ldots, G_m(s)G_m^{-1}(-j\omega_k), \ldots G_2^{-1}(-j\omega_k)\right] \times \frac{G_1^{-1}(-j\omega_k)}{s+j\omega_k}.$$

It is clear that

$$\frac{G_1^{-1}(s)}{s+j\omega_k} - \frac{G_1^{-1}(-j\omega_k)}{s+j\omega_k} = \frac{2\mathrm{Re}(z_1)\eta_1\eta_1^*}{(z_1 - s)(z_1 + j\omega_k)} \in \mathcal{H}_2^\perp$$

and

$$\Delta_k(s) \in \mathcal{H}_2.$$

Thus

$$\begin{aligned}\left\|\frac{G_{in}^{-1}(s)}{s+j\omega_k} - \frac{G_{in}^{-1}(-j\omega_k)}{s+j\omega_k}\right\|_2^2 &= \left\|\frac{2\mathrm{Re}(z_1)}{(z_1-s)(z_1+j\omega_k)}\right\|_2^2 + \|\Delta_k(s)\|_2^2 \\ &= \left\|\frac{2\mathrm{Re}(z_1)}{(z_1-s)(z_1+j\omega_k)}\right\|_2^2 + \left\|\frac{[G_2(s), \ldots, G_m(s)]^{-1}}{s+j\omega_k} - \frac{[G_2(-j\omega_k), \ldots, G_m(-j\omega_k)]^{-1}}{s+j\omega_k}\right\|_2^2 \\ &= \sum_{i=1}^{m}\left\|\frac{2\mathrm{Re}(z_i)}{(z_i-s)(z_i+j\omega_k)}\right\|_2^2 \\ &= \sum_{i=1}^{m}\left(\frac{1}{z_i+j\omega_k} + \frac{1}{z_i^*-j\omega_k}\right).\end{aligned}$$

Analogously, we have

$$\begin{aligned}\left\|\frac{G_{in}^{-1}(s)}{s-j\omega_k} - \frac{G_{in}^{-1}(j\omega_k)}{s-j\omega_k}\right\|_2^2 &= \sum_{i=1}^{m}\left\|\frac{2\mathrm{Re}(z_i)}{(z_i-s)(z_i-j\omega_k)}\right\|_2^2 \\ &= \sum_{i=1}^{m}\left(\frac{1}{z_i-j\omega_k} + \frac{1}{z_i^*+j\omega_k}\right).\end{aligned}$$

Consequently, in view of (A-4), we have shown that

$$\|e_{1k}(s)\|_2^2 = 2\sum_{i=1}^{m}\left(\frac{1}{z_i+j\omega_k} + \frac{1}{z_i-j\omega_k}\right). \quad \text{(A-8)}$$

Next, to calculate $\|e_{2k}(s)\|_2^2$, we construct the inner matrix function

$$T_k(s) = \begin{bmatrix}\frac{s-j\omega_k}{\sqrt{2}(s+\omega_k)}I & \frac{s+j\omega_k}{\sqrt{2}(s+\omega_k)}I \\ \frac{s+j\omega_k}{\sqrt{2}(s+\omega_k)}I & -\frac{s-j\omega_k}{\sqrt{2}(s+\omega_k)}I\end{bmatrix}$$

and denote

$$M_k(s) = \frac{G_{in}^*(-j\omega_k)(s-j\omega_k)^2 + G_{in}^*(j\omega_k)(s+j\omega_k)^2}{(s-\omega_k)(s+j\omega_k)(s-j\omega_k)}.$$

Pre-multiplying the matrix function $T_k(s)$ to $e_{2k}(s)$, its $\mathcal{H}_2$ norm remains invariant. Hence

$$\begin{aligned}\|e_{2k}(s)\|_2^2 &= \left\| T_k(s)\left\{ \begin{bmatrix} \frac{G_{in}^{-1}(-j\omega_k)}{s+j\omega_k} \\ \frac{G_{in}^{-1}(j\omega_k)}{s-j\omega_k} \end{bmatrix} - N_0(s)Q_k(s)\begin{bmatrix} \frac{1}{s+j\omega_k}I \\ \frac{1}{s-j\omega_k}I \end{bmatrix} \right\} \right\|_2^2 \\ &= \frac{1}{2}\left\| M_k(s) - N_0(s)Q_k(s)\frac{2(s-\omega_k)}{(s+j\omega_k)(s-j\omega_k)} \right\|_2^2 \\ &\quad + \frac{1}{2}\left\| \frac{G_{in}(-j\omega_k) - G_{in}(j\omega_k)}{s+\omega_k} \right\|_2^2.\end{aligned}$$

Select

$$\begin{aligned}Q_k(s) &= \frac{1+j}{2}\frac{s-j\omega_k}{s+\omega_k}N_0^{\dagger}(-j\omega_k)G_{in}^*(-j\omega_k) \\ &\quad + \frac{1-j}{2}\frac{s+j\omega_k}{s+\omega_k}N_0^{\dagger}(j\omega_k)G_{in}^*(j\omega_k) + \frac{s^2+\omega_k^2}{2(s+\omega_k)^2}\tilde{Q}_k(s)\end{aligned}$$

where $\tilde{Q}_k(s) \in \mathcal{RH}_\infty$ and $N_0^{\dagger}(s)$ is the pseudo-inverse of $N_0(s)$. As in the earlier part of this proof, conducting an orthogonal decomposition of $M_k(s)$ into the sum of two terms in $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$, respectively, and selecting $\tilde{Q}_k(s) \in \mathcal{RH}_\infty$ appropriately, we arrive at the conclusion

$$\begin{aligned}\inf_{Q_k(s)\in\mathcal{RH}_\infty} &\left\| M_k(s) - N_0(s)Q_k(s)\frac{2(s+\omega_k)}{(s+j\omega_k)(s-j\omega_k)} \right\|_2^2 \\ &= \left\| \frac{G_{in}^*(-j\omega_k) - G_{in}^*(j\omega_k)}{s-\omega_k} \right\|_2^2.\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}\inf_{Q_k(s)\in\mathcal{RH}_\infty} \|e_{2k}(s)\|_2^2 &= \frac{1}{2}\left\| \frac{G_{in}^*(-j\omega_k) - G_{in}^*(j\omega_k)}{s-\omega_k} \right\|_2^2 \\ &\quad + \frac{1}{2}\left\| \frac{G_{in}^*(-j\omega_k) - G_{in}^*(j\omega_k)}{s+\omega_k} \right\|_2^2 \\ &= \frac{\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2}{2\omega_k}.\end{aligned}$$

To complete the proof, we note from [8] that a state-space realization (A, B, C, D) for $G_{in}(s)$ is given by

$$\begin{aligned}A &= \begin{bmatrix} -z_1^* & -2\gamma_{12}\eta_1^*\eta_2 & \cdots & -2\gamma_{1m}\eta_1^*\eta_m \\ 0 & -z_2^* & \cdots & -2\gamma_{2m}\eta_1^*\eta_m \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -z_m^* \end{bmatrix}, \\ B &= -\sqrt{2}Z[\eta_1, \eta_2, \ldots, \eta_m]^* \\ C &= \sqrt{2}[\eta_1 \quad \eta_2 \quad \cdots \quad \eta_m]Z, \\ D &= I.\end{aligned} \tag{A-9}$$

Since

$$\begin{aligned}&G_{in}(-j\omega_k) - G_{in}(j\omega_k) \\ &= C\left[(-j\omega_k I - A)^{-1} - (j\omega_k I - A)^{-1}\right]B \\ &= 2j\omega_k C(-j\omega_k I - A)^{-1}(j\omega_k I - A)^{-1}B \\ &= 2j\omega_k C\left(\omega_k^2 I + A^2\right)^{-1}B\end{aligned}$$

and

$$\begin{aligned}&\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 \\ &= 4\omega_k^2\left\| C\left(\omega_k^2 I + A^2\right)^{-1}B \right\|_F^2 \\ &= 4\omega_k^2 \mathrm{Tr}\left(B^*\left[\left(\omega_k^2 I + A^2\right)^{-1}\right]^* C^*C\left(\omega_k^2 I + A^2\right)^{-1}B\right)\end{aligned}$$

the result follows by noting that

$$BB^* = C^*C = 2ZRZ,$$

and hence that

$$\begin{aligned}&\|G_{in}(-j\omega_k) - G_{in}(j\omega_k)\|_F^2 \\ &= 16\omega_k^2 \mathrm{Tr}\left(\left[\left(\omega_k^2 I + A^2\right)^{-1}\right]^* ZRZ\left(\omega_k^2 I + A^2\right)^{-1}ZRZ\right) \\ &= 16\omega^2\left\| R^{1/2}Z\left(\omega_k^2 I + A^2\right)^{-1}ZR^{1/2} \right\|_F^2.\end{aligned}$$

The proof is now completed. □

REFERENCES

[1] J. Chen, L. Qiu, and O. Toker, "Limitations on maximal tracking accuracy," *IEEE Trans. Automat. Control*, vol. 45, no. 2, pp. 326–331, Feb. 2000.

[2] G. C. Goodwin, M. E. Salgado, and J. I. Yuz, "Performance limitations for linear feedback systems in the presence of plant uncertainty," *IEEE Trans. Automat. Control*, vol. 48, no. 8, pp. 1312–1319, Aug. 2003.

[3] L. B. Jemaa and E. J. Davison, "Performance limitations in the robust servomechanism problem for discrete-time LTI systems," *IEEE Trans. Automat. Control*, vol. 48, no. 8, pp. 1299–1311, Aug. 2003.

[4] W. Su, L. Qiu, and J. Chen, "Fundamental performance limitations in tracking sinusoidal signals," *IEEE Trans. Automat. Control*, vol. 48, no. 8, pp. 1371–1380, Aug. 2003.

[5] W. Su, L. Qiu, and J. Chen, "On performance limitation in tracking a sinusoid," *IEEE Trans. Automat. Control*, vol. 51, no. 8, pp. 1320–1325, Aug. 2006.

[6] W. Su, L. Qiu, and J. Chen, "Fundamental limit of discrete-time systems in tracking multi-tone sinusoidal signals," *Automatica*, vol. 43, no. 1, pp. 15–30, Jan. 2007.

[7] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: MIT Press, 1985.

[8] J. Chen, "Logarithmic integrals, interpolation bounds, and performance limitations in MIMO feedback systems," *IEEE Trans. Automat. Control*, vol. AC-45, no. 6, pp. 1098–1114, Jun. 2000.

[9] J. E. Wall, Jr., J. C. Doyle, and C. A. Harvey, "Tradeoffs in the design of multivariable feedback systems," in *Proc. 18th Allerton Conf.*, 1980, pp. 715–725.

[10] Z. Zhang, "Loop Transfer Recovery for Nonminimum Phase Plants and Ill-Conditioned Plants," Ph.D. dissertation, , Univ. Michigan, Ann Arbor, MI, 1990.