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An Improved Bound on the Real Stability Radius*

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Abstract

In this paper, we give a new lower bound on the real stability radius of a real stable matrix. We also conjecture that this new lower bound is equal to the exact value of the real stability radius.

1 Introduction

One of the long standing open problems in robust control is the computation of the real stability radius of a real stable matrix [3, 6]. The real stability radius of $A \in \mathbb{R}^{n\times n}$ is defined as

$$r_{\mathbb{R}}(A) = \inf\{\overline{\sigma}(\Delta A) : \Delta A \in \mathbb{R}^{n\times n} \text{ and } A + \Delta A \text{ is unstable}\}$$

where $\overline{\sigma}(\cdot)$ is the largest singular value.

A closely related concept is the complex stability radius of a complex matrix. The complex stability radius of $A \in \mathbb{C}^{n\times n}$ is defined as

$$r_{\mathbb{C}}(A) = \inf\{\overline{\sigma}(\Delta A) : \Delta A \in \mathbb{C}^{n\times n} \text{ and } A + \Delta A \text{ is unstable}\}.$$

The computation of $r_{\mathbb{C}}(A)$ turns out to be easy. It is now well-known that for stable $A \in \mathbb{C}^{n\times n}$

$$r_{\mathbb{C}}(A) = \inf_{\omega\in\mathbb{R}} \underline{\sigma}(A - j\omega I_n)$$

where $\underline{\sigma}(\cdot)$ is the smallest singular value [4, 7, 2, 5]. This infimum can be computed by using a bisection algorithm [1].

It is clear that for a real matrix, the complex stability radius gives a lower bound of the real stability radius. In the following, we will always assume that A is a real $n \times n$ stable matrix. Hence,

$$r_{\mathbb{R}}(A) \geq r_{\mathbb{C}}(A). \tag{1}$$

Some other lower bounds of the real stability radius are given in [6] as follows

$$r_{\mathbb{R}}(A) \geq \min\left\{\underline{\sigma}(A), \frac{1}{2}\sigma_{n^2-1}(A \otimes I_n + I_n \otimes A)\right\} \tag{2}$$

$$r_{\mathbb{R}}(A) \geq \frac{1}{2}\underline{\sigma}(A \vee I_n + I_n \vee A) \tag{3}$$

$$r_{\mathbb{R}}(A) \geq \min\left\{\underline{\sigma}(A), \frac{1}{2}\underline{\sigma}(A \wedge I_n + I_n \wedge A)\right\}. \tag{4}$$

where $\otimes$, $\vee$, $\wedge$ denote the Kronecker (tensor) product, symmetrical tensor product and skew-symmetric tensor product respectively [6]. In (2) and in the rest of this paper, we assume that singular values are ordered decreasingly and that $\sigma_k(\cdot)$ denotes the k-th sigular value.

Inequalities (1)–(4) give easily computable lower bounds to the real stability radius. Moreover, (1)–(4) are actually equalities if A is normal, and (2) and (4) are equalities if A is 2×2. The tightness of (1)–(4) for a general stable matrix A, however, is hard to judge.

In this short paper, we present another lower bound which certainly improves (1) and likely improves (2)–(4). In fact, for all the examples in which we have tested the new lower bound, we have also managed to find destabilizing perturbations whose norms are equal to the respective new lower bounds. This suggests that the new lower bound may turn out to be equal to the real stability radius. Unfortunately, we can neither prove nor disprove this conjecture at this time.

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2 Main results

Let $A \in \mathbb{R}^{n\times n}$ be stable. For the convenience of analysis, define

$$r_{\mathbb{R}\omega}(A) = \inf\{\overline{\sigma}(\Delta A) : \Delta A \in \mathbb{R}^{n\times n} \text{ and } A + \Delta A \text{ has a pair of imaginary eigenvalues}\}.$$

It is clear that

$$r_{\mathbb{R}}(A) = \min\{\underline{\sigma}(A), r_{\mathbb{R}\omega}(A)\}.$$

For $\omega \in (0, \infty)$, let $B(\omega)$ be a 2×2 complex matrix with eigenvalue $j\omega$ and $-j\omega$. Then the rank of

$$B(\omega) \otimes I_n + I_2 \otimes (A + \Delta A)$$

is at most $2n - 2$ if $A + \Delta A$ has eigenvalues at $j\omega$ and $-j\omega$. This implies that $\overline{\sigma}(\Delta A) = \overline{\sigma}(I_2 \otimes \Delta A)$ is at least $\sigma_{2n-1}[B(\omega) \otimes I_n + I_2 \otimes A]$. Therefore

$$r_{\mathbb{R}\omega}(A) \geq \inf_{\omega\in(0,\infty)} \sup_{B(\omega)} \sigma_{2n-1}[B(\omega) \otimes I_n + I_2 \otimes A]. \tag{5}$$

The right hand side of (5) involves a complicated constrained minimax problem. However, it can be simplified as follows. Since $B(\omega)$ has eigenvalues $j\omega$ and $-j\omega$, then there exists a unitary matrix U such that

$$U^* B(\omega) U = \begin{bmatrix} j\omega & x \\ 0 & -j\omega \end{bmatrix}$$

where $x \in [0, \infty)$. Since $B(\omega) \otimes I_n + I_2 \otimes A$ and $U^* B(\omega) U \otimes I_n + I_2 \otimes A$ have the same singular values, it follows that

$$\begin{aligned} &\sup_{B(\omega)} \sigma_{2n-1}[B(\omega) \otimes I_n + I_2 \otimes A] \\ &= \sup_{x\in[0,\infty)} \sigma_{2n-1}\left(\begin{bmatrix} j\omega & x \\ 0 & -j\omega \end{bmatrix} \otimes I_n + I_2 \otimes A\right) \\ &= \sup_{x\in[0,\infty)} \sigma_{2n-1}\begin{bmatrix} A + j\omega I_n & xI_n \\ 0 & A - j\omega I_n \end{bmatrix}. \end{aligned}$$

Therefore

$$r_{\mathbb{R}\omega}(A) \geq \inf_{\omega\in(0,\infty)} \sup_{x\in[0,\infty)} \sigma_{2n-1}\begin{bmatrix} A + j\omega I_n & xI_n \\ 0 & A - j\omega I_n \end{bmatrix}. \tag{6}$$

The right hand side of (6) is a much easier minimax problem. Now let us denote

$$\beta(A) = \inf_{\omega\in(0,\infty)} \sup_{x\in[0,\infty)} \sigma_{2n-1}\begin{bmatrix} A + j\omega I_n & xI_n \\ 0 & A - j\omega I_n \end{bmatrix}$$

and

$$\alpha(A) = \min\{\underline{\sigma}(A), \beta(A)\}.$$

Then we have arrived at our main result.

Theorem 1 $r_{\mathbb{R}}(A) \geq \alpha(A)$.

At this moment, we are unable to say much analytically about the new lower bound $\alpha(A)$. However, its computation is a feasible task since it involves a minimax problem with only two real variables. A few simple facts are given in the following.

Fact 1 $\alpha(A) \geq r_{\mathbb{C}}(A)$.

Fact 1 follows easily from the fact that $r_{\mathbb{C}}(A) \leq \underline{\sigma}(A)$ and

$$\begin{aligned} r_{\mathbb{C}}(A) &= \inf_{\omega\in\mathbb{R}} \underline{\sigma}(A - j\omega I_n) \\ &= \inf_{\omega\in(0,\infty)} \sigma_{2n-1}\begin{bmatrix} A+j\omega I_n & 0 \\ 0 & A-j\omega I_n \end{bmatrix} \\ &\leq \inf_{\omega\in(0,\infty)} \sup_{x\in[0,\infty)} \sigma_{2n-1}\begin{bmatrix} A+j\omega I_n & xI_n \\ 0 & A-j\omega I_n \end{bmatrix} \end{aligned}$$

Fact 2 *If A is normal or if A is* 2×2, *then* $\alpha(A) = r_{\mathbb{R}}(A)$.

The proof of Fact 2 is tedious but straightforward. It is omitted here.

3 Examples

In the numerous examples studied, the lower bound $\alpha(A)$ obtained has always been *equal* to the real stability radius $r_{\mathbb{R}}(A)$. In these examples, the destabilizing perturbation matrices, whose norms are equal to $\alpha(A)$, are found by a global optimization method. The following three examples are representative:

Example 1

Let matrix A be

$$\begin{bmatrix} 7.90\times 10^1 & 2.00\times 10^1 & -3.00\times 10^1 & -2.00\times 10^1 \\ -4.10\times 10^1 & -1.20\times 10^1 & 1.70\times 10^1 & 1.30\times 10^1 \\ 1.67\times 10^2 & 4.00\times 10^1 & -6.00\times 10^1 & -3.80\times 10^1 \\ 3.35\times 10^1 & 9.00\times 10^0 & -1.45\times 10^1 & -1.10\times 10^1 \end{bmatrix}.$$

This matrix is stable with eigenvalues $-1 \pm 10i$ and $-1 \pm i$. The complex stability radius $r_{\mathbb{C}}(A) = 8.234\times 10^{-2}$.

The solution of the minimax problem is $\beta(A) = 1.538\times 10^{-1}$ with $\omega = 1.0497$ and $x = 1.5549$. Thus

$$\begin{aligned} \alpha(A) &= \min\{\underline{\sigma}(A), \beta(A)\} \\ &= \min\{2.038\times 10^{-1}, 1.538\times 10^{-1}\} \\ &= 1.538\times 10^{-1}. \end{aligned}$$

We also find that the following ΔA matrix

$$\begin{bmatrix} -4.815\times 10^2 & 6.989\times 10^2 & 1.091\times 10^1 & -6.492\times 10^2 \\ 8.846\times 10^2 & 7.207\times 10^2 & 3.627\times 10^2 & 7.927\times 10^2 \\ -4.382\times 10^2 & 2.191\times 10^2 & -6.423\times 10^2 & -2.426\times 10^2 \\ -9.346\times 10^2 & 6.663\times 10^2 & -2.857\times 10^2 & 8.791\times 10^2 \end{bmatrix}$$

is a destabilizing perturbation matrix with $A + \Delta A$ having eigenvalues on the imaginary axis and $\overline{\sigma}(\Delta A) = 1.539\times 10^{-1}$.

Example 2

Consider the matrix

$$A = \begin{bmatrix} -93.72 & -9520 & -121400 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This matrix is stable with eigenvalues $-39.609 \pm 82.476i$ and -14.502. The complex stability radius $r_{\mathbb{C}}(A) = 5.4696\times 10^{-2}$.

The solution of the minimax problem is $\beta(A) = 6.7545\times 10^{-2}$ with $\omega = 35.896$ and $x = 0$. Thus

$$\begin{aligned} \alpha(A) &= \min\{\underline{\sigma}(A), \beta(A)\} \\ &= \min\{9.9694\times 10^{-1}, 6.7545\times 10^{-2}\} \\ &= 6.7545\times 10^{-2}. \end{aligned}$$

We also find that

$$\Delta A = \begin{bmatrix} 2.4570\times 10^{-5} & -3.0951\times 10^{-4} & 4.8768\times 10^{-2} \\ -5.7705\times 10^{-3} & 2.9900\times 10^{-2} & 1.6763\times 10^{-2} \\ -6.7337\times 10^{-2} & -2.4890\times 10^{-3} & -1.4116\times 10^{-3} \end{bmatrix}$$

is a destabilizing perturbation matrix with $A + \Delta A$ having eigenvalues on the imaginary axis and $\overline{\sigma}(\Delta A) = 6.7584\times 10^{-2}$.

Example 3

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 100 \\ -10 & -1 & 2 \\ -1 & 1 & -110 \end{bmatrix}$$

This matrix is stable with eigenvalues $-0.90593 \pm 4.3984i$ and -109.19. The complex stability radius $r_{\mathbb{C}}(A) = 5.0928\times 10^{-1}$.

The solution of the minimax problem is $\beta(A) = 7.6696\times 10^{-1}$ with $\omega = 4.4190$ and $x = 10$. Thus

$$\begin{aligned} \alpha(A) &= \min\{\underline{\sigma}(A), \beta(A)\} \\ &= \min\{1.4703, 7.6696\times 10^{-1}\} \\ &= 7.6696\times 10^{-1}. \end{aligned}$$

We also find that

$$\Delta A = \begin{bmatrix} 5.6912\times 10^{-1} & -2.6767\times 10^{-2} & 1.3494\times 10^{-1} \\ -1.5527\times 10^{-2} & 7.6475\times 10^{-1} & 1.6390\times 10^{-2} \\ 5.1400\times 10^{-1} & 5.2745\times 10^{-2} & -1.5247\times 10^{-1} \end{bmatrix}$$

is a destabilizing perturbation matrix with $A + \Delta A$ having eigenvalues on the imaginary axis and $\overline{\sigma}(\Delta A) = 7.6703\times 10^{-1}$.

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