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Complex and Real Performance Radii and Their Computation*

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Abstract

This paper considers the problem of robust performance of linear time-invariant system in $\mathcal{H}_\infty$ norm. The concepts of complex and real performance radii are introduced to describe the smallest size of dynamic or parametric perturbations to a feedback system that either destabilize the system or destroy a performance bound in certain closed loop transfer matrix of the system. An algorithm to compute the complex performance radius is given. For the real performance radius, a lower bound, which often turns out to be exact, is obtained.

1 Introduction

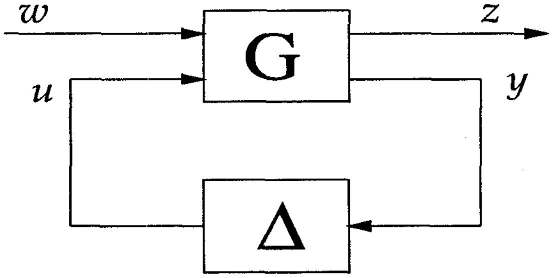

Figure 1: Uncertain control system

This paper concerns the robust $\mathcal{H}_\infty$ performance of a linear time-invariant (LTI) system under dynamic or parametric perturbation. Consider the uncertain system shown in Figure 1. Let $\mathcal{RH}_\infty$ denote the ring of real rational functions in $\mathcal{H}_\infty$. Assume that $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathcal{RH}_\infty^{(q+p)\times(l+m)}$ is a transfer matrix representing an LTI system and $\Delta \in \mathbb{F}^{m\times p}$, where $\mathbb{F}$ is either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. The transfer matrix from w to z is then given by the following linear fractional transformation:

$$\mathcal{F}(G,\Delta) = G_{11} + G_{12}(I-\Delta G_{22})^{-1}\Delta G_{21}.$$

The system is said to be internally stable if $(I-\Delta G_{22})^{-1}$ exists and belongs to $\mathcal{H}_\infty^{m\times m}$.

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Now assume that $\|G_{11}\|_\infty < 1$. Define the performance radius of G to be

$$pr_{\mathbb{F}}(G) := \sup\{r : \mathcal{F}(G,\Delta) \text{ is internally stable and } \|\mathcal{F}(G,\Delta)\|_\infty < 1, \forall\, \|\Delta\| < r, \Delta \in \mathbb{F}^{m\times p}\}. \quad (1)$$

Here the norm of Δ is the spectral norm, i.e., the largest singular value. $pr_{\mathbb{C}}(G)$ will be called the complex performance radius of G and $pr_{\mathbb{R}}(G)$ the real performance radius. The purpose of this paper is to study the computation of $pr_{\mathbb{C}}(G)$ and $pr_{\mathbb{R}}(G)$. A complete solution to the problem of computing $pr_{\mathbb{C}}(G)$ is given in this paper. Roughly speaking, $pr_{\mathbb{C}}(G)$ can be computed via the computation of an $\mathcal{H}_\infty$ norm and a frequency sweep of a function which can be evaluated by performing a one dimensional convex optimization. The problem of computing $pr_{\mathbb{R}}$ is however not completely solved in this paper. Instead a lower bound is obtained. This lower bound can be obtained via the computation of a real stability radius and a frequency sweep of a function which can be evaluated by performing a two dimensional nonlinear minimization. It is also shown that the function to be minimized has only one local minimum. Numerical experience shows that this lower bound is often tight.

It is clear that $pr_{\mathbb{R}}(G)$ gives a measure to the robust performance of the system shown in Figure 1 under parametric uncertainty. It can be shown using standard techniques [2] that

$$pr_{\mathbb{C}}(G) = \sup\{r : \mathcal{F}(G,\Delta) \text{ is internally stable and } \|\mathcal{F}(G,\Delta)\|_\infty < 1, \forall\, \|\Delta\|_\infty < r, \Delta \in \mathcal{H}_\infty^{m\times p}\}.$$

Hence $pr_{\mathbb{C}}(G)$ gives a measure to the robust performance of the feedback system under LTI dynamic perturbation.

The robust performance measure $pr_{\mathbb{F}}(G)$ is also connected to the robust stability of LTI systems under structured perturbations, which has been studied extensively in the μ framework, see e.g., [3, 13, 6, 11]. Consider the uncertain system shown in Figure 2. The small gain theorem implies that this uncertain system is internally stable for all $\Delta \in \mathcal{RH}_\infty^{m\times p}$ with $\|\Delta\|_\infty < r$ and $\tilde{\Delta} \in \mathcal{RH}_\infty^{l\times q}$ with $\|\tilde{\Delta}\|_\infty < 1$ if and only if $pr_{\mathbb{C}}(G) \geq r$. In the μ framework, algorithms are available to determine if the uncertain system in Figure 2 is internally stable for all $\Delta \in \mathcal{RH}_\infty^{m\times p}$ with $\|\Delta\|_\infty < r$ and $\tilde{\Delta} \in \mathcal{RH}_\infty^{l\times q}$ with $\|\tilde{\Delta}\|_\infty < r$. Notice the slight

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difference between the formulation of the complex performance radius and that of μ. Of course, a moment thought reveals that the μ algorithm can be iteratively used to find $pr_{\mathbb{C}}(G)$. In this paper, we propose an algorithm specifically tailored to $pr_{\mathbb{C}}(G)$ so that it is computed directly without this additional iteration.

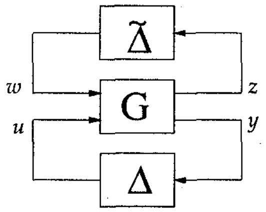

Figure 2: Augmented uncertain control system

The complex performance radius, though not been called so, is also studied in [5]. An algorithm for its computation, free from the additional iteration, is given in [5]. The algorithm is based on the quasi-convex univariate minimization of the largest generalized eigenvalue of a semi-definite matrix pair. Our algorithm, however, is different. It is based on the convex univariate minimization of the largest eigenvalue of a Hermitian matrix. Due to the better convexity property of our minimization problem, faster and more reliable line search methods can be applied. Also notice that generalized eigenvalue problems has higher computational complexity than Hermitian eigenvalue problems (with roughly the same size).

The emphasis of this paper is the real performance radius. Again the small gain theorem implies that the uncertain system in Figure 2 is internally stable for all $\Delta \in \mathbb{R}^{m\times p}$ with $\|\Delta\| < r$ and $\tilde{\Delta} \in \mathcal{H}_\infty^{l\times q}$ with $\|\tilde{\Delta}\|_\infty < 1$ if and only if $pr_{\mathbb{R}}(G) \geq r$. This type of robust stability problem with one complex full block and one real full block can also be studied in the μ framework. However, no algorithm is readily available to compute the corresponding μ value. The past literature in μ favours to model parametric perturbations in the form of scalar times identity [6], which usually ends up with exponential time algorithms. Our conviction is that modelling parameter perturbations in the form of full matrices may in some cases yield easier solutions. Although we are not able to solve completely the computation problem of the real performance radius at this moment, we believe that it is potentially solvable.

The computation of $pr_{\mathbb{F}}$ can also be used to find the worst $\mathcal{H}_\infty$ performance when the perturbation bound is given.

The paper is organized as follows. Section 2 is for the preliminary development. We will convert the computation of the performance radius to well-defined linear algebra problems. Sections 3 gives a formula for the complex performance radius. Section 4 gives a lower bound of the real performance radius and Section 5 gives some properties of the lower bound. In Section 6, two examples are presented. Section 7 is the conclusion.

In the following, we define some notation used in this paper. For $X \in \mathbb{C}^{m\times p}$, the real and imaginary parts of X are denoted by $\operatorname{Re} X$ and $\operatorname{Im} X$ respectively. The singular values of X are denoted by $\sigma_i(X)$, assuming nonincreasing order. The largest singular value of X is also denoted by $\bar{\sigma}(X)$. We always set $\|X\| = \bar{\sigma}(X)$. If X is Hermitian, then the eigenvalues of X are denoted by $\lambda_i(X)$, also assuming nonincreasing order.

Due to space limitation, some proofs are omitted. For details, see [8].

2 Development

Recall from [4, 2, 7] the definition of the stability radius of $F \in \mathcal{RH}_\infty^{p\times m}$:

$$r_{\mathbb{F}}(F) = \inf\{\|\Delta\| : \Delta \in \mathbb{F}^{m\times p} \text{ and } (I-\Delta F)^{-1} \notin \mathcal{H}_\infty\}.$$

For $G \in \mathcal{RH}_\infty^{(q+p)\times(l+m)}$ with $\|G_{11}\|_\infty < 1$, define

$$p_{\mathbb{F}}(G) := \inf_{\omega\in[0,\infty]} \inf\left\{\|\Delta\| : \Delta \in \mathbb{F}^{m\times p},\right.$$
$$\left.\det\left\{I - \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M(\mathrm{j}\omega)\right\} = 0\right\},$$

where

$$M(\mathrm{j}\omega) = \begin{bmatrix} S(\mathrm{j}\omega) & N(\mathrm{j}\omega) \\ N^*(\mathrm{j}\omega) & R(\mathrm{j}\omega) \end{bmatrix}, \tag{2}$$

and

$$\begin{aligned} S &= G_{12}^\sim[I - G_{11}G_{11}^\sim]^{-1}G_{12}, & (3)\\ N &= G_{22}^\sim + G_{12}^\sim G_{11}[I - G_{11}^\sim G_{11}]^{-1}G_{21}^\sim, & (4)\\ R &= G_{21}[I - G_{11}^\sim G_{11}]^{-1}G_{21}^\sim. & (5) \end{aligned}$$

Note that $S(\mathrm{j}\omega) \geq 0$ and $R(\mathrm{j}\omega) \geq 0$ for all $\omega \in [-\infty, \infty]$.

In fact, M equals to the transfer matrix from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in Figure 3. By the definition of star product [16], we also have $M = [(JG^\sim J) \star G]J$, where $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

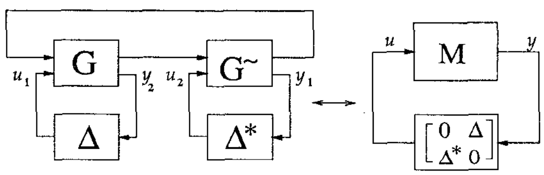

Figure 3: System theoretic interpretation of M.

Proposition 1 $pr_{\mathbb{F}}(G) = \min\{r_{\mathbb{F}}(G_{22}), p_{\mathbb{F}}(G)\}$.

The computation of $r_{\mathbb{F}}(G_{22})$ has been studied for a long time and the following formulas are now well-known [4, 2, 7, 12]:

$$r_{\mathbb{C}}(G_{22}) = \|G_{22}\|_\infty^{-1}, \tag{6}$$

$$r_{\mathbb{R}}(G_{22}) = \left\{ \sup_{\omega\in[0,\infty]} \mu_{\mathbb{R}}(G_{22}(j\omega)) \right\}^{-1}, \qquad (7)$$

where

$$\mu_{\mathbb{R}}(X) = \inf_{\gamma\in(0,1]} \sigma_2 \left(\begin{bmatrix} \operatorname{Re} X & -\gamma \operatorname{Im} X \\ \gamma^{-1}\operatorname{Im} X & \operatorname{Re} X \end{bmatrix} \right).$$

Hence, we only need to focus on the computation of $p_{\mathbb{F}}(G)$ in this paper.

For Hermitian matrix $M = \begin{bmatrix} S & N \\ N^* & R \end{bmatrix}$ with $S \in \mathbb{C}^{m\times m} \geq 0$ and $R \in \mathbb{C}^{p\times p} \geq 0$, define

$$\psi_{\mathbb{F}}(M) := \inf \Big\{ \|\Delta\| : \Delta \in \mathbb{F}^{m\times p}, \det\left\{ I - \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0 \Big\}, \qquad (8)$$

then it follows that $p_{\mathbb{F}}(G) = \inf_{\omega\in[0,\infty]} \psi_{\mathbb{F}}[M(j\omega)]$. Therefore, if $\psi_{\mathbb{F}}(M)$ can be obtained for each $M = \begin{bmatrix} S & N \\ N^* & R \end{bmatrix}$ with $S \geq 0$ and $R \geq 0$, then $pr_{\mathbb{F}}(G)$ can be computed by a frequency sweep.

The next two sections are dedicated to the computation of $\psi_{\mathbb{C}}$ and $\psi_{\mathbb{R}}$ respectively.

3 Complex performance radius

It follows from the development in Section 2 that $pr_{\mathbb{C}}(G)$ can be obtained via the computation of $\psi_{\mathbb{C}}(M)$ for each $M \in \begin{bmatrix} S & N \\ N^* & R \end{bmatrix}$ with $S \geq 0$ and $R \geq 0$. Hence, this section is dedicated to the computation of $\psi_{\mathbb{C}}(M)$. In the following, we show that $\psi_{\mathbb{C}}(M)$ is the infimum of a convex univariate function.

Note that

$$\det\left\{ I - \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = \det\left\{ I - \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} \begin{bmatrix} S/\gamma & N \\ N^* & \gamma R \end{bmatrix} \right\}. \qquad (9)$$

Theorem 1 *Let $M = \begin{bmatrix} S & N \\ N^* & R \end{bmatrix}$ satisfy $S \geq 0$ and $R \geq 0$. Then*

$$\psi_{\mathbb{C}}(M) = \left\{ \inf_{\gamma>0} \lambda_1 \begin{bmatrix} S/\gamma & N \\ N^* & \gamma R \end{bmatrix} \right\}^{-1}.$$

Furthermore, $\lambda_1 \begin{bmatrix} S/\gamma & N \\ N^* & \gamma R \end{bmatrix}$ is a convex function of γ on $(0,\infty)$.

Several lemmas are needed for the proof of Theorem 1.

Lemma 1 ([1, p. 149]) *Let $F(\gamma) \in \mathbb{C}^{n\times n}$ be a Hermitian matrix function analytic on an open set $\Gamma \subset \mathbb{R}$. Then there exist a unitary matrix function $\tilde{V}(\gamma) = [\tilde{v}_1(\gamma), \cdots, \tilde{v}_n(\gamma)] \in \mathbb{C}^{n\times n}$ and a diagonal matrix function $\tilde{\Lambda}(\gamma) = \operatorname{diag}\left[\tilde{\lambda}_1(\gamma), \cdots, \tilde{\lambda}_n(\gamma)\right] \in \mathbb{C}^{n\times n}$, both analytic on Γ, such that $F(\gamma) = \tilde{V}(\gamma)\tilde{\Lambda}(\gamma)\tilde{V}^*(\gamma)$. Furthermore,*

$$\frac{d\tilde{\lambda}_i(\gamma)}{d\gamma} = \tilde{v}_i{}^*(\gamma)\frac{dF(\gamma)}{d\gamma}\tilde{v}_i(\gamma). \qquad (10)$$

Lemma 2 *Let $F(\gamma) \in \mathbb{C}^{n\times n}$ be a Hermitian matrix function analytic on an open set $\Gamma \subset \mathbb{R}$. Let $\lambda_1(\gamma) \geq \lambda_2(\gamma) \geq \cdots \geq \lambda_n(\gamma)$ be its ordered eigenvalues. If $\lambda_i(\gamma)$ has a local extremum at $\gamma_0 \in \Gamma$, then $F(\gamma_0)$ has an eigenvector $v \in \mathbb{C}^n$ corresponding to $\lambda_i(\gamma_0)$ such that $v^*\frac{dF(\gamma_0)}{d\gamma}v = 0$.*

Lemma 3 *Let $F = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in \mathbb{C}^{(m+p)\times(m+p)}$ be a Hermitian matrix. Suppose $X \geq 0$ and $Z \geq 0$, then for $i = 1, 2, \cdots, \min\{m,p\}$, $\lambda_i(F) \geq -\lambda_{m+p-i+1}(F)$.*

Proof of Theorem 1:

Denote $F(\gamma) = \begin{bmatrix} S/\gamma & N \\ N^* & \gamma R \end{bmatrix}$. Then from Lemma 3, $\lambda_1[F(\gamma)] \geq -\lambda_{m+p}[F(\gamma)]$ for all $\gamma > 0$, Hence $\lambda_1[F(\gamma)] = \bar{\sigma}[F(\gamma)]$. We see from (9) that

$$\psi_{\mathbb{C}}(M) \geq \left\{ \inf_{\gamma>0} \lambda_1[F(\lambda)] \right\}^{-1}.$$

The fact that $\bar{\sigma}[F(\gamma)]$ is convex follows from [14].

Let $\lambda_0 = \inf_{\gamma>0} \lambda_1[F(\gamma)]$. The rest of the proof is to show that $\psi_{\mathbb{C}}(M) \leq \lambda_0^{-1}$.

If $\lambda_0 = 0$, we certainly have $\psi_{\mathbb{C}}(M) \leq \lambda_0^{-1}$. So we assume that $\lambda_0 > 0$. There are two cases.

Case 1: $\inf_{\gamma>0} \lambda_1[F(\gamma)]$ is attained at $\gamma_0 \in (0,\infty)$.

By Lemma 2, there exist $v = \begin{bmatrix} x \\ y \end{bmatrix}$, $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^p$ such that $F(\gamma_0)v = \lambda_0 v$, i.e.,

$$\frac{1}{\gamma_0}Sx + Ny = \lambda_0 x, \qquad (11)$$

$$N^*x + \gamma_0 Ry = \lambda_0 y, \qquad (12)$$

and

$$v^*\frac{dF(\gamma_0)}{d\gamma}v = -\frac{1}{\gamma_0^2}x^*Sx + y^*Ry = 0.$$

Multiplying (11) and (12) from the left by x^* and y^* respectively, subtracting the resulting equations, and noting that x^*Ny must be real, we get

$$\lambda_0(x^*x - y^*y) = \frac{1}{\gamma_0}x^*Sx - \gamma_0 y^*Ry = 0.$$

Hence, $x^*x = y^*y$. Now construct $\Delta = (\lambda_0)^{-1}xy^*/x^*x$, then it is easy to verify that $\|\Delta\| = (\lambda_0)^{-1}$, and

$$\left\{ I - \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} \begin{bmatrix} S/\gamma_0 & N \\ N^* & \gamma_0 R \end{bmatrix} \right\} \begin{bmatrix} x \\ y \end{bmatrix} = 0,$$

which means that $\det \left\{ I - \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0$. From the definition of $\psi_{\mathbb{C}}(M)$, we have $\psi_{\mathbb{C}}(M) \leq (\lambda_0)^{-1}$.

Case 2: $\lambda_0 = \lim_{\gamma\to 0} \lambda_1[F(\gamma)]$ or $\lambda_0 = \lim_{\gamma\to\infty} \lambda_1[F(\gamma)]$.

This case occurs only when $S = 0$ or $R = 0$. Assume $S = 0$, then

$$\det \left\{ I - \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = \det(I - \Delta N^*) \det(I - \Delta^* N),$$

which shows $\psi_{\mathbb{C}}(M) = [\bar{\sigma}(N)]^{-1}$.

On the other hand, since $R \geq 0$, we have

$$\lambda_0 = \inf_{\gamma > 0} \lambda_1 \begin{bmatrix} 0 & N \\ N^* & \gamma R \end{bmatrix} = \bar{\sigma}(N).$$

Therefore, $\psi_{\mathbb{C}}(M) = \lambda_0^{-1}$ in this case. □

4 A lower bound of the real performance radius

Recall from Section 2 that $pr_{\mathbb{R}}(G) = \min\{r_{\mathbb{R}}(G_{22}), p_{\mathbb{R}}(G)\}$. The computation of $r_{\mathbb{R}}(G_{22})$ can be done using formula (7). The computation of $p_{\mathbb{R}}(G)$ depends on the computation of $\psi_{\mathbb{R}}(M)$ for a given Hermitian matrix $M = \begin{bmatrix} S & N \\ N^* & R \end{bmatrix}$ with $S \geq 0$ and $R \geq 0$. Unfortunately, we are not able to compute $\psi_{\mathbb{R}}(M)$ at this moment. In this section, a lower bound of $\psi_{\mathbb{R}}(M)$ will be given.

Let $S_r = \operatorname{Re} S$, $S_i = \operatorname{Im} S$, $R_r = \operatorname{Re} R$, $R_i = \operatorname{Im} R$, $N_r = \operatorname{Re} N$, $N_i = \operatorname{Im} N$. Then, $S_r = S_r^T$, $R_r = R_r^T$, $S_i = -S_i^T$, $R_i = -R_i^T$ and $S_r \geq 0$, $R_r \geq 0$.

We follow the idea in [12] to convert the problem of computing $\psi_{\mathbb{R}}(M)$, one with complex data and realness constraint, into a pure real problem. Let

$$P_0 = \begin{bmatrix} \operatorname{Re} M & -\operatorname{Im} M \\ \operatorname{Im} M & \operatorname{Re} M \end{bmatrix}, \Delta_a = \begin{bmatrix} 0 & \Delta & 0 & 0 \\ \Delta^T & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta \\ 0 & 0 & \Delta^T & 0 \end{bmatrix}.$$

The following lemma will be frequently used.

Lemma 4 *Given Hermitian matrix $Z = X + jY$ with $X, Y \in \mathbb{R}^{n\times n}$. Let $Q = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$, then $\lambda_{2i-1}(Q) = \lambda_{2i}(Q) = \lambda_i(Z)$*

For $\alpha, \beta > 0$, define scaling matrix

$$D(\alpha,\beta) = \operatorname{diag}\left[\sqrt{\alpha\beta} I, \frac{1}{\sqrt{\alpha\beta}} I, \sqrt{\frac{\beta}{\alpha}} I, \sqrt{\frac{\alpha}{\beta}} I\right], \quad (13)$$

then $D^{-1}(\alpha,\beta)\Delta_a D^{-1}(\alpha,\beta) = \Delta_a$ and

$$P(\alpha,\beta) := D(\alpha,\beta) P_0 D(\alpha,\beta) = \begin{bmatrix} \alpha\beta S_r & N_r & -\beta S_i & -\alpha N_i \\ N_r^T & \frac{1}{\alpha\beta} R_r & \frac{1}{\alpha} N_i^T & -\frac{1}{\beta} R_i \\ \beta S_i & \frac{1}{\alpha} N_i & \frac{\beta}{\alpha} S_r & N_r \\ -\alpha N_i^T & \frac{1}{\beta} R_i & N_r^T & \frac{\alpha}{\beta} R_r \end{bmatrix}. \quad (14)$$

Assume $P_0 = P(1,1)$ has π positive and ν negative eigenvalues. Lemma 4 says that π and ν are even. Also observe that $\pi \geq \nu$ which follows from applying Lemma 3 and Lemma 4 to a similarly permuted version of P_0:

$$\begin{bmatrix} S_r & -S_i & N_r & -N_i \\ S_i & S_r & N_i & N_r \\ N_r^T & N_i^T & R_r & -R_i \\ -N_i^T & N_r^T & R_i & R_r \end{bmatrix}$$

By the law of inertia, $P(\alpha,\beta)$ also has π positive and ν negative eigenvalues for all $\alpha, \beta > 0$ and its second eigenvalue is always nonnegative.

Another interesting property of $P(\alpha,\beta)$ is that

$$\lambda_i[P(\alpha,\beta)] = \lambda_i[P(\frac{1}{\alpha},\beta)] \quad (15)$$

for $i = 1, 2, \cdots, 2(m+p), \alpha, \beta \neq 0$. This property can be obtained by a similarity permutation of P.

Theorem 2 $\psi_{\mathbb{R}}(M) \geq \left\{\inf_{\alpha\in(0,1],\beta>0} \lambda_2[P(\alpha,\beta)]\right\}^{-1}$.

To prove Theorem 2, we need two lemmas.

Lemma 5 ([15, p. 203]) *Let $M, \Delta \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Denote the eigenvalues of M as $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ and the eigenvalues of $M + \Delta$ as $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$. Then, $|\xi_i - \eta_i| \leq \bar{\sigma}(\Delta)$.*

Lemma 6 *For $A \in \mathbb{C}^{m\times p}$ and $B \in \mathbb{C}^{p\times m}$,*

$$m - \operatorname{rank}(I + AB) = p - \operatorname{rank}(I + BA).$$

Proof of Theorem 2:

It follows from (15) that

$$\inf_{\alpha\in(0,1],\beta>0} \lambda_2[P(\alpha,\beta)] = \inf_{\alpha,\beta>0} \lambda_2[P(\alpha,\beta)].$$

Since $\operatorname{rank}(P_0) = \pi + \nu$, it follows from Lemma 4 that $\operatorname{rank}(M) = (\pi+\nu)/2$. Suppose that M is decomposed as $M = U_M \Lambda_M U_M^*$, where $U_M^* U_M = I, U_M \in \mathbb{C}^{(m+p)\times(\frac{\pi+\nu}{2})}$, and $\Lambda_M \in \mathbb{R}^{(\frac{\pi+\nu}{2})\times(\frac{\pi+\nu}{2})}$ is diagonal and nonsingular. Then

$$P_0 = \begin{bmatrix} \operatorname{Re} M & -\operatorname{Im} M \\ \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} = U \Lambda U^T$$

where

$$U = \begin{bmatrix} \operatorname{Re} U_M & -\operatorname{Im} U_M \\ \operatorname{Im} U_M & \operatorname{Re} U_M \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_M & 0 \\ 0 & \Lambda_M \end{bmatrix}.$$

Let $X(\alpha,\beta) = D(\alpha,\beta)U$, then $X(\alpha,\beta)$ has full column rank. Carry out the Gram-Schmidt ortho-normalization to the columns of $X(\alpha,\beta)$, we get $X(\alpha,\beta) = V(\alpha,\beta)R(\alpha,\beta)$, where $V^T(\alpha,\beta)V(\alpha,\beta) = I$ and $R(\alpha,\beta)$ is nonsingular. It is easy to see from the orthonormalization process that the maps from X to V and R are analytic when X has full column rank. Hence, $V(\alpha,\beta)$ and $R(\alpha,\beta)$ are analytic in (α,β).

Let $E(\alpha,\beta) = R(\alpha,\beta)\Lambda R^T(\alpha,\beta)$, then $E(\alpha,\beta)$ is analytic and nonsingular. From $P_0 = U\Lambda U^T$ and (14) we get

$$P(\alpha,\beta) = V(\alpha,\beta)E(\alpha,\beta)V^T(\alpha,\beta). \quad (16)$$

Since $D(1,1) = I$, we get $V(1,1) = U$, $E(1,1) = \Lambda$.

Since $V(\alpha,\beta)$ is orthonormal, the eigenvalues of $E(\alpha,\beta)$ are equal to the non-zero eigenvalues of $P(\alpha,\beta)$.

Note that

$$\begin{aligned}\text{rank}\,(I - \Delta_a P_0) &= \text{rank}\,\{I - \Delta_a P(\alpha,\beta)\} \\ &= \text{rank}\,\left\{I - \Delta_a V(\alpha,\beta)E(\alpha,\beta)V^T(\alpha,\beta)\right\}.\end{aligned}$$

By Lemma 6, we obtain

$$\begin{aligned}2(m+p) - \text{rank}\,(I - \Delta_a P_0) = &\pi+\nu \\ &-\text{rank}\left\{E^{-1}(\alpha,\beta) - V^T(\alpha,\beta)\Delta_a V(\alpha,\beta)\right\}. \quad (17)\end{aligned}$$

Denote

$$H(\alpha,\beta,\Delta) = E^{-1}(\alpha,\beta) - V^T(\alpha,\beta)\Delta_a V(\alpha,\beta), \quad (18)$$

then by (17), rank $[H(\alpha,\beta,\Delta)]$ is independent of (α,β), and

$$\begin{aligned}&\text{rank}\,[H(\alpha,\beta,\Delta)] \\ &\quad = \pi + \nu - \{2(m+p) - \text{rank}\,(I - \Delta_a P_0)\}. \quad (19)\end{aligned}$$

In the following, we will show that if $\|\Delta\| < \{\inf_{\alpha,\beta>0} \lambda_2[P(\alpha,\beta)]\}^{-1}$, then rank $[H(\alpha,\beta,\Delta)] = \pi + \nu$, which leads to $\psi_{\mathbb{R}}(M) \geq \{\inf_{\alpha,\beta>0} \lambda_2[P(\alpha,\beta)]\}^{-1}$ by (19).

Since $E(\alpha,\beta)$ and $V(\alpha,\beta)$ are analytic, it follows that for a fixed Δ, the eigenvalues of $H(\alpha,\beta,\Delta)$ are continuous in (α,β). Since the rank of $H(\alpha,\beta,\Delta)$ is independent of (α,β), we conclude that for a fixed Δ, the inertia of $H(\alpha,\beta,\Delta)$ are independent of (α,β). Consequently, it can be denoted by $\{\pi_\Delta, \nu_\Delta, \zeta_\Delta\}$. Furthermore, π_Δ, ν_Δ and ζ_Δ are even numbers, since by Lemma 4 the eigenvalues of $H(1,1,\Delta)$ have even multiplicity.

The eigenvalues of $H(\alpha,\beta,0) = E^{-1}(\alpha,\beta)$, which are $\lambda_i^{-1}[P(\alpha,\beta)]$, $i = 1,\ldots,\pi$ and $i = 2(m+p) - \nu + 1, \ldots, 2(m+p)$, satisfy

$$\begin{aligned}\cdots &\geq \lambda_2^{-1}[P(\alpha,\beta)] \geq \lambda_1^{-1}[P(\alpha,\beta)] > 0 \\ 0 &> \lambda_{2(m+p)}^{-1}[P(\alpha,\beta)] \geq \lambda_{2(m+p)-1}^{-1}[P(\alpha,\beta)] \geq \cdots.\end{aligned}$$

Now consider the inertia of $H(\alpha,\beta,\Delta)$ under perturbation. If $\|\Delta\| < \{\inf_{\alpha,\beta>0} \lambda_2[P(\alpha,\beta)]\}^{-1}$, there exists α_0, β_0 such that $\|\Delta\| < \lambda_2^{-1}[P(\alpha_0,\beta_0)]$. Note that the eigenvalues of $P(\alpha,\beta)$ are the same as those of

$$\begin{bmatrix} \alpha\beta S_r & -\beta S_i & N_r & -\alpha N_i \\ \beta S_i & \frac{\beta}{\alpha} S_r & \frac{1}{\alpha} N_i & N_r \\ N_r^T & \frac{1}{\alpha} N_i^T & \frac{1}{\alpha\beta} R_r & -\frac{1}{\beta} R_i \\ -\alpha N_i^T & N_r^T & \frac{1}{\beta} R_i & \frac{\alpha}{\beta} R_r \end{bmatrix} \quad (20)$$

Hence, if $\nu \geq 2$, then Lemma 3 implies that $\|\Delta\| < -\lambda_{2(m+p)-1}^{-1}[P(\alpha_0,\beta_0)]$. Since $\|V^T(\alpha,\beta)\Delta_a V(\alpha,\beta)\| \leq \|\Delta\|$, it follows from Lemma 5 that $\pi_\Delta > \pi - 2$ and $\nu_\Delta > \nu - 2$. Since π_Δ and ν_Δ are even numbers, we must have $\pi_\Delta = \pi$, $\nu_\Delta = \nu$. This is also true if $\nu = 0$. Therefore if $\|\Delta\| < \{\inf_{\alpha,\beta>0} \lambda_2 P[(\alpha,\beta)]\}^{-1}$, then rank $[H(\alpha,\beta,\Delta)] = \pi + \nu$, which shows rank$(I - \Delta_a P_0) = 2(m+p)$. Then it follows from Lemma 4 that $\psi_{\mathbb{R}}(M) \geq \{\inf_{\alpha,\beta>0} \lambda_2[P(\alpha,\beta)]\}^{-1}$. □

Remark : Let $\sigma_0 = \inf_{\alpha,\beta>0} \sigma_2[P(\alpha,\beta)]$, then it is easy to see from the above proof that another lower bound for $\psi_{\mathbb{R}}(M)$ is σ_0^{-1}. It is obvious that $\sigma_0 \geq \inf_{\alpha,\beta>0} \lambda_2[P(\alpha,\beta)]$. If $\sigma_0 = \inf_{\alpha,\beta>0} \sigma_2[P(\alpha,\beta)]$ is obtained at (α_0,β_0), and σ_0 is a singular value of $P(\alpha_0,\beta_0)$ with multiplicity greater than one, then σ_0^{-1} may be strictly smaller than $\{\inf_{\alpha,\beta>0} \lambda_2[P(\alpha,\beta)]\}^{-1}$, if this multiplicity is caused by the intersection of $\lambda_2[P(\alpha,\beta)]$ and $-\lambda_{2(m+p)}[P(\alpha,\beta)]$, as is illustrated in Section 6.

5 Properties of the lower bound

A lower bound of $\psi_{\mathbb{R}}(M)$, given in terms of a two dimensional nonlinear minimization problem, is derived in last section. Two questions remain to be answered:

1. How tight is the lower bound?
2. Is the minimization problem numerically tractable?

The first question is partially answered in this section; we give sufficient condition under which the lower bound is tight (Theorem 3). The second issue is completely answered; we show that the function $\lambda_2[P(\alpha,\beta)]$, which is to be minimized in the lower bound, is unimodal in the area $(0,1] \times (0,\infty)$. The definition of a multivariate function being unimodal is nonstandard. We adopt the following definition: a real valued function is said to be unimodal if the inverse image of $(-\infty, y)$ is connected for all $y \in \mathbb{R}$.

Theorem 3 *If $\lambda_2[P(\alpha,\beta)]$ has a local minimum λ_0 at (α_0,β_0) with $(\alpha_0,\beta_0) \in (0,1) \times (0,\infty)$ and if λ_0 is a simple eigenvalue of $P(\alpha_0,\beta_0)$, then $\psi_{\mathbb{R}}(M) = \lambda_0^{-1}$.*

Theorem 4 *$\lambda_2[P(\alpha,\beta)]$ is unimodal in the area $\alpha \in (0,1], \beta > 0$.*

Numerical experience shows that the condition in Theorem 3 is often satisfied. Numerical examples also show that $\lambda_2[P(\alpha,\beta)]$ has rather weak convexity property: it is not quasi-convex. For fixed α_0, the univariate function $\lambda_2[P(\alpha_0,\beta)]$ is not unimodal in general. Nevertheless, it is enough that $\lambda_2[P(\alpha,\beta)]$ is unimodal. Many nonsmooth local optimization methods can be applied to find the global infimum of $\lambda_2[P(\alpha,\beta)]$. Numerical experience shows that the simplex method works quite well.

6 Examples

Example 1: Assume that a state space description of G is given by

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] =$$

$$\left[\begin{array}{cccc|cccccc} 79 & 20 & -30 & -20 & -0.5 & -0.35 & 0 & 0 & 0.3 & 0 \\ -41 & -12 & 17 & 13 & 0 & 0.15 & 0.2 & 0.4 & 0 & 0.2 \\ 167 & 40 & -60 & -38 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 33.5 & 9 & -14.5 & -11 & 0 & 0.3 & 0 & 0 & 0 & 0.2 \\ \hline 0.25 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0.1 & 0 & 0.1 & 0 & 0.2 & 0 & 0 \\ 0.4 & 0 & 0.5 & 0 & 0.2 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & -0.2 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The perturbation matrix Δ is 3 by 3. The computation result is:

$$r_{\mathbb{C}}(G_{22}) = 0.5006; \quad p_{\mathbb{C}}(G) = 0.1700$$
$$r_{\mathbb{R}}(G_{22}) = 1.0432; \quad p_{\mathbb{R}}(G) \geq 0.3998$$

Hence $pr_{\mathbb{C}}(G) = 0.1700$ and $pr_{\mathbb{R}}(G) \geq 0.3998$.

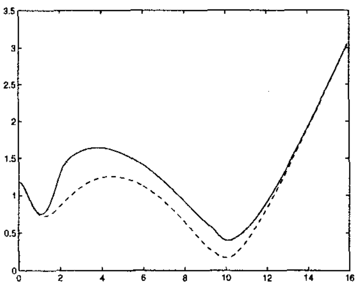

Figure 4: The solid line is a lower bound for $\psi_{\mathbb{R}}[M(\mathrm{j}\omega)]$ and the dased line is $\psi_{\mathbb{C}}[M(\mathrm{j}\omega)]$

In Figure 4, the dashed line is $\psi_{\mathbb{C}}[M(\mathrm{j}\omega)]$ and the solid line is the lower bound for $\psi_{\mathbb{R}}[M(\mathrm{j}\omega)]$. For this example, the condition in Theorem 3 is satisfied at all ω. Therefore, the solid line in Figure 4 is actually the plot of $\psi_{\mathbb{R}}[M(\mathrm{j}\omega)]$ and the real performance radius is exactly computed as $pr_{\mathbb{R}}(G) = 0.3998$

It is remarked in section 4 that another lower bound of $\psi_{\mathbb{R}}(M)$ is $\{\inf_{\alpha,\beta>0} \sigma_2 [P(\alpha,\beta)]\}^{-1}$. As a comparison, $\{\inf_{\alpha,\beta>0} \sigma_2 [P(\alpha,\beta)]\}^{-1}$, where $P(\alpha,\beta)$ is formed from $M(\mathrm{j}10.11)$, is computed and the value is 0.2491, whereas the lower bound given by Theorem 2 is 0.3998. This shows that the lower bound of $\psi_{\mathbb{R}}(M)$ given by $\{\inf_{\alpha,\beta>0} \sigma_2 [P(\alpha,\beta)]\}^{-1}$ is more conservative.

Example 2: It is of interest to know how tight the lower bound in Theorem 2 is. A sufficient condition is given in Theorem 3 for this lower bound to be the exact value of $\psi_{\mathbb{R}}(M)$. Hence the probability that the condition in Theorem 3 is satisfied gives an indication on how often the lower bound is tight. 4000 complex matrices $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ are randomly generated with $\bar{\sigma}(G_{11}) < 1$ and M is computed from G by using (2)-(5). The number of G matrices for which the condition of Theorem 3 is satisfied is 3661. This shows that the probability of the lower bound being tight is over 90%.

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