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# An $\mathcal{H}_\infty$ Approach to Robust Adaptive Control *

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## Abstract

Robust adaptive control is investigated using the $\mathcal{H}_\infty$ approach. An equivalent measure to the $\mathcal{H}_\infty$-norm is adopted to quantify the unmodeled dynamics associated with adaptive estimation of the nominal model based on time-domain measurement data. Such an equivalent description for modeling errors allows $\mathcal{H}_\infty$ optimization to be successfully used in adaptive control to achieve robust stability and performance comparable to $\mathcal{H}_\infty$ control. The proposed adaptive control consists of the total least-squares (TLS) algorithm for adaptive model estimation, and $\mathcal{H}_\infty$-loopshaping for adaptive controller design. Our results show that the proposed adaptive control system admits robust stability and performance asymptotically, provided that the estimated plant model converges.

## 1 Introduction

In our earlier work [7], a novel notion termed as *uncertainty equivalence principle* is proposed that enables the quantification of the modeling errors in an equivalent measure to the $\mathcal{H}_\infty$ norm based on the minimum *a priori* information on the underlying physical process. It is noted that the modeling errors are inevitable in engineering practice for any nominal model adaptively estimated using the time domain data. The equivalent quantification of the modeling error allows $\mathcal{H}_\infty$ optimization to be successfully applied to adaptive control. Hence the work presented in [7] offers an $\mathcal{H}_\infty$ approach to robust adaptive control and provides new insights to coping with the modeling errors. However the results presented in [7] are limited to stable systems only. Further study for $\mathcal{H}_\infty$ based robust adaptive control is necessary.

In this paper we continue our investigation on robust adaptive control based again on *uncertainty equivalence principle*, contrast to the conventional *certainty equivalence principle*. Roughly speaking, the modeling error in adaptive estimation can not be quantified at each time instant in terms of $\mathcal{H}_\infty$-norm based on real time data which represent only one time sample path. It turns out that even though the $\mathcal{H}_\infty$-norm of the error system can not be quantified, the output signal of the error system can be guaranteed to satisfy the same energy amplification constraint as the $\mathcal{H}_\infty$-norm, thereby providing an equivalent description of the dynamics uncertainty and enabling applications of $\mathcal{H}_\infty$ optimization in adaptive control to achieve equivalent stability margin and performance comparable to those achievable in $\mathcal{H}_\infty$ control. Moreover we will propose a specific adaptive feedback control system and prove its asymptotic stability and performance under some mild condition. The proposed adaptive control system employs the total least-squares (TLS) algorithm for adaptive model estimation capable of quantifying and minimizing the modeling error in terms of the normalized coprime factors of the physical system. It employs robust stabilization of uncertain systems described by normalized coprime factors for adaptive controller design leading to the performance index reminiscent of $\mathcal{H}_\infty$-based loopshaping [11, 19]. Simulation results are presented to illustrate the robustness of the proposed adaptive feedback control system. The notations of our paper are standard and will be made clear as we proceed.

## 2 Preliminaries

Denote $\ell^2_+$ as the collection of all the causal signals (which can be vector-valued for each time instance $t$) having bounded energy. Then for any $s(t) \in \ell^2_+$, its $\ell_2$-norm is defined by

$$\|s\|_2 := \sqrt{\sum_{t=0}^{\infty} \|s(t)\|^2} = \sqrt{\sum_{t=0}^{\infty} s'(t)s(t)} < \infty.$$

Assume that the uncertainty represented by its transfer function $\Delta(z)$ is $\ell^2$-BIBO stable. Then its $\mathcal{H}_\infty$-norm $\|\Delta\|_\infty$ is bounded, determined by its frequency response. Let $\mu = \{\mu(t)\}_{t=0}^{\infty} \in \ell^2_+$ be the input. Then the output $\nu = \{\nu(t)\}_{t=0}^{\infty} \in \ell^2_+$. Moreover

$$\delta = \|\Delta\|_\infty := \sup_{\omega \in \mathbf{R}} \overline{\sigma}\left(\Delta(e^{j\omega})\right) = \sup_{\mu \in \ell^2_+} \frac{\|\nu\|_2}{\|\mu\|_2} \qquad (1)$$

with $\overline{\sigma}(\cdot)$ the maximum singular value. That is, $\mathcal{H}_\infty$-norm is the square-root of the worst-case energy amplification, or $\ell^2$-gain. The collection of all stable rational transfer functions is denoted by $\mathbf{R}\mathcal{H}_\infty$, and its closure is denoted by $\mathcal{H}_\infty$.

For any signal $s = \{s(t)\}_{t=0}^{\infty} \in \ell^2_+$, we define $\pi_T, T \geq 0$, as the projection operator satisfying

$$\pi_T\left[\{s(t)\}_{t=0}^{\infty}\right] = \{s(t)\}_{t=0}^{T}, \quad \|s\|_{[0,T]} = \|\pi_T[s]\|_2.$$

By slight abuse of notation, $\ell^2$-gain can also be defined over the finite time horizon by

$$\delta_T = \|\Delta\|_{\infty,[0,T]} := \sup_{\mu \in \ell^2_+} \frac{\|\nu\|_{[0,T]}}{\|\mu\|_{[0,T]}}.$$

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For any input/output pair $(\mu, \nu)$, there hold $\|\nu\|_{[0,T]} \leq \delta_T \|\mu\|_{[0,T]}$ for $T \geq 0$, and $\delta = \lim_{T\to\infty} \delta_T$. However it is difficult to determine $\delta_T$ for each $T \geq 0$ using only the time-domain measurement data, which requires the presence of the worst-case input/output signals. Indeed for adaptive feedback control, only one time sample-path is available. If $\Delta(z)$ represents the unmodeled dynamics at time $t = T$, then it is in general infeasible to estimate $\delta_T$, based on the time-domain measurement data, due to the possible absence of the worst-case input/output signals. Moreover the modeling error in adaptive control varies with respect to time $t$, and depends on the input and output of the true system, due to the use of the adaptive estimation algorithm. Hence $\Delta$ represents a time-varying nonlinear system. For this reason it is more suitable to be denoted by a nonlinear operator $\Delta_{\mathcal{N}}(\cdot)$, even though the true plant is linear and time-invariant, and the nominal plant model is parameterized by fixed order linear time-invariant systems. Such a time-varying nonlinear system is said to be $\ell^2$-BIBO stable, if

$$\|\Delta_{\mathcal{N}}\|_{\infty,[0,T]} := \sup_{\mu\in\ell^2_+} \frac{\|\Delta_{\mathcal{N}}(\mu)\|_{[0,T]}}{\|\mu\|_{[0,T]}} < \infty \;\forall\; T \geq 0. \tag{2}$$

The transition from $\Delta(z)$ to $\Delta_{\mathcal{N}}(\cdot)$ is important. Although the frozen model uncertainty at each time $t$ is linear, and may be represented by $\Delta(z)$, its $\mathcal{H}_\infty$-norm $\|\Delta\|_\infty$ can not be quantified based on input/output pair $(\{\mu(t)\}_{t=0}^T, \{\nu(t)\}_{t=0}^T)$ in general. But if an adaptive estimation algorithm can ensure $\|\Delta_{\mathcal{N}}(\mu)\|_{[0,T]} \leq \epsilon\|\mu\|_{[0,T]}$ for some $\epsilon > 0$, and any $\{\mu(t)\}_{t=0}^T$, and $T \geq 0$, then $\epsilon$ can serve as an upper bound for $\|\Delta_{\mathcal{N}}(\mu)\|_{[0,T]}$. This is essentially the *uncertainty equivalence principle* introduced in [7].

For adaptive estimation there is no guarantee for the $\mathcal{H}_\infty$-norm of the frozen model uncertainty $\Delta(z)$ at each time instant due to the lack of the worst-case signal. But if the upper bound $\epsilon$ under which $\|\nu\|_{[0,T]} \leq \epsilon\|\mu\|_{[0,T]}$ can be estimated and ensured for each pair $(\{\mu(t)\}_{t=0}^T, \{\nu(t)\}_{t=0}^T)$, and each $T \geq 0$, then the underlying adaptive feedback control system design can be approached via $\mathcal{H}_\infty$ based robust control. Hence if the adaptive feedback controller is designed appropriately such that some suitable feedback transfer matrix is internally stable with upper bound $\gamma$ minimized, and $\gamma\epsilon << 1$ at each time instant $t = T \geq 0$, then both stability and performance of the adaptive feedback control system can be ensured. This is illustrated in our earlier work [7] for adaptive control of stable systems with demonstrated robust stability. In this paper we propose a different adaptive feedback control system that is applicable to unstable plants. In the following section we investigate how the equivalent uncertainty can be quantified by employing the TLS (total least-squares) algorithm for model estimation, and how a robust adaptive feedback controller can be designed based on the $\mathcal{H}_\infty$ loopshaping method. The results will be demonstrated with a simulation example with comparisons to a traditional adaptive control system. Due to the space limit, all the proofs are skipped.

# 3 Proposed Adaptive Control Algorithm

The RLS algorithm has been the center piece for adaptive estimation. Its role in robust estimation has been investigated in [5, 20]. However for robust adaptive control of unstable plants, the TLS algorithm rather than the LS algorithm does a better job for adaptive estimation of the left coprime factors of the plant model in terms of the size of the equivalent modeling error. In this section we consider a different adaptive control system which consists of the TLS algorithm for estimation of the normalized coprime factors of the true system and design of robust stabilizing controllers for coprime factor uncertain systems.

## 3.1 Adaptive Model Estimation via TLS

Consider an $m$-input, $p$-output, and possibly unstable linear time-invariant system represented by its transfer matrix $P(z)$ which may have infinitely many poles but admits normalized coprime factorizations: $P(z) = \tilde{M}^{-1}\tilde{N}(z) = N(z)M^{-1}(z)$ where $\tilde{M}(z), \tilde{N}(z), M(z), N(z) \in \mathcal{H}_\infty$ have continuous frequency responses and satisfy

$$\begin{aligned} \tilde{M}(z)\tilde{M}'(z^{-1}) + \tilde{N}(z)\tilde{N}(z^{-1}) &= I \\ M'(z^{-1})M(z) + N'(z^{-1})N(z) &= I \end{aligned} \tag{3}$$

with $I$ the identity matrix. Let

$$\hat{M}(z) = \sum_{i=0}^{\kappa} \alpha_i z^{-i}, \quad \hat{N}(z) = \sum_{i=0}^{\kappa} \beta_i z^{-i}, \tag{4}$$

be approximants to the left normalized coprime factors $\tilde{M}(z)$ and $\tilde{N}(z)$, respectively. Let

$$\Delta_{\tilde{M}}(z) = \tilde{M}(z) - \hat{M}(z), \quad \Delta_{\tilde{N}}(z) = \tilde{N}(z) - \hat{N}(z), \tag{5}$$

be the associated modeling errors. Then by the input/output relation $\tilde{M}(q)y(t) = \tilde{N}(q)u(t)$,

$$e(t) = \begin{bmatrix} \hat{M}(q) & -\hat{N}(q) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \Delta(q) \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}$$

with $\Delta(z) = \begin{bmatrix} -\Delta_{\tilde{M}}(z) & \Delta_{\tilde{N}}(z) \end{bmatrix}$ and $q$ the unit advance operator. In the above equation, noise free measurements are assumed for simplicity. Then the nominal plant is given by $\hat{P}(z) = \hat{M}^{-1}(z)\hat{N}(z)$. We shall assume that

$$\delta_\kappa^* := \inf_{\hat{M},\hat{N}} \|\Delta\|_\infty = \inf_{\hat{M},\hat{N}} \left\| \begin{bmatrix} \tilde{M} - \hat{M} & \tilde{N} - \hat{N} \end{bmatrix} \right\|_\infty$$

is a decreasing sequence of $\kappa$ with limit zero where the infimum is taken over all possible $\hat{M}(z)$ and $\hat{N}(z)$ as parameterized in (4). It is easy to verify that the error function can be written as

$$e(t) = \Theta\psi(t), \quad \Theta = \begin{bmatrix} \alpha_0 & \cdots & \alpha_\kappa & \beta_0 & \cdots & \beta_\kappa \end{bmatrix}.$$

An ideal adaptive estimation algorithm is the one which minimizes

$$\|e\|_{[0,T]} := \sqrt{\sum_{t=0}^{T} e'(t)e(t)} = \sqrt{\text{trace}\left\{\sum_{t=0}^{T} e(t)e'(t)\right\}}$$

for each $T > 0$, subject to the condition that

$$\hat{M}(z)\hat{M}'(z^{-1}) + \hat{N}(z)\hat{M}'(z^{-1}) = I. \tag{6}$$

But this is very difficult to achieve, if not impossible. Thus we are led to consider replacing (6) by the set

$$\mathcal{S}_\Theta(r) = \left\{\Theta = \begin{bmatrix} \alpha_0 & \cdots & \alpha_\kappa & \beta_0 & \cdots & \beta_\kappa \end{bmatrix}: \Theta\Theta' = R^2,\ 1 \pm \overline{\sigma}(I-R) = r\right\}.$$

The matrix $R$ in the above equation is assumed to be symmetric and non-negative definite. The constraint $1 \pm \overline{\sigma}(I-R) = r$ is equivalent to $\overline{\sigma}(I-R) = |1-r|$. If $r = 1$, then the parameter set is simply denoted by $\mathcal{S}_\Theta$. It is obvious that those parameter vectors $\Theta$ satisfying (6) belong to $\mathcal{S}_\Theta$. Our objective is to search for $\Theta \in \mathcal{S}_\Theta$ which minimizes $\|e\|_{[0,T]}$ for each $T \geq 0$. The key is quantification of the equivalent uncertainty bound associated with such a minimization scheme in hope that it is comparable to $\delta_\kappa^*$.

Let $E_T = \begin{bmatrix} e(0) & e(1) & \cdots & e(T) \end{bmatrix} \in \mathbf{R}^{p\times(T+1)}$ for each $T \geq 0$. Then

$$E_T = \Theta\Psi_T, \quad \Psi_T = \begin{bmatrix} \psi(0) & \psi(1) & \cdots & \psi(T) \end{bmatrix}.$$

Define $\|E_T\|_2 := \|e\|_{[0,T]}$ and

$$\Theta_T^* := \arg\min_{\Theta\in\mathcal{S}_\Theta} \|E_T\|_2 = \arg\min_{\Theta\in\mathcal{S}_\Theta} \|\Theta\Psi_T\|_2. \quad (7)$$

Then the following result can be established.

**Theorem 3.1** *Let $0 < \delta_\kappa^* < 1$ and $\Theta = \Theta_T^*$ as in (7) for each $T \geq 0$. Then for each input/output pair $(\{y(t)\}, \{u(t)\})$ where $t = 0, 1, \cdots, T$, there holds*

$$\inf_{\Theta\in\mathcal{S}_\Theta} \|e\|_{[0,T]} = \|\Theta_T^*\Psi_T\|_2 \leq \frac{\delta_\kappa^*}{1-\delta_\kappa^*} \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{[0,T]}.$$

Theorem 3.1 provides an equivalent uncertainty bound for the frozen time error model $\Delta(z) = \hat{G}(z) - G(z)$ with $\Theta_T = \Theta_T^*$ at time $T$ where the measurement data are over the time horizon $[0,\ T]$. The computation of the optimal solution $\Theta_T^*$ as in (7) requires to compute the $p$ eigenvectors of the $N \times N$ matrix $\Psi_T\Psi_T'$ corresponding to the $p$ smallest eigenvalues. There exist numerically efficient algorithms to compute $\Theta_T^*$, given $\Theta_{T-1}^*$ which requires only $\mathcal{O}(N^2)$ comparable to the computational complexity for the RLS algorithm. Such algorithms are referred to as subspace tracking which track the subspace spanned by $\Theta_T^*$ for each $T > 0$. See [13, 15] for details.

Because we are concerned with the under modeling issue, $\alpha_0^*$ consisting of the first $p \times p$ elements of $\Theta_T^*$ is nonsingular generically especially when $T >> N$. In this case, $\Theta_T^*$ as in (7) can be written as

$$\begin{aligned} \Theta_T^* &= \alpha_0^* \begin{bmatrix} I & a_1^* & \cdots & a_\kappa^* & b_0^* & \cdots & b_\kappa^* \end{bmatrix}, \\ a_i^* &= (\alpha_0^*)^{-1}\alpha_i^*, \quad b_i^* = (\alpha_0^*)^{-1}\beta_i^*, \end{aligned} \quad (8)$$

for $i = 0, 1, \cdots, \kappa$. Denote $\phi(t)$ as the regressor vector. Let

$$\begin{aligned} Y_T &= \begin{bmatrix} y(0) & \cdots & y(T) \end{bmatrix}, \\ \Phi_T &= \begin{bmatrix} \phi(0) & \cdots & \phi(T) \end{bmatrix}, \\ \hat{\Theta}_T &= \begin{bmatrix} a_1 & \cdots & a_\kappa & b_0 & \cdots & b_\kappa \end{bmatrix}. \end{aligned}$$

Then by assuming that $\alpha_0$ is nonsingular, the error can be written equivalently into

$$\hat{E}_T = \alpha_0^{-1}E_T = Y_T - \hat{\Theta}_T\Phi_T. \quad (9)$$

With $a_i^*$ and $b_i^*$ as in (8) and in light of [17],

$$\hat{\Theta}_{\rm TLS} = \begin{bmatrix} a_1^* & \cdots & a_\kappa^* & b_0^* & \cdots & b_\kappa^* \end{bmatrix} \quad (10)$$

is in fact the TLS solution to $Y_T \approx \hat{\Theta}\Phi_T$ that minimizes

$$J_{\rm TLS}(T) := \text{trace}\left\{\hat{E}_T' \left(I + \hat{\Theta}_T\hat{\Theta}_T'\right)^{-1} \hat{E}_T\right\}. \quad (11)$$

Therefore the optimal solution $\Theta_T^*$ as defined in (7) is equivalent to the TLS solution to (11) and there holds

$$\Theta_T^* = \left(I + \hat{\Theta}_{\rm TLS}\hat{\Theta}_{\rm TLS}'\right)^{-1/2} \begin{bmatrix} I & \hat{\Theta}_{\rm TLS} \end{bmatrix} \in \mathcal{S}_\Theta.$$

At this moment, it is beneficial to compare the TLS solution $\hat{\Theta}_{\rm TLS}$ with the LS solution $\hat{\Theta}_{\rm LS}$ that minimizes

$$J_{\rm LS}(T) := \text{trace}\left\{\hat{E}_T'\hat{E}_T\right\}. \quad (12)$$

Because of the lack of the analytical expression for $\hat{\Theta}_{\rm TLS}$ in the case $p > 1$, we restrict our discussion to single output systems or $p = 1$. The following result characterizes the relations between the equivalent uncertainty bounds of the modeling errors in using the TLS and LS algorithms.

**Theorem 3.2** *Let $\hat{\Theta}_{\rm TLS}$ and $\hat{\Theta}_{\rm LS}$ be* TLS *and* LS *solutions which minimize* (11) *and* (12), *respectively. Then for $T > N$, there holds generically*

$$\begin{aligned} \|e_{\rm TLS}\|_{[0,T]} &\leq \|e_{\rm LS}\|_{[0,T]} \leq \frac{\xi\delta_\kappa^*}{1-\delta_\kappa^*} \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{[0,T]}, \\ \xi &= \left[1 + \frac{\underline{\sigma}(\Psi_T)}{\underline{\sigma}(\Phi_T)}\right] \sqrt{\frac{1 + \hat{\Theta}_{\rm TLS}\hat{\Theta}_{\rm TLS}'}{1 + \hat{\Theta}_{\rm LS}\hat{\Theta}_{\rm LS}'}} \end{aligned}$$

The modeling results in this subsection can be generalized to the case when noise and disturbance are present. However due to the space limit, they are not presented here. Interested readers can obtain the full version of our paper by contacting either of the authors.

### 3.2 Controller Design via $\mathcal{H}_\infty$ Control

The previous subsection establishes an equivalent uncertainty bound associated with the model $\hat{G}(z) = \begin{bmatrix} \hat{M}(z) & -\hat{N}(z) \end{bmatrix}$ estimated at sampling time $t = T$. Based on the uncertainty equivalence principle, $\hat{G}(z)$ is now treated as a frozen time model with equivalent uncertainty bound $\|\Delta\|_\infty \leq \varepsilon$ for some $\varepsilon > 0$ quantified in the previous subsection and a robust controller will be synthesized to stabilize the set of frozen time uncertain systems described by the normalized coprime factors of the true system. This problem is completely solved for continuous-time multivariable systems (cf. [4, 11, 19]) based on which $\mathcal{H}_\infty$ loopshaping is developed. It has successful applications to MIMO (multi-input/multi-output) feedback control system design. However there lacks a corresponding solution for discrete-time multivariable systems that will be investigated in this subsection.

Given the estimated model $\hat{G}(z)$ at time $T$, the nominal plant is given by $P_0(z) = \hat{M}^{-1}(z)\hat{N}(z)$. We assume that it admits an $n$th order state-space realization as

$$P_0(z) = D_0 + C_0(zI - A_0)^{-1}B_0 =: \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right], \quad (13)$$

where $(A_0, B_0)$ is stabilizable and $(C_0, A_0)$ is detectable. Because $\hat{G}(z) = \begin{bmatrix} \hat{M}(z) & -\hat{N}(z) \end{bmatrix}$ has an FIR form, a minimal realization can be easily obtained for $\hat{G}(z)$ from which the realization of $P_0(z)$ is also available. It is reasonable to assume that $\hat{G}(z)$ is strictly minimum phase due to its good approximation to the normalized coprime factors of the true system that ensures stabilizability and detectability of the realization $(A_0, B_0, C_0)$. Denote $R_0 = I + D_0'D_0$. Let $X \geq 0$ be the stabilizing solution to the following algebraic Riccati equation (ARE):

$$\begin{aligned} X &= A_0'XA_0 + C_0'C_0 - F_0'(R_0 + B_0'XB_0)F_0, \\ F_0 &= -(R_0 + B_0'XB_0)^{-1}(B_0'XA_0 + D_0'C_0) \end{aligned}$$

Then the normalized right coprime factors of $P_0(z) = N_0(z)M_0^{-1}(z)$ are given by

$$\begin{aligned} G_0(z) &= \begin{bmatrix} M_0(z) \\ N_0(z) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{c|c} A_0 + B_0F_0 & B_0\Omega_0^{-1} \\ \hline F_0 & \Omega_0^{-1} \\ C_0 + D_0F_0 & D_0\Omega_0^{-1} \end{array}\right] \end{aligned}$$

Our controller design aims to computing $G_0^{\text{inv}}(z)$, stable left inverse of $G_0(z)$, that has the minimum $\mathcal{H}_\infty$ norm. In fact $G_0^{\text{inv}}(z)$ is parameterized by

$$\left[\begin{array}{c|c} A - BD^+C + LD_\perp'C & -BD^+ + LD_\perp' \\ \hline \left(D^+ + Q(z)D_\perp'\right)C & \left(D^+ + Q(z)D_\perp'\right) \end{array}\right] \quad (14)$$

where $L$ is stabilizing and $Q(z)$ is stable.

**Theorem 3.3** *Consider the stable and proper left inverse $G_0^{\text{inv}}(z)$ as in (14). There exist stabilizing gains $L$ and $Q$ such that $\|G_0^{\text{inv}}\|_\infty < \gamma$, if and only if the following* ARE

$$\Sigma = (A - B\Pi D'C)\Sigma\Psi^{-1}(A - B\Pi D'C)' + B\Pi B' \quad (15)$$

*with $\Psi = I + C'(I - D\Pi D')C\Sigma$ and $\Pi = \gamma^2(\gamma^2 D'D - I)^{-1}$ has a stabilizing solution $\Sigma \geq 0$. In this case, a left inverse satisfying $\|G_0^{\text{inv}}\|_{\infty,[0,T]} < \gamma$ is specified by*

$$\begin{bmatrix} L \\ Q \end{bmatrix} = \begin{bmatrix} BD^+C - A \\ -D^+C \end{bmatrix} \Sigma C'D_\perp(I + D_\perp'C\Sigma C'D_\perp)^{-1}.$$

*That is, $Q(\cdot)$ can be chosen as a constant gain.*

Theorem 3.3 provides a procedure for computing the robust controller as required in this subsection. Let $G_0^{\text{inv}}(z)$ be the stable and proper left inverse of $G_0(z)$ whose $\mathcal{H}_\infty$-norm is bounded by $\gamma > 0$. Partition $G_0^{\text{inv}}(z) = \begin{bmatrix} V_0(z) & U_0(z) \end{bmatrix}$ conformally with that of $G_0(z)$. Then

$$G_0^{\text{inv}}(z)G_0(z) = V_0(z)M_0(z) + U_0(z)N_0(z) = I \quad (16)$$

and $K_0(z) = V_0^{-1}(z)U_0(z)$ is the feedback controller that stabilizes the uncertain system described by normalized coprime factors [4, 11, 19] so long as the $\mathcal{H}_\infty$-norm of the uncertainty is no more than $\varepsilon < \gamma^{-1}$. In fact a two degree freedom controller can be employed as in the following figure to achieve not only robust stability but also good tracking performance [19]:

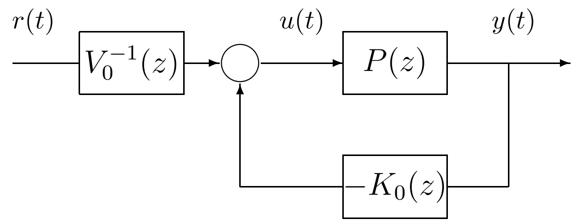


**Fig. 1** Feedback system with $K_0(z) = V_0^{-1}(z)U_0(z)$

### 3.3 Asymptotic Stability and Performance

Our proposed adaptive control system employs the TLS algorithm to adaptively estimate the left coprime factors, and the full information $\mathcal{H}_\infty$ control to adaptively tune the feedback controller based on the estimated model. The key issue is the asymptotic stability, and performance of the feedback system associated with the true plant. The following presents the main result for our proposed adaptive feedback control system in this subsection.

**Theorem 3.4** *Consider the adaptive feedback control system proposed in this section where the true plant is linear and time-invariant. Assume that the adaptively estimated left coprime factors model $\hat{G}_t(q)$ converges to $\hat{G}(q)$ that is strictly minimum phase with the equivalent uncertainty bound $\varepsilon_0$. Suppose further that the adaptively tuned feedback controller $K_{0t}(q)$ is designed according to* Theorem 3.3 *for some $\gamma > 0$ that converges to $K_0(q)$ as $t \to \infty$. Then the proposed adaptive feedback control system in this section is asymptotically stable provided that $\delta_0 = \gamma\varepsilon_0\|\hat{R}^{-1}\|_\infty < 1$, and admits an equivalent and asymptotic robust performance measured by*

$$\left\| \begin{bmatrix} P \\ I \end{bmatrix} (I + K_0P)^{-1} \begin{bmatrix} K_0 & I \end{bmatrix} \right\|_\infty < \frac{\gamma}{1 - \delta_0} \quad (17)$$

*where $\hat{R}(z)$ is the left spectral factor of $\hat{G}\hat{G}^\sim$, i.e., $\hat{R}\hat{R}^\sim = \hat{G}\hat{G}^\sim$.*

The convergence assumption in Theorem 3.4 may not hold in its generality [8] which deserves further investigation in the future. However it is argued that the disturbance $\{d(t)\}$ and the noise $\{\eta(t)\}$ are persistently exciting generically that tend to ensure the convergence of the adaptively estimated left coprime factors of the nominal model. Such an assumption holds for many adaptive estimation algorithms (cf. Chapter 3 in [6]), and is implied by

$$\underline{\sigma}(\Phi_T) = \underline{\sigma}(\begin{bmatrix} \phi(0) & \phi(1) & \cdots & \phi(T) \end{bmatrix}) \to \infty \quad (18)$$

as $T \to \infty$.

It should also be pointed out that the convergence for estimated model says nothing about its limit due to the dependence on the input/output signals. This is the main reason why the asymptotic stability and performance can be established. In the case when the computational complexity is high in computing the ARE solution to (15), the DRE can be used in adaptive controller design for which asymptotic stability and equivalence performance in Theorem 3.4 still hold true. We skip the details for this case.

## 4 An Illustrative Example

The previous section introduces a new adaptive feedback control system based on the proposed uncertainty equivalence principle. To illustrate this adaptive control system, we consider a common process control example with the nominal plant described by the continuous-time transfer function

$$P_0(s) = \frac{Ke^{-hs}}{s(s+\frac{1}{T_0})}, \quad K = 40, \quad h = 0.2\text{s}, \quad T_0 = 1.$$

With the sampling frequency 10 Hertz, the discretized model admits transfer function

$$P_0(z) = \frac{z^{-3}(0.1934 + 0.1872z^{-1})}{1 - 1.9048z^{-1} + 0.9048z^{-2}}. \quad (19)$$

Often the model uncertainty is inevitable in high frequency range or close to the half sampling frequency. For this reason we introduce a simple second order multiplicative uncertainty in high frequency range as given by

$$P(z) = P_0(z)\left[1 + \Delta_m(z)\right], \quad \Delta_m(z) = \frac{K_u}{z^2 + 0.8z + 0.9}$$

where $0.1 \leq K_u \leq 0.5$. The following figure shows the magnitude frequency response of $P_0(z)$ (solid line) and also $P(z)$ for $K_u = 0.1$ (dashed line) and for $K_u = 0.5$ (dot-dashed line) where normalized frequency is employed.

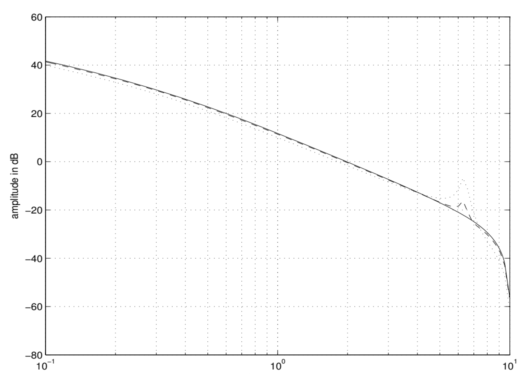


**Fig. 2** Magnitude frequency response

Normally for process control there is some *a priori* information available. For this example we assume that the gain $K$ in the nominal system is between 10 and 50 and the time constant $T_0$ is between 1 and 5. Hence we take

$$\hat{P}_0(s) = \frac{25e^{-sh}}{s+3} \quad \Longrightarrow \quad \hat{P}_0(z) = \frac{0.4683z + 0.4381}{z^2 - 1.818z + 0.818}$$

as the assumed nominal model that provides the initial guess of the true nominal parameters. For convenience the RLS algorithm is used for estimation of the plant model that has a similar performance to the TLS algorithm, and the proposed robust control algorithm is employed to tune the feedback controller adaptively. With $K_u = 0.1$ and the reference signal a rectangular wave of $\pm 1$ and period of 10 seconds, the output of the true plant model is shown on the top of Fig. 3.

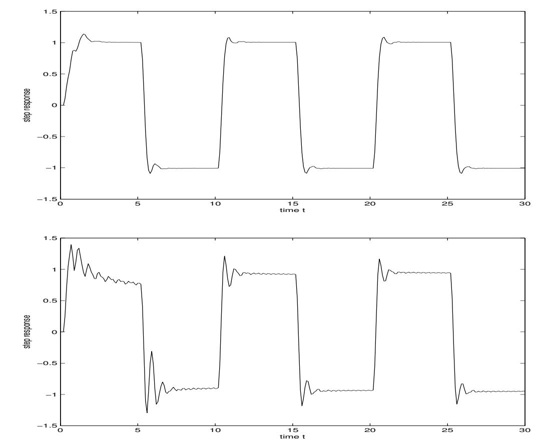


**Fig. 3** Step responses with $K_u = 0.1$ (top: robust control algorithm; bottom: MDPP algorithm)

As a comparison the MMDP (minimum degree pole placement) algorithm is also simulated. This algorithm is probably the most sophisticated self tuning algorithm [1]. To make a fair comparison, we assume that the continuous-time reference model has the characteristic polynomial $\lambda(s) = s^2 + 14.14s + 100$. The complete characteristic polynomial for the corresponding discretized feedback system can be obtained. The resulting adaptive control system based on MDPP has a similar nominal performance to that with the robust controller designed earlier. In fact the two have almost identical output responses for the case $K_u = 0$ or zero uncertainty. Moreover the gain and phase margins for the two nominal loop transfer functions are very close with only $3^o$ difference in phase margin and 1.4 dB difference in gain margin. However the output response based on MDPP algorithm is worse in presence of uncertainty that is shown at the bottom of Fig. 3. This is in spite of the fact that the two different adaptive control systems employ the same estimation algorithm. If the multiplicative uncertainty increases to $K_u = 0.5$, the performance of the two different adaptive control systems diverges much further as shown next in Fig. 4. The two output responses show that our proposed robust adaptive control is much more resilient to uncertainties than the MDPP based adaptive control which behaves much more poorly. We would like to comment that the use of TLS in place of RLS algorithm in estimation has small impact on the performance which is consistent with the result in Subsection 3.1.

Theoretically MMDP algorithm covers the robust control algorithm as employed in this section by choosing a suitable characteristic polynomial for the closed-loop system. However it is difficult to build the robustness into the characteristic polynomial directly and before hand that is why the MDPP algorithm compares less favorably with the robust control algorithm. One may wish to self-tune the characteristic polynomial in order to boost the robustness of the conventional adaptive control but it remains unclear how such self-tunes can be accomplished.

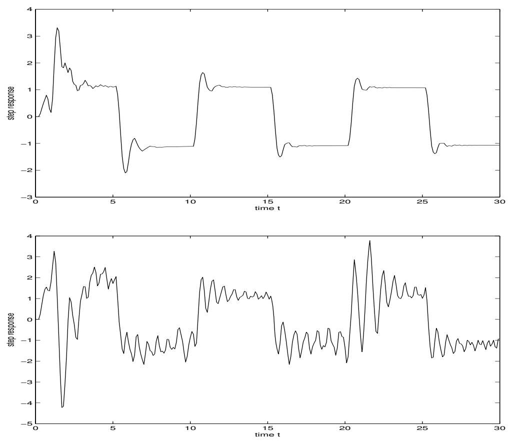

**Fig. 4** Step responses with $K_u = 0.5$ (top: robust control algorithm; bottom: MDPP algorithm)

## 5 Conclusion

Robust adaptive control has been tackled in this paper. A new adaptive control system is proposed with asymptotic stability and performance established under some mild conditions consistent with $\mathcal{H}_\infty$ loopshaping. It is seen that the successful unification of adaptive control and $\mathcal{H}_\infty$-based robust control empowers robust adaptive control, enabling the proposed adaptive feedback control systems to achieve robust stability and performance comparable to those achievable by $\mathcal{H}_\infty$ control. The results in this paper shed some light to new direction for robust adaptive control, and compare favorably with some conventional adaptive control in a simulation study.

A distinguished feature of our proposed robust adaptive control is the use of state-space realizations in tuning the controller parameters and in implementing the feedback adaptive controller. It unifies the treatment for single-input/single-output and MIMO systems naturally. It is commented that polynomial method can also be developed to design the required robust controller in Subsection 3.2. Indeed as shown in [11], the Corona problem for the normalized coprime factor model can be solved by the optimal Hankel-norm approximation for which a polynomial approach is available in [10]. While this paper presents our initial work on $\mathcal{H}_\infty$-based robust adaptive control, problems such as convergence of the adaptively estimated left coprime factors and near cancellation of the left coprime factors deserve further attention in future research.

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