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Best Achievable Tracking Performance in Sampled-Data Control Systems

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Abstract

In this paper we study the problem of tracking a step reference signal using sampled-data control systems. We investigate the best achievable tracking performance, where the performance is deemed best for it is the minimal attainable by all possible sampled-data stabilizing controllers. Our primary objective is to investigate the fundamental tracking performance limit in sampled-data systems, and to understand whether and how sampling and hold in a sampled-data system may impose intrinsic barriers to performance. For this purpose we derive an analytical expression for the optimal tracking performance. The result shows that a performance loss is generally incurred in a sampled-data system, in comparison to the tracking performance achievable by continuous-time controllers. This loss of performance, as so demonstrated by the expression, can be attibuted to the non-minimum behaviors and the aliasing effects generated by samplers and hold devices.
Keywords: performance limitation, $\mathcal{H}_2$ control, sampled-data feedback system, non-minimum phase zeros

1 Introduction

For a given plant, the optimal tracking ability, measured by the minimal tracking error between its output and a reference input to be tracked via a stabilizing controller, depends on the plant, the class of controllers, as well as the reference signal. When the plant and the reference input signal are given, and the controller has been designed, the implementation mode of the controller, i.e. by means of an analog or a digital implementation will also lead to different tracking performance. In this paper, we consider the tracking performance problem for sampled-data systems, in which the plant operates in continuous time while the controller in discrete time.

We consider single-input, single-output (SISO), linear time invariant (LTI) plants. As a benchmark test signal, the reference input will be the unit step signal. The controller will consist of a zero order hold (ZOH), a discrete-time controller, and an ideal sampler. The tracking performance is defined as the integral square of the error between the output of the plant and the unit step input, and the tracking capacity is measured by the minimal error achievable by all possible sampled-data stabilizing controllers. Our main objective is to find out what may affect the tracking performance in a sampled-data system, and whether any limit to this performance may exist, and if any, how it arises.

The tracking capability of feedback systems is an important problem and has been the subject of research for many years. For SISO and stable plants, the ability to track step signals with an LTI stabilizing controller is considered in [11, 12, 13]. It has been shown that the tracking capability is completely determined by the location of non-minimum phase zeros of the plant, whether in continuous time or in discrete time. Recently, these studies have been extended to multi-input multi-output, possibly unstable systems [6, 14], wherein it has been found that the tracking performance is determined by both the location and the directional properties of the unstable poles and non-minimum phase zeros in the plant, and that the effects of pole and zero locations as well as their directional properties can be completely described via closed form expressions. A similar conclusion holds with respect to other benchmark test signals than the step signal, including complex sinusoids, real sinusoids, and ramp signals.

Problems concerning tracking with sampled-data systems have been widely studied as well; see, e.g., [9, 4, 1, 7] and the references therein. These problems become substantially more difficult, and closed form expressions for tracking performance are not yet available. Among several issues which are unique to sampled-data tracking systems, one important problem is whether the tracking performance in a sampled-data system may be worse than that of the corresponding continuous-time system. If this is the case, why then is a performance loss incurred in the sampled-data system? Furthermore, what may be the cause contributing to this loss of performance? Would the loss be fundamental to the sampled-data implementation of the controller? If so, can the loss be recovered with arbitrarily fast samplers? These issues form the primary objectives in the present paper.

We adopt a frequency-domain lifting approach in our analysis. This analysis, in essence, amounts to converting a hybrid continuous-time/discrete-time system into one

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solely operating in discrete time, while fully taking into account the continuous-time inter-sampling characteristics. Thus, a key step in our analysis is to construct an equivalent LTI discrete-time system, and to reduce the problem to one of the tracking problems in the equivalent discrete-time system; the solution for the latter is readily available. It is worth pointing out that this can be achieved using both time- and frequency-domain lifting techniques. A distinguishing advantage with the frequency-domain lifting, however, lies in that it preserves well the structure of the zeros relative to the lifted plant, and this will prove critical.

While the problem being considered herein amounts to a sampled-data $\mathcal{H}_2$ control problem and can be tackled numerically (See, e.g., [7].), our result differs from computational solutions. Specifically, our main result is an analytical closed form expression for the minimal tracking error, which serves to answer the aforementioned issues and questions. First, it shows that the non-minimum phase zeros of the continuous-time plant will continue to impose a limit to the tracking performance, in exactly the same manner as they do when the controller is one of continuous-time rather than that of sampled-data. Secondly, it is well-known that with the use of samplers, a sampled-data system invariably involves discretization in one way or another, and such a discretization is likely to lead to additional non-minimum phase zeros, i.e., zeros outside the unit disc, in the resultant discrete-time plant. Our result shows that such zeros will also have a negative effect on the tracking performance. Third, in order to remove sampling noise and prevent sampling aliases, a continuous-time pre-filter is generally needed in a sampled-data system; likewise, discretization of this per-filter may also generate non-minimum phase zeros, and hence they too will affect the tracking performance. Finally, for any physically realizable sampled-data system, aliasing is inevitable. Our result demonstrates further a relationship between the aliasing and the tracking performance. In summary, it is seen that sampling and hold results in undesirable "byproducts" unfound in continuous-time systems, which contribute to a degradation of tracking performance, while any of the performance limitations due to the continuous-time plant remains unchanged. In simple terms, the expression for the minimal tracking error will now contain more terms, and each of these additional terms represents an additional limit on the best achievable tracking accuracy, resulting from sampling and hold operations.

The remainder of this paper is organized as follows. In Section 2, we formulate the problem and present preliminary results required in the subsequent development. In Section 3, we carry out the frequency-domain lifting procedure on the system, which leads to our main result in Section 4, that is, the closed form expression for the optimal tracking performance. Section 5 provides a brief outline of the proof for this result. The paper concludes in Section 6.

The notation used throughout this paper is fairly standard. For any complex number z, we denote its complex conjugate by $\bar{z}$. For any vector u, we denote its transpose by u^T. For any signal $u(t)$, we denote its Laplace transform by $U(s)$. The transpose of a matrix A is denoted by A^T. The trace of a matrix A is denoted by $\mathrm{Tr}(A)$. All the vectors and matrices involved in the sequel are assumed to have compatible dimensions, which for simplicity, will be omitted. Let the open right half plane (RHP) be denoted by $\mathbb{C}_+ := \{s : \mathrm{Re}(s) > 0\}$, the open left half plane (LHP) by $\mathbb{C}_- := \{s : \mathrm{Re}(s) < 0\}$, and the imaginary axis by $\mathbb{C}_0$. Moreover, let $\|\cdot\|$ denote the Euclidean vector norm and define

$$\mathcal{L}_2(\mathbb{C}_0) := \left\{ f : f(s) \text{ measurable in } \mathbb{C}_0, \ \|f\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(j\omega)\|^2 d\omega\right)^{1/2} < \infty \right\}.$$

Similarly, let $\mathbb{D}$ and $\mathbb{T}$ denote the open unit disc and unit circle respectively, and define

$$\mathcal{L}_2(\mathbb{T}) := \left\{ f : f(z) \text{ measurable in } \mathbb{T}, \ \|f\|_2 := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|f(e^{j\theta})\|^2 d\theta\right)^{1/2} < \infty \right\}.$$

Finally, we define

$$\mathcal{H}_\infty(\mathbb{D}) := \{f : f(z) \text{ bounded and analytic in } \mathbb{D}\},$$

$$\mathcal{H}_\infty(\mathbb{C}_+) := \{f : f(z) \text{ bounded and analytic in } \mathbb{C}_+\}.$$

A subset of $\mathcal{H}_\infty(\mathbb{D})$, denoted by $\mathcal{RH}_\infty$, is the set of all proper stable rational transfer functions in the discrete-time sense.

2 Problem Formulation

The generic tracking scheme under consideration in this paper is the single-input, single-output unity feedback system depicted in Figure 1, in which P represents the plant model and K the compensator. The signals r and y are the reference input and the system output, respectively. For a given reference signal r, the compensator K is designed such that the output y tracks r. The signal e represents the tracking error response. In general, both P and K

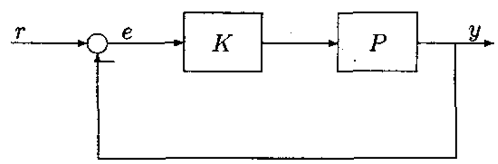

Figure 1: The general unity feedback system

may be continuous-time or discrete-time systems. Correspondingly, the signals r and y may be continuous-time or discrete-time signals. In all cases, we assume that P is stable and linear time-invariant, whose transfer function is given by $P(s)$ if it is continuous-time and $P(z)$ if it is discrete-time. The reference input will be taken as the unit step signal; in the continuous-time case,

$$r(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}, \qquad R(s) = \frac{1}{s}, \tag{1}$$

while in the discrete-time case,

$$r(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}, \qquad R(z) = \frac{z}{z-1}. \tag{2}$$

Assume that the system is initially at rest. The problem then is to seek to determine the best tracking performance achievable by all possible compensators that stabilize the plant. Here we measure the tracking performance by the energy of the tracking error response. When e is a continuous-time signal, the tracking error, denoted as J_c, is quantified by the integral

$$J_c := \int_0^\infty |e(t)|^2 dt = \int_0^\infty |y(t) - r(t)|^2 dt . \tag{3}$$

Analogously, for a discrete-time signal, the error criterion is defined by

$$J_d := \sum_{k=0}^{\infty} |e(k)|^2 = \sum_{k=0}^{\infty} |y(k) - r(k)|^2.$$

Clearly, the tracking performance depends upon the types of the compensator to be employed. Under the general setup alluded to above, of main interest in the present paper are sampled-data controllers. A sampled-data controller K consists of a discrete-time compensator, followed by a hold device and preceded by a sampler, together with a possible low-pass, anti-aliasing filter. To emphasize, we explicitly draw the sampled-data tracking scheme in Figure 2. Here we take S_T as an ideal point sampler, and H_T

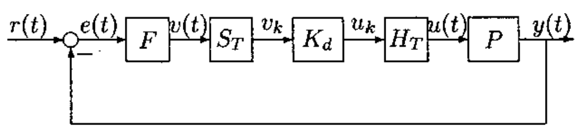

Figure 2: The sampled-data tracking system

a zero-order hold (ZOH), which are synchronized and are of the sampling period $T > 0$. Thus, the sampled sequence $\{v_k\}_{k=0}^{\infty}$ is given by

$$v_k := v(kT), \qquad k = 0, 1, \cdots,$$

and the ZOH yields as its output the signal

$$u(t) := u_k, \qquad kT \leq t < (k+1)T.$$

The transfer function of the zero-order hold is given by

$$H(s) = \frac{1 - e^{-sT}}{s} . \tag{4}$$

Throughout this paper we shall assume that

Assumption 1:

1) $P(s)$ is rational, proper, stable, and $P(0) \neq 0$,

2) $F(s)$ is rational, proper, stable, and minimum-phase,

3) $K_d(z)$ is rational and proper.

These assumptions are non-restrictive except the stability assumption on $P(s)$, which may be removed if we use two-parameter controllers. For example, $P(0) \neq 0$ is a standard requirement to insure J_c to be finite.

With the given filter F and the sampling rate T, the best tracking performance attainable by sampled-data controllers is

$$J_{sd}^* := \inf_{K_d(z) \text{ stabilizes } P(s)} J_c,$$

where K_d is to be taken from all possible discrete-time compensators that, together with F, S_T and H_T, stabilize the continuous-time plant P.

Let $(FP)_d(z)$ denote the *zero-order hold equivalent discretization* of $F(s)P(s)$. The stability assumption on $F(s)$ and $P(s)$ guarantees that $(FP)_d(z)$ be stable. Hence, the set of all stabilizing controllers for the sampled-data system depicted in Figure 2 is given by

$$\mathcal{K}_s := \{K_d(z) = Q(z)[1 - (FP)_d(z)Q(z)]^{-1} \mid Q(z) \in \mathcal{RH}_\infty\} \tag{5}$$

The tracking problem in this paper thus amounts to determining

$$J_{sd}^* = \inf_{K_d(z) \in \mathcal{K}_s} J_c = \inf_{Q(z) \in \mathcal{RH}_\infty} J_c . \tag{6}$$

Similarly, for a continuous-time and discrete-time system as depicted in Figure 1, the minimal achievable error is given by

$$J_c^* := \inf_{K(s) \text{ stabilizes } P(s)} J_c ,$$

$$J_d^* := \inf_{K(z) \text{ stabilizes } P(z)} J_d ,$$

respectively. These two cases have been studied extensively. With r being the unit step signal, the following results are known.

Fact 1: *Let r be given in (1), and suppose that $P(s)$ is stable. Furthermore, assume that $P(0) \neq 0$. Then,*

$$J_c^* = \sum_{i=1}^{N_c} \frac{2\text{Re}(z_i)}{|z_i|^2} = \sum_{i=1}^{N_c} \frac{2}{z_i} . \tag{7}$$

where $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_c$, are the non-minimum phase zeros of $P(s)$.

Fact 2: *Let r be given in (2), and suppose that $P(z)$ is stable. Furthermore, assume that $P(1) \neq 0$. Then,*

$$J_d^* = \sum_{i=1}^{N_d} \frac{|s_i|^2 - 1}{|s_i - 1|^2} = \sum_{i=1}^{N_d} \frac{s_i + 1}{s_i - 1} , \tag{8}$$

where $s_i \in \overline{\mathbb{D}}^c$, $i = 1, \cdots, N_d$, are the non-minimum phase zeros of $P(z)$, and $\overline{\mathbb{D}}^c$, the complement of $\overline{\mathbb{D}}$, represents the exterior of the closed unit disc.

Our aim in this paper is to derive a corresponding analytical solution for the sampled-data tracking problem. Both these results will play a role in our subsequent analysis.

3 Frequency Domain Lifting

Let us first define the sampling frequency and the Nyquist frequency by

$$\omega_s = \frac{2\pi}{T}, \qquad \omega_N = \frac{\pi}{T}, \tag{9}$$

respectively. We shall refer to the frequency range $\Omega_N = [-\omega_N, \omega_N]$ as the baseband. For a continuous-time transfer function $G(S)$, define the discrete-time transfer function

$$G_d(z) = \mathcal{Z}[G(s)] = \mathcal{Z}\left[\mathcal{S}_T\left[\mathcal{L}^{-1}[G(s)]\right]\right], \tag{10}$$

where $\mathcal{Z}$ is the $\mathcal{Z}$-transform operator, $\mathcal{S}_T$ the sampling operator, and $\mathcal{L}^{-1}$ the inverse Laplace transform operator. It is a well-known fact [8] that

$$G_d(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s+jk\omega_s). \tag{11}$$

Clearly, the frequency response $G_d(e^{j\omega T})$ of a sampled-data system, which consists of the fundamental harmonics $G(j\omega)$ and the high-frequency harmonics $G(s+jk\omega_s)$, is a $j\omega_s$-periodic function. For simplicity, we shall write $G_k(s)$ to represent $G(s+jk\omega_s)$:

$$G_k(s) = G(s+jk\omega_s). \tag{12}$$

It is also known that

$$G_d(z) = \mathcal{Z}[G(S)] = (1-z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right].$$

Thus, we may write

$$\begin{aligned}(FP)_d(z) &= \mathcal{Z}[F(s)P(s)H(s)] \\ &= (1-z^{-1})\mathcal{Z}\left[\frac{F(s)P(s)}{s}\right],\end{aligned}$$

$$(FP)_d(e^{j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_k(j\omega)P_k(j\omega)F_k(j\omega). \tag{13}$$

Consider then the sampled-data system given in Figure 2. It follows that the system output can be represented as [10, 9]

$$\begin{aligned}&Y(j\omega)\\ =\ & P(j\omega)H(j\omega)S_d(e^{j\omega T})K_d(e^{j\omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(j\omega)R_k(j\omega)\end{aligned}$$

where

$$S_d(z) := [I + K_d(z)(FP)_d(z)]^{-1}.$$

Using the parametrization of $\mathcal{K}_s$, we have

$$E(j\omega) = R(j\omega) - P(j\omega)H(j\omega)Q(e^{j\omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} F_k(j\omega)R_k(\omega) \tag{14}$$

By means of the Parseval identity

$$\|e(t)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|E(j\omega)\|_2^2\, d\omega,$$

we may express J_{sd}^* in terms of $E(j\omega)$. To facilitate the analysis of J_{sd}^*, we perform a frequency-domain lifting on $E(j\omega)$. This is discussed below.

Consider a frequency-domain signal $Y(j\omega)$ in $\mathcal{L}_2(\mathbb{C}_0)$. Define the function sequence

$$Y_k(j\omega) = Y(j\omega + jk\omega_s) \quad k = 0,\ 1,\ \cdots, \quad \omega \in \Omega_N\ . \tag{15}$$

Furthermore, arrange the sequence $Y_k(j\omega)$ as an infinite-dimensional vector $\mathcal{Y}(j\omega)$, defined as

$$\mathcal{Y}(j\omega) = [\cdots,\ Y_1(j\omega),\ Y_0(j\omega),\ Y_{-1}(j\omega),\ \cdots]^T. \tag{16}$$

In other words, the function $\mathcal{Y}(j\omega)$ is an infinite-dimensional signal defined on Ω_N which takes values in $\mathcal{L}_2(\mathbb{C}_0)$. The space of such functions is a Hilbert space under the norm

$$\|\mathcal{Y}(j\omega)\|_2 = \left[\sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\int_{\Omega_N} \|Y_k(j\omega)\|^2 d\omega\right]^{\frac{1}{2}}, \tag{17}$$

which forms an isometry with the space $\mathcal{L}_2(\mathbb{C}_0)$, since

$$\|Y(j\omega)\|_2 = \|\mathcal{Y}(j\omega)\|_2. \tag{18}$$

Before lifting $E(j\omega)$, we first analyze the plant $P(s)$. With the assumption that $P(s)$ is stable, we may factorize $P(s)$ in the form of

$$P(s) = L(s)P^{(m)}(s), \tag{19}$$

where $P^{(m)}(s)$ is stable and represents the minimum phase part of $P(s)$, and $L(s)$ is an all-pass factor containing all the non-minimum phase zeros of $P(s)$ in $\mathbb{C}_+$. It then follows from a direct calculation that (14) can be equivalently written as

$$\begin{aligned}&\mathcal{E}(j\omega)\\ =\ & \left[I - \frac{1}{T}\mathcal{L}(j\omega)\mathcal{P}^{(m)}\mathcal{H}(j\omega)Q(e^{j\omega T})\mathcal{F}^T(j\omega)\right]\mathcal{R}(j\omega),\end{aligned} \tag{20}$$

where

$$\begin{aligned}\mathcal{L}(j\omega) &:= diag[\ldots, L_1(j\omega), L_0(j\omega), L_{-1}(j\omega), \ldots],\\ \mathcal{P}^{(m)}\mathcal{H}(j\omega) &:= [\ldots, P_1^{(m)}(j\omega)H_1(j\omega),\\ &\qquad P_0^{(m)}(j\omega)H_0(j\omega), P_{-1}^{(m)}(j\omega)H_{-1}(j\omega)\ldots,]^T.\end{aligned}$$

By doing so, the signal $E(j\omega)$ is lifted to its infinite-dimensional representation $\mathcal{E}(j\omega)$.

4 The Main Result

In light of the norm defined in (17) and the ensuing equality (18), the best sampled-data tracking performance can be expressed as

$$\begin{aligned}&J_{sd}^*\\ =\ & \inf_{Q\in\mathcal{Q}} \left\| \left[I - \frac{1}{T}\mathcal{L}(j\omega)\mathcal{P}^{(m)}\mathcal{H}(j\omega)Q(e^{j\omega T})\mathcal{F}^T(j\omega)\right]\mathcal{R}(j\omega)\right\|_2^2.\end{aligned}$$

The following theorem gives an analytical expression of J_{sd}^*, which is the main result of this paper.

Theorem 1: *Consider the sampled-data system depicted in Figure 2. Let the reference input r be the unit step signal given in (1), and the holder be a ZOH. Then under Assumption 1, the best achievable tracking performance is given by*

$$J_{sd}^* = \sum_{i=1}^{m_c} \frac{2}{z_i} + T \sum_{i=1}^{m_d} \frac{\sigma_i + 1}{\sigma_i - 1} + \frac{T^2}{2\pi} \int_0^{\omega_N} \frac{\log \left\{ T^2 \frac{\sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2}{\left|\sum_{k=-\infty}^{\infty} P_k(j\omega)H_k^2(j\omega)\right|^2} \right\}}{1 - \cos \omega T} d\omega, \tag{21}$$

where $z_i \in \mathbb{C}_+$, $i = 1, \cdots, m_c$, *and* $\sigma_i \in \overline{\mathbb{D}}^c$, $i = 1, \cdots, m_d$, *are respectively the non-minimum phase zeros of* $P(s)$ *and the non-minimum phase zeros of the ZOH equivalent discretized transfer function*

$$\hat{P}_d(z) := z(\tilde{P}H)_d(z)F_d(z) , \tag{22}$$

with $(\tilde{P}H)_d(z) = \mathcal{Z}[P^m(s)H(s)]$ *and* $F_d(z) = \mathcal{Z}[F(s)]$.

According to Theorem 1, the minimal tracking error consists of three parts. The first term in (21) is caused by the non-minimum phase zeros of the continuous-time plant. This term coincides with the minimal tracking error attainable by continuous-time controllers. Thus, Theorem 1 shows that the influence on the tracking performance by these zeros remains in complete existence, when the continuous-time controller is replaced by a sampled-data controller. In other words, sampled-data controllers cannot improve tracking performance.

In fact, sampled-data controllers will only worsen the tracking performance. The second term in (21) is attributed to the non-minimum phase zeros of the discretized plant and the discretized anti-aliasing filter, which occur due to sampling, and hence are the undesirable "byproducts" due to the use of sampled-data controllers. Since sampled-data systems are prone to such zeros (even when the continuous-time plant is minimum phase) even if the sampling period may be sufficiently small [2], the tracking performance will thus become worse than that of the corresponding continuous-time system. Note that these terms will be diminishing when the sampling period T tends to zero, since the limiting zeros are finite.

The third term in (21) captures the aliasing effects resulted from sampling and hold operations. This is seen by its explicit dependence on the high-frequency harmonics of $P(s)$ and $H(s)$. While to suppress such effects a pre-filter is introduced in the system, they can never be eradicated; indeed, this term is independent of the anti-aliasing prefilter $F(s)$. It can be shown as well that this term will tend to zero as T goes to zero. As a result, when the sampling period T becomes infinitely small, the best tracking error of the sampled-data system will approach that of the continuous-time system. The following example further demonstrates this point.

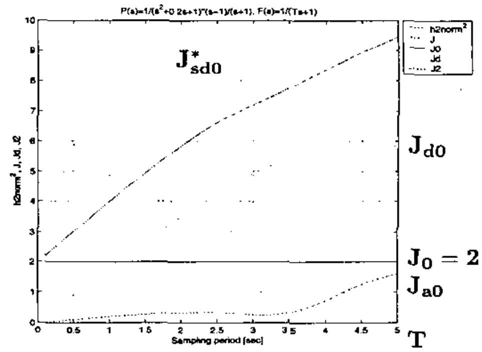

Figure 3: $J_{sd0}^*(T)$ in the example

Example: Consider the sampled-data system given in Figure 2, with

$$P(s) = \frac{(1-s)}{s(1+s)(s^2+0.2s+1)}, \quad F(s) = \frac{1}{\alpha s + 1}.$$

where α is an arbitrary positive number. The top plot in Figure 3 shows J_{sd0}^*, which can be confirmed to coincide with the values computed by numerical method. $J_{sd0}^*(T)$ consists of $J_0 = 2$, $J_{d0}(T)$, the second term in (21), and $J_{a0}(T)$, the third term in (21), as illustrated in the figure.

5 Proof of Theorem 1

A series of lemmas below lead to the proof of Theorem 1.

Lemma 1:

$$J_{sd}^* = J_0 + J_m^* , \tag{23}$$

where

$$J_0 := \sum_{i=1}^{m_c} \frac{2Re(z_i)}{|z_i|^2} = \sum_{i=1}^{m_c} \frac{2}{z_i} ,$$

$$J_m^* := \inf_{Q \in \mathcal{RH}_\infty} \| \left[I - \frac{1}{T} \mathcal{P}^{(m)} \mathcal{H}(j\omega) Q(e^{j\omega T}) \mathcal{F}^T(j\omega) \right] \mathcal{R}(j\omega) \|_2^2 .$$

In order to evaluate J_m^*, let us define

$$C_d(e^{j\omega T}) := \frac{1}{T} \sum_{k=-\infty}^{\infty} |P_k(j\omega)H_k(j\omega)|^2 = \frac{1}{T} \mathcal{PH}^H(j\omega)\mathcal{PH}(j\omega) ,$$

where

$$|P_k(j\omega)H_k(j\omega)|^2 = P_k(-j\omega)H_k(-j\omega)P_k(j\omega)H_k(j\omega).$$

We may then carry out the spectral factorization

$$C_d(e^{j\omega T}) = \theta_d^{\sim}(e^{j\omega T})\theta_d(e^{j\omega T}) , \tag{24}$$

with $\theta_d(z) \in \mathcal{H}_\infty(\mathbb{D})$, $\theta_d^{-1}(z) \in \mathcal{H}_\infty(\mathbb{D})$. We may assume further that the sign of $\theta_d(1)$ is the same as that of $P(0)$, with no loss of generality.

Lemma 1 shows that we can completely separate from other effects the effect by the continuous-time non-minimum zeros. Therefore, we assume that $P(s)$ is minimum phase in the following lemmas, and the notation $(PH)_d^m(z)$ will be used instead of $(\tilde{P}H)_d^m(z)$.

The following lemma evaluates J_m^*.

Lemma 2: *Suppose that $P(s)$ is minimum phase and that $\theta_d(z)$ is defined by (24). Then, we have*

$$J_m^* = J_{m1}^* + J_{m2}, \tag{25}$$

where

$$J_{m1}^* := \inf_{Q(z)\in\mathcal{RH}_\infty} J_{m1}(Q)$$

with

$$J_{m1}(Q) = \left\| \frac{(PH)_d(z^{-1})\theta_d^{-1}(z^{-1}) - zF_d(z)Q(z)}{z-1} \right\|_2^2 ,$$

$$J_{m2} = \frac{T}{2\pi}\int_0^{\omega_s} \frac{T - \left|\frac{(PH)_d(e^{j\omega T})}{\theta_d(e^{j\omega T})}\right|^2}{(1-e^{j\omega T})(1-e^{-j\omega T})}\, d\omega .$$

The next lemma evaluates J_{m1}^*.

Lemma 3: *Suppose that $P(s)$ is minimum phase. Let $(PH)_d^m(z)$ denote the minimum phase part of $(PH)_d(z)$, and $\theta_d(z)$ be defined by (24). Then, we have*

$$J_{m1}^* := \inf_{Q(z)\in\mathcal{RH}_\infty} J_{m1}(Q) = J_d + J_{s2} , \tag{26}$$

where

$$J_d := T\sum_{i=1}^{m_d} \frac{\sigma_i + 1}{\sigma_i - 1} ,$$

$$J_{s2} = \frac{T}{2\pi}\int_0^{\omega_s} \left| \frac{\frac{(PH)_d^m(e^{j\omega T})}{\theta_d(e^{j\omega T})} - \sqrt{T}}{e^{j\omega T} - 1} \right|^2 d\omega .$$

Finally, it follows from a somewhat lengthy calculation, using the Hilbert transform and the Schawtz formula, that the sum of J_{s2} and J_{m2} is equal to the third term in (21).

6 Conclusion

We have investigated the optimal performance in tracking a step reference signal via sampled-data systems. We employed a frequency-domain lifting approach, and obtained an analytical closed form result. It is shown that each of the following factors will negatively affect the tracking performance:

- the non-minimum phase zeros of the continuous-time plant;
- the non-minimum phase zeros of the discretized plant and the discretized pre-filter due to sampling;
- the aliasing effects of sideband harmonics.

While the first source inherits completely and exactly from the continuous-time tracking problem, the rest results from the use of a sampled-data controller, and is seen as a tradeoff for other advantages offered by sampled-data controllers.

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