Bounds on the Real Stability Radius

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Abstract. In this paper, we give a new lower bound on the real stability radius of a real stable matrix. We also formulate a nonlinear programming problem which can be used to obtain upper bounds for the real stability radius. Computational experience suggests that the new lower bound may in general turn out to be equal to the exact value of the real stability radius.

1. INTRODUCTION

One of the long standing open problems in robust control is the computation of the real stability radius of a real stable matrix [1, 2]. Here we say that a matrix is stable if the real parts of its eigenvalues are negative. The real stability radius of $A \in \mathbf{R}^{n\times n}$ is defined as

$$r_{\mathbf{R}}(A) = \inf\{\overline{\sigma}(\Delta A) : \Delta A \in \mathbf{R}^{n\times n} \text{ and } A + \Delta A \text{ is unstable}\}$$

where $\overline{\sigma}(\cdot)$ is the largest singular value.

A closely related concept is the complex stability radius of a complex matrix. The complex stability radius of $A \in \mathbf{C}^{n\times n}$ is defined as

$$r_{\mathbf{C}}(A) = \inf\{\overline{\sigma}(\Delta A) : \Delta A \in \mathbf{C}^{n\times n} \text{ and } A + \Delta A \text{ is unstable}\}.$$

The computation of $r_{\mathbf{C}}(A)$ turns out to be easy. It is now well-known that for stable $A \in \mathbf{C}^{n\times n}$

$$r_{\mathbf{C}}(A) = \inf_{\omega\in\mathbf{R}} \underline{\sigma}(A - j\omega I_n) \tag{1}$$

where $\underline{\sigma}(\cdot)$ is the smallest singular value [3, 4, 5, 6]. The infimum in the right hand side of (1) can be computed by using a bisection algorithm [7].

It is clear that for a real matrix, the complex stability radius gives a lower bound of the real stability radius. In the following, we will always assume that A is a real $n \times n$ stable matrix. Hence,

$$r_{\mathbf{R}}(A) \geq r_{\mathbf{C}}(A). \tag{2}$$

Some other lower bounds of the real stability radius are given in [2] as follows

$$r_{\mathbf{R}}(A) \;\geq\; \min\left\{\underline{\sigma}(A), \frac{1}{2}\sigma_{n^2-1}(A\otimes I_n + I_n \otimes A)\right\} \tag{3}$$

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$$r_{\mathbb{R}}(A) \geq \frac{1}{2}\underline{\sigma}(A \vee I_n + I_n \vee A) \tag{4}$$

$$r_{\mathbb{R}}(A) \geq \min\left\{\underline{\sigma}(A), \frac{1}{2}\underline{\sigma}(A \wedge I_n + I_n \wedge A)\right\}. \tag{5}$$

where $\otimes$, $\vee$, $\wedge$ denote the Kronecker (tensor) product, symmetric tensor product and skew-symmetric tensor product respectively. In (3) and in the rest of this paper, we assume that singular values are ordered decreasingly and that $\sigma_k(\cdot)$ denotes the k-th singular value.

Inequalities (2)–(5) give easily computable lower bounds to the real stability radius. Moreover, (2)–(5) are actually equalities if A is normal, and (2) and (5) are equalities if A is 2×2. The tightness of (2)–(5) for a general stable matrix A, however, is hard to judge.

The current paper is an enlarged version of [8]. In this paper, we present a new lower bound which certainly improves (2) and likely improves (3)–(5). We also formulate a nonlinear programming problem which can be used to obtain upper bounds for the real stability radius. For all the examples in which we have tested the new lower bound, the upper bounds obtained from the nonlinear programming solution coincide with the respective lower bounds. This suggests that the new lower bound may in general turn out to be equal to the real stability radius. Unfortunately, we can neither prove nor disprove this conjecture at this time.

2. THE MAIN RESULT — A LOWER BOUND

Let $A \in \mathbb{R}^{n\times n}$ be stable. For the convenience of analysis, define

$$r_{\mathbb{R}_\omega}(A) = \inf\{\overline{\sigma}(\Delta A) : \Delta A \in \mathbb{R}^{n\times n} \text{ and } A + \Delta A \text{ has a pair of imaginary eigenvalues}\}.$$

It is clear that

$$r_{\mathbb{R}}(A) = \min\{\underline{\sigma}(A), r_{\mathbb{R}_\omega}(A)\}.$$

For $\omega \in (0,\infty)$, let $B(\omega)$ be a 2×2 complex matrix with eigenvalue $j\omega$ and $-j\omega$. Then the rank of

$$B(\omega) \otimes I_n + I_2 \otimes (A + \Delta A)$$

is at most $2n - 2$ if $A + \Delta A$ has eigenvalues at $j\omega$ and $-j\omega$. This implies that $\overline{\sigma}(\Delta A) = \overline{\sigma}(I_2 \otimes \Delta A)$ is at least $\sigma_{2n-1}[B(\omega) \otimes I_n + I_2 \otimes A]$. Define

$$\beta(A) = \inf_{\omega\in(0,\infty)} \sup_{B(\omega)} \sigma_{2n-1}[B(\omega) \otimes I_n + I_2 \otimes A]. \tag{6}$$

Then $r_{\mathbb{R}_\omega}(A) \geq \beta(A)$.

The right hand side of (6) involves a complicated constrained minimax problem. However, it can be simplified in two ways. First, notice that there exists a unitary matrix U such that

$$U^*B(\omega)U = \begin{bmatrix} j\omega & x \\ 0 & -j\omega \end{bmatrix}$$

where $x \in [0,\infty)$. Since $B(\omega)\otimes I_n + I_2\otimes A$ and $U^*B(\omega)U\otimes I_n + I_2\otimes A$ have the same singular values, it follows that

$$\begin{aligned}\sup_{B(\omega)} \sigma_{2n-1}[B(\omega)\otimes I_n + I_2\otimes A] &= \sup_{x\in[0,\infty)} \sigma_{2n-1}\left(\begin{bmatrix} j\omega & x \\ 0 & -j\omega\end{bmatrix}\otimes I_n + I_2\otimes A\right) \\ &= \sup_{x\in[0,\infty)} \sigma_{2n-1}\begin{bmatrix} A+j\omega I_n & xI_n \\ 0 & A-j\omega I_n\end{bmatrix}.\end{aligned}$$

Therefore

$$\beta(A) = \inf_{\omega\in(0,\infty)}\sup_{x\in[0,\infty)} \sigma_{2n-1}\begin{bmatrix} A+j\omega I_n & xI_n \\ 0 & A-j\omega I_n\end{bmatrix}. \tag{7}$$

The right hand side of (7) is a much easier minimax problem.

To obtain the second way to simplify the right hand side of (6), we need the following lemma, which will be proved in Appendix A.

Lemma 1 *There exists a unitary matrix V such that*

$$V^*B(\omega)V = \begin{bmatrix} 0 & \gamma\omega \\ -\frac{\omega}{\gamma} & 0\end{bmatrix}$$

where $\gamma \in (0,1]$.

By using a similar argument as above, we then obtain

$$\begin{aligned}\sup_{B(\omega)} \sigma_{2n-1}[B(\omega)\otimes I_n + I_2\otimes A] &= \sup_{\gamma\in(0,1]} \sigma_{2n-1}\left(\begin{bmatrix} 0 & \gamma\omega \\ -\frac{\omega}{\gamma} & 0\end{bmatrix}\otimes I_n + I_2\otimes A\right) \\ &= \sup_{\gamma\in(0,1]} \sigma_{2n-1}\begin{bmatrix} A & \gamma\omega I_n \\ -\frac{\omega}{\gamma}I_n & A\end{bmatrix}.\end{aligned}$$

Therefore

$$\beta(A) = \inf_{\omega\in(0,\infty)}\sup_{\gamma\in(0,1]} \sigma_{2n-1}\begin{bmatrix} A & \gamma\omega I_n \\ -\frac{\omega}{\gamma}I_n & A\end{bmatrix}. \tag{8}$$

An advantage of the right hand side of (8) over that of (7) is that it involves only real numbers. It is of interest to notice that the effect of γ in (8) is equivalent to the diagonal similarity scaling used in the μ-analysis [9].

On summarizing, we obtain our main result.

Theorem 1

$$r_{\mathbb{R}}(A) \geq \min\{\underline{\sigma}(A), \beta(A)\}$$

where

$$\begin{aligned}\beta(A) &= \inf_{\omega\in(0,\infty)}\sup_{x\in[0,\infty)} \sigma_{2n-1}\begin{bmatrix} A+j\omega I_n & xI_n \\ 0 & A-j\omega I_n\end{bmatrix} \\ &= \inf_{\omega\in(0,\infty)}\sup_{\gamma\in(0,1]} \sigma_{2n-1}\begin{bmatrix} A & \gamma\omega I_n \\ -\frac{\omega}{\gamma}I_n & A\end{bmatrix}.\end{aligned}$$

Let us denote $\min\{\underline{\sigma}(A), \beta(A)\}$ by $\alpha(A)$. A few simple facts are given in the following.

Fact 1 $\alpha(A) \geq r_{\mathbf{C}}(A)$.

Fact 1 follows easily from the fact that $r_{\mathbf{C}}(A) \leq \underline{\sigma}(A)$ and

$$\begin{aligned} r_{\mathbf{C}}(A) &= \inf_{\omega \in \mathbf{R}} \underline{\sigma}(A - j\omega I_n) \\ &= \inf_{\omega \in (0,\infty)} \sigma_{2n-1} \begin{bmatrix} A + j\omega I_n & 0 \\ 0 & A - j\omega I_n \end{bmatrix} \\ &\leq \inf_{\omega \in (0,\infty)} \sup_{x \in [0,\infty)} \sigma_{2n-1} \begin{bmatrix} A + j\omega I_n & x I_n \\ 0 & A - j\omega I_n \end{bmatrix} \end{aligned}$$

Fact 2 *If A is normal, then $\alpha(A) = r_{\mathbf{R}}(A)$.*

Fact 2 follows from Fact 1 and the well-known fact that $r_{\mathbf{C}}(A) = r_{\mathbf{R}}(A)$ when A is normal [2].

3. AN UPPER BOUND

To verify the tightness of the new lower bound on the real stability radius, we try to find the minimum of $\overline{\sigma}(\Delta A)$ with the constraint that $A + \Delta A$ has imaginary eigenvalues by using nonlinear programming technique. There are many ways to formulate such a nonlinear programming problem. The following formulation is used in our computation:

Minimize	$\overline{\sigma}(\Delta A)$
with respect to	$\Delta A \in \mathbf{R}^{n \times n}$ and $\omega \in \mathbf{R}$
subject to	$\underline{\sigma}[(A + \Delta A)^2 + \omega^2 I] = 0$.

This nonlinear optimization problem may have local optima which are not global. Each local optimum certainly gives an upper bound to $r_{\mathbf{R}}(A)$. For all the examples we have attempted to date, we are able to obtain the global optimum which in fact coincides with the lower bound $\alpha(A)$.

4. EXAMPLES

Some examples have been provided in [8]. Here we give some new examples. In all of the following examples, $\beta(A)$ is computed according to (7).

Example 1

The following matrix A is taken from a model of a large flexible space structure with no rigid body mode.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -0.01 & 0 & 0 \\ 0 & -2 & 0 & 0 & -0.01 & 0 \\ 0 & 0 & -10 & 0 & 0 & -0.01 \end{bmatrix} .$$

This matrix is stable with eigenvalues which have a real part -5×10^{-3} and various imaginary parts. The complex stability radius is given by $r_{\mathbb{C}}(A) = 4.7140 \times 10^{-3}$.

The solution to the minimax problem is $\beta(A) = 5 \times 10^{-3}$ with $\omega = 1.4142$ and $x = 1$. Thus

$$\alpha(A) = \min\{\underline{\sigma}(A), \beta(A)\} = \min\{0.99501, 5 \times 10^{-3}\} = 5 \times 10^{-3}.$$

On observing that the real part of the eigenvalues of A is -5×10^{-3}, we easily conclude that for this A matrix, $\alpha(A) = r_{\mathbb{R}}(A)$.

Example 2

Let

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

This matrix is stable with eigenvalues $-1 \pm j1$ and $-1 \pm j1$. The complex stability radius $r_{\mathbb{C}}(A) = 0.61803$.

The solution of the minimax problem is $\beta(A) = 0.61803$ with $\omega = 1$ and $x = 0$. Thus

$$\alpha(A) = \min\{\underline{\sigma}(A), \beta(A)\} = \min\{1, 0.61803\} = 0.61803$$

which is the same as the complex stability radius.

We also find that

$$\Delta A = \begin{bmatrix} 2.8025 \times 10^{-1} & 1.3680 \times 10^{-1} & 1.6842 \times 10^{-1} & -2.1707 \times 10^{-1} \\ 4.5081 \times 10^{-2} & 2.2008 \times 10^{-1} & -6.8052 \times 10^{-2} & 2.5301 \times 10^{-1} \\ 4.4415 \times 10^{-1} & -8.4699 \times 10^{-2} & 2.7901 \times 10^{-1} & 1.3196 \times 10^{-1} \\ -2.3748 \times 10^{-2} & 4.7929 \times 10^{-1} & 3.7897 \times 10^{-2} & 2.2996 \times 10^{-1} \end{bmatrix}$$

is a destabilizing perturbation matrix with $A + \Delta A$ having eigenvalues on the imaginary axis and $\overline{\sigma}(\Delta A) = 0.61806$.

Example 3

Consider the matrix

$$A = \begin{bmatrix} -1 & 1000 & 0.001 \\ -1 & -1 & 0 \\ 1 & 1 & -100 \end{bmatrix}$$

This matrix is stable with eigenvalues $-1 \pm j31.623$ and -100. The complex stability radius $r_{\mathbb{C}}(A) = 6.3179 \times 10^{-2}$.

The solution of the minimax problem is $\beta(A) = 0.99829$ with $\omega = 3.1624$ and $x = 997.31$. Thus

$$\alpha(A) = \min\{\underline{\sigma}(A), \beta(A)\} = \min\{1.0009, 0.99829\} = 0.99829.$$

We also find that

$$\Delta A = \begin{bmatrix} 9.9872 \times 10^{-1} & -1.6258 \times 10^{-2} & 6.8691 \times 10^{-3} \\ 1.5120 \times 10^{-2} & 9.9814 \times 10^{-1} & -3.8562 \times 10^{-2} \\ -1.9686 \times 10^{-1} & -5.5692 \times 10^{-2} & -6.8466 \times 10^{-1} \end{bmatrix}$$

is a destabilizing perturbation matrix with $A+\Delta A$ having eigenvalues on the imaginary axis and $\overline{\sigma}(\Delta A) = 0.99983$. There is a small but noticable gap between the lower bound and the upper bound. We believe that it is due to the numerical problem in computing the upper bound.

5. CONCLUDING REMARKS

In this paper, we have derived a lower bound on the real stability radius of a real stable matrix and we conjecture that this lower bound is actually equal to the real stability radius. In addition to proving or disproving this conjecture, some other questions, mainly concerning the computation of $\beta(A)$, are yet to be answerd. For example, computational experience shows that for fixed $\omega \in (0,\infty)$,

$$\sigma_{2n-1}\begin{bmatrix} A+j\omega I_n & xI_n \\ 0 & A-j\omega I_n \end{bmatrix} \quad \text{and} \quad \sigma_{2n-1}\begin{bmatrix} A & \gamma\omega I_n \\ -\frac{\omega}{\gamma}I_n & A \end{bmatrix}$$

are functions with only one peak in intervals $[0,\infty)$ and $(0,1]$ respectively. If we can prove this, we would be able to improve the computational complexity of $\beta(A)$. Another question is as follows: can we switch the order of the "inf" and the "sup" in (7) and (8) and what benefit can it provide if such a switch is possible.

Finally, we would like to point out that the same idea used in this paper can be easily adapted to obtain a lower bound for the discrete time real stability radius of a real matrix. In fact, there is not much technical difficulty to extent the results in this paper to study the real stability radius with respect to an essentially arbitrary stability region in the complex plane provided only that the stability region is symmetric to the real axis.

APPENDIX A. PROOF OF LEMMA 1

It is enough to show that for each $x \in [0,\infty)$ there exists a unitary matrix V such that

$$V^*\begin{bmatrix} j\omega & x \\ 0 & -j\omega \end{bmatrix}V = \begin{bmatrix} 0 & \gamma\omega \\ -\frac{\omega}{\gamma} & 0 \end{bmatrix}$$

for some $\gamma \in (0,1]$. We will accomplish this by finding explicitly a unitary matrix W for each $\gamma \in (0,1]$ such that

$$W^*\begin{bmatrix} 0 & \gamma\omega \\ -\frac{\omega}{\gamma} & 0 \end{bmatrix}W = \begin{bmatrix} j\omega & x \\ 0 & -j\omega \end{bmatrix}$$

and showing that x takes every value in $[0,\infty)$ as γ varies in $(0,1]$. In fact, a choice of such a W is given by the following Hermitian matrix

$$W = \frac{1}{\sqrt{1+\gamma^2}}\begin{bmatrix} \gamma & -j \\ j & -\gamma \end{bmatrix}$$

with $x = \frac{1-\gamma^4}{\gamma(1+\gamma^2)}\omega$. $\square$

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