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CONNECTION OF MULTIPLICATIVE OR RELATIVE PERTURBATION IN COPRIME FACTORS AND GAP METRIC UNCERTAINTY

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Abstract. In this paper, it is shown that an uncertain system described by a certain $\mathcal{L}_\infty$ multiplicative or relative perturbation in its coprime factors is the same as the one described by a gap or ν-gap metric ball. Hence all of the stability robustness results for gap or ν-gap metric uncertainty carries over to this type of coprime factor perturbation. Uncertain systems described by $\mathcal{H}_\infty$ multiplicative or relative perturbations in coprime factors are also studied in this paper. Necessary and sufficient conditions for robust stability of a feedback system with coprime factors of both the plant and the controller subject to simultaneous $\mathcal{H}_\infty$ multiplicative or relative perturbations are obtained.

Keywords. Robust control, robust stability, uncertain linear systems, coprime factorization, gap metric.

1. INTRODUCTION

In studying controller reduction with multiplicative or relative error bound in coprime factors, a robust stability condition was derived in (Gu, 1995) for a feedback system whose plant is subject to $\mathcal{H}_\infty$ norm bounded multiplicative or relative perturbation in the coprime factors. The condition obtained is exactly the same as that for the gap metric or ν-gap metric uncertainty studied in (Georgiou and Smith, 1990; Vinnicombe, 1993). There thus appears to be an inherent connection between the two different types of uncertainties that is missed in (Gu, 1995). This paper aims to clarify the missing connection between the multiplicative or relative perturbation in coprime factors and the gap metric or ν-gap metric uncertainty. This is made possible by extending the $\mathcal{H}_\infty$ perturbation studied in (Gu, 1995) to certain $\mathcal{L}_\infty$ perturbations. With the connection established, it becomes easy to analize the robust stability of feedback systems with both the plant and the controller subject to simultaneous but independent multiplicative or relative perturbations in coprime factors.

The gap metric was introduced to control literature in (Zames and El-Sakkary, 1980). Its power and elegance have been demonstrated in subsequent studies, see, e.g., (Georgiou, 1988; Georgiou and Smith, 1990; Qiu and Davison, 1992a; Sefton and Ober, 1993). A new metric, called ν-gap metric, was invented in (Vinnicombe, 1993) and was shown to be advantageous over gap metric. The optimal robust stabilizing controller with re-

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spect to gap or ν-gap plant uncertainty has been shown to have some nice properties and has formed the basis of the loop shaping design method in (McFarlane and Glover, 1990). In the gap or ν-gap based robust control theory, normalized coprime factorizations have been playing a crucial role. In particular, one main result in this theory states that a set of systems in a gap metric ball is equal to a set of systems formed by $\mathcal{H}_\infty$ norm bounded additive perturbations on normalized coprime factors (Georgiou and Smith, 1990). In this paper, by connecting the gap or the ν-gap with perturbations on coprime factors that are not necessarily normalized, we provide more insight into this theory, make the theory more convenient and versatile, and pave the way for the extension of the theory to the cases when normalized coprime factorizations are not desirable, such as infinite dimensional systems (Georgiou and Smith, 1992; Treil, 1994), or to cases when normalized coprime factorizations are not possible, such as systems with Banach input output spaces (Qiu, 1995).

The notation used in this paper is standard. $\mathcal{L}_2^m$ denotes $\mathcal{C}^m$ valued Lebesgue 2-space defined on the imaginary axis. $\mathcal{H}_2^m$ denotes the $\mathcal{C}^m$ valued Hardy 2-space defined on the right half of the complex plane. $\mathcal{L}_\infty^{p\times m}$ and $\mathcal{H}_\infty^{p\times m}$ denote the $\mathcal{C}^{p\times m}$ valued Lebesgue and Hardy ∞-spaces respectively. $\mathcal{RL}_\infty^{p\times m}$ and $\mathcal{RH}_\infty^{p\times m}$ consist of real rational members of $\mathcal{L}_\infty^{p\times m}$ and $\mathcal{H}_\infty^{p\times m}$ respectively. Sometimes we simply write $\mathcal{L}_2$, $\mathcal{L}_\infty$, $\mathcal{RL}_\infty$, etc. if the dimensions are irrelevant or can be deduced from the context. For $G \in \mathcal{L}_\infty$, we write $G^\sim(s) = \overline{G(-\bar{s})}^T$. For $G \in \mathcal{RL}^{m\times m}$ with $G^{-1} \in \mathcal{RL}^{m\times m}$, the winding number of G, denoted by wno$G$, is defined to be the excess of the number of zeros over the number of poles of G in the open right half of the complex plain. For a matrix $A \in \mathcal{C}^{p\times m}$, the largest and the smallest singular values are denoted by $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$ respectively. In the following, a diagonal matrix $\mathrm{diag}(a_1, a_2, \cdots, a_n)$ is not necessarily square and its dimensions are deducible from the context.

2. UNCERTAINTY DESCRIPTIONS

The systems considered in this paper are assumed to be linear time-invariant and finite dimensional, so they can be identified with real rational transfer matrices. The set of such transfer matrices of size $p \times m$ is denoted by $\mathcal{P}^{p\times m}$. A system is said to be stable if its transfer matrix belongs to $\mathcal{H}_\infty$. The feedback system shown in Figure 1, or simply a pair (P, C), is said to be stable if the transfer matrix from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$, which is given by

$$\begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1},$$

is stable.

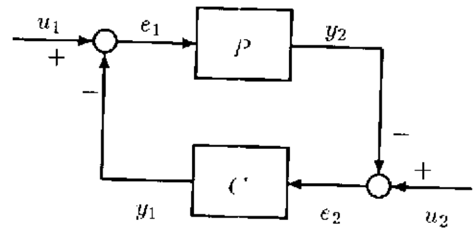

Fig. 1. Feedback Control System

Often in practical situations, the exact transfer matrix P of a physical plant is unknown but belongs to a neighborhood of a known nominal transfer matrix P_0. In this case, a feedback controller C_0 is designed based on the nominal plant P_0. However, the implemented controller C may not be exactly C_0 due to the need for controller reduction, finite wordlength effect, etc., but belongs to a neighborhood of C_0. Hence an important problem is whether or not the feedback system in Figure 1 remains stable when only (P_0, C_0) is known to be stable. This is referred to as robust stability. There are many ways to define neighborhoods of systems. In general, different definitions lead to different conditions for robust stability. Some of the most elegant results on robust control were obtained by using the gap metric and the ν-gap metric to describe uncertainty (Georgiou, 1988; Georgiou and Smith, 1990; Qiu and Davison, 1992*a*; Sefton and Ober, 1993; Vinnicombe 1993).

The gap metric and the ν-gap metric can be defined using Hilbert space geometric language. The definitions adopted below, which are actually computation formulas derived in (Sefton and Ober, 1993) and (Vinnicombe, 1993) respectively, appear to be more elementary for control researchers. It is well-known that each member of $\mathcal{P}^{p\times m}$ admits right and left coprime factorizations:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad \tilde{M}, \tilde{N}, M, N \in \mathcal{H}_\infty.$$

The coprime factorizations can be made normalized, i.e., satisfying

$$M^\sim M + N^\sim N = I \text{ and } \tilde{M}\tilde{M}^\sim + \tilde{N}\tilde{N}^\sim = I.$$

Let $P_1, P_2 \in \mathcal{P}^{p\times m}$ and $P_1 = N_1 M_1^{-1}$, $P_2 = N_1 M_1^{-1}$ be normalized coprime factorizations. The gap metric between P_1 and P_2 is defined as

$$\delta(P_1, P_2) = \inf_{Q, Q^{-1} \in \mathcal{RH}_\infty} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty. \quad (1)$$

The ν-gap metric between P_1 and P_2 is defined as

$$\delta_\nu(P_1, P_2) = \inf_{\substack{Q, Q^{-1} \in \mathcal{RL}_\infty \\ \text{wno} G = 0}} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty . \quad (2)$$

A gap ball and a ν-gap ball are then given by

$$\mathcal{B}(P_0, r) = \{P : \delta(P, P_0) < r\} \quad (3)$$
$$\mathcal{B}_\nu(P_0, r) = \{P : \delta_\nu(P, P_0) < r\} \quad (4)$$

which give open neighborhoods of P_0 and can be used as uncertain system descriptions.

Assume that $P_0 \in \mathcal{P}^{m \times p}$ and $P_0 = N_0 M_0^{-1}$ is a right coprime factorization. The following neighborhoods of P_0 are introduced in (Gu, 1995).

$$\mathcal{C}_{mul}(P_0, r) = \left\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \right. \\ \left. \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty < r \right\} \quad (5)$$

$$\mathcal{C}_{rel}(P_0, r) = \left\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \right. \\ \left. \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty < r \right\} \quad (6)$$

It is shown in (Gu, 1995) that if these neighborhoods are used describe the uncertainty of the plant of a feedback system, the necessary and sufficient conditions for the robust stability of the feedback system are the same and they are exactly the same as in the case when the uncertainty is described by gap metric or the ν-gap metric. This hints a connection between gap metric ball or ν-gap metric ball and the sets given in (5) – (6). In this paper, we will show that the gap ball are actually more closely related to the following enlarged sets.

$$\mathcal{C}'_{mul}(P_0, r) \\ = \left\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \in \mathcal{H}_\infty, \right. \\ \left. M \text{ and } N \text{ are coprime}, \Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty < r \right\} (7)$$

$$\mathcal{C}'_{rel}(P_0, r) \\ = \left\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \in \mathcal{H}_\infty, \right. \\ \left. M \text{ and } N \text{ are coprime}, \Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty < r \right\} . (8)$$

To connect to the ν-gap metric, we need to enlarge the sets further. First, we need to extend the concept of winding number to nonsquare transfer matrices. Let $G \in \mathcal{L}_\infty^{n \times m}$ have Smith-McMillan form

$$\text{diag}(\gamma_1, \gamma_2, \cdots, \gamma_{\min\{n,m\}}).$$

If all $\gamma_1, \gamma_2, \cdots, \gamma_{\min\{n,m\}}$ are nonzero and $\mathcal{L}_\infty$ invertible, then define

$$\text{wno} G = \text{wno}\, \gamma_1 \gamma_2 \cdots \gamma_{\min\{n,m\}}.$$

Now define,

$$\mathcal{C}''_{mul}(P_0, r) = \left\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \right. \\ \left. \text{wno} \begin{bmatrix} M \\ N \end{bmatrix} = 0, \Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty < r \right\} (9)$$

$$\mathcal{C}''_{rel}(P_0, r) = \left\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \right. \\ \left. \text{wno} \begin{bmatrix} M \\ N \end{bmatrix} = 0, \Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty < r \right\} (10)$$

Notice that the definitions (5) – (10) do not depend on the particular coprime factorization used in their definitions. To be absolute rigorous, we need to require in the sets (5) – (10) that M^{-1} exists. Also notice that the perturbation matrices Δ in (7) – (10) are not required to be stable. It is clear that

$$\mathcal{C}_{mul}(P, r) \subset \mathcal{C}'_{mul}(P, r) \subset \mathcal{C}''_{mul}(P, r) \quad (11)$$
$$\mathcal{C}_{rel}(P, r) \subset \mathcal{C}'_{rel}(P, r) \subset \mathcal{C}''_{rel}(P, r). \quad (12)$$

This is because matrices M and N in (5) – (6) are always right coprime and matrices M and N in (9) – (10) have $\text{wno} \begin{bmatrix} M \\ N \end{bmatrix} = 0$.

3. CONNECTIONS

In this section, two theorems are stated and proved which completely establish the connections between sets $\mathcal{B}(P_0, r)$, $\mathcal{C}'_{mul}(P_0, r)$, $\mathcal{C}'_{rel}(P_0, r)$ and between $\mathcal{B}_\nu(P_0, r)$, $\mathcal{C}''_{mul}(P_0, r)$, $\mathcal{C}''_{rel}(P_0, r)$.

Theorem 1 $\mathcal{B}(P_0, r) = \mathcal{C}'_{mul}(P_0, r) = \mathcal{C}'_{rel}(P_0, r)$.

Proof: The theorem is trivially true when $r > 1$, so we assume $r \leq 1$ in the following. We first prove $\mathcal{B}(P_0, r) = \mathcal{C}'_{mul}(P_0, r)$. Use a normalized coprime factorization $P_0 = N_0 M_0^{-1}$ in the definition of $\mathcal{C}'_{mul}(P, r)$. If $\|\Delta\|_\infty < r$, then $\left\|\Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}\right\|_\infty < r$. Since M and N are right coprime, there exists $Q \in \mathcal{RH}_\infty$ with $Q^{-1} \in \mathcal{RH}_\infty$ such that $\begin{bmatrix} MQ^{-1} \\ NQ^{-1} \end{bmatrix}$ is an isometry. Hence

$$\|\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} MQ^{-1} \\ NQ^{-1} \end{bmatrix} Q\|_\infty = \|\Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}\|_\infty < r$$

and it follows from definition (1) that $\mathcal{C}'_{mul}(P_0, r) \subset \mathcal{B}(P_0, r)$. Now assume $P \in \mathcal{B}(P_0, r)$. Then P has normalized coprime factorization NM^{-1} such that $\|\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q\| < r$ for some $Q \in \mathcal{RH}_\infty$ with $Q^{-1} \in \mathcal{RH}_\infty$. Let

$$\Delta = -\left(\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q\right) \begin{bmatrix} M_0^\sim & N_0^\sim \end{bmatrix}.$$

Then $\begin{bmatrix} MQ \\ NQ \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}$ and $\|\Delta\|_\infty = \|\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q\|_\infty < r$. By using this Δ, we see that $P \in \mathcal{C}'_{mul}(P_0, r)$. This proves $\mathcal{B}(P_0, r) \subset \mathcal{C}'_{mul}(P_0, r)$.

By the definitions (7) (8), it is obvious that

$$P \in \mathcal{C}'_{rel}(P_0, r) \Leftrightarrow P_0 \in \mathcal{C}'_{mul}(P, r).$$

Since

$$P \in \mathcal{B}(P_0, r) \Leftrightarrow P_0 \in \mathcal{B}(P, r) \Leftrightarrow P_0 \in \mathcal{C}'_{mul}(P, r),$$

it follows $\mathcal{B}(P_0, r) = \mathcal{C}'_{rel}(P_0, r)$

Theorem 2 $\mathcal{B}_\nu(P_0, r) = \mathcal{C}''_{mul}(P_0, r) = \mathcal{C}''_{rel}(P_0, r)$.

Proof: Again the case $r > 1$ is trivial, so we assume $r \leq 1$. We first prove $\mathcal{B}_\nu(P_0, r) = \mathcal{C}''_{mul}(P_0, r)$. Use a normalized coprime factorization $P_0 = N_0 M_0^{-1}$ in the definition of $\mathcal{C}''_{mul}(P, r)$. If $\|\Delta\|_\infty < r$, then $\|\Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}\|_\infty < r$. Since $\text{wno}\begin{bmatrix} M \\ N \end{bmatrix} = 0$, there exists $Q \in \mathcal{RL}_\infty$ with $Q^{-1} \in \mathcal{RL}_\infty$ and $\text{wno}Q = 0$ such that MQ^{-1} and NQ^{-1} belong to $\mathcal{RH}_\infty$ and are right coprime. Hence

$$\|\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} MQ^{-1} \\ NQ^{-1} \end{bmatrix} Q\| = \|\Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}\| < r$$

and it follows from (2) that $P \in \mathcal{B}_\nu(P_0, r)$. This shows $\mathcal{C}''_{mul}(P_0, r) \subset \mathcal{B}_\nu(P_0, r)$. Now assume $P \in \mathcal{B}_\nu(P_0, r)$. Then P has normalized coprime factorization NM^{-1} such that $\|\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q\| < r$ for some $Q \in \mathcal{RL}_\infty$ with $Q^{-1} \in \mathcal{RL}_\infty$ and $\text{wno}Q = 0$. Let

$$\Delta = -\left(\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q\right) \begin{bmatrix} M_0^\sim & N_0^\sim \end{bmatrix}.$$

Then $\begin{bmatrix} MQ \\ NQ \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}$, $\text{wno}\begin{bmatrix} MQ \\ NQ \end{bmatrix} = 0$, and $\|\Delta\|_\infty = \|\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q\|_\infty < r$. By using this Δ, we see that $P \in \mathcal{C}'_{mul}(P_0, r)$. This proves $\mathcal{B}(P_0, r) \subset \mathcal{C}'_{mul}(P_0, r)$.

Note that if $\begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}$ and $\text{wno}\begin{bmatrix} M \\ N \end{bmatrix} = 0$, then there exists $Q \in \mathcal{RL}_\infty$ with $Q^{-1} \in \mathcal{RL}_\infty$ and $\text{wno}Q = 0$ such that MQ and NQ are right coprime. Hence $\begin{bmatrix} M_0 Q \\ N_0 Q \end{bmatrix} = (I + \Delta) \begin{bmatrix} MQ \\ NQ \end{bmatrix}$ and $\text{wno}\begin{bmatrix} M_0 Q \\ N_0 Q \end{bmatrix} = 0$. This shows that

$$P \in \mathcal{C}''_{rel}(P_0, r) \Leftrightarrow P_0 \in \mathcal{C}''_{mul}(P, r).$$

Since

$$P \in \mathcal{B}_\nu(P_0, r) \Leftrightarrow P_0 \in \mathcal{B}_\nu(P, r) \Leftrightarrow P_0 \in \mathcal{C}''_{mul}(P, r),$$

it follows $\mathcal{B}_\nu(P_0, r) = \mathcal{C}''_{rel}(P_0, r)$.

In the remaining part of this section, we take a close look at $\mathcal{C}_{mul}(P_0, r)$ and $\mathcal{C}_{rel}(P_0, r)$.

For $P_0 \in \mathcal{P}^{p \times m}$, let $P_0 = \tilde{M}^{-1} \tilde{N}$ be a left coprime factorization. Define

$$\begin{aligned} &\tilde{\mathcal{C}}_{mul}(P_0, r) \\ &= \{P = \tilde{M}^{-1} \tilde{N} : \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix} (I + \Delta) \\ &\qquad \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty < r\} \end{aligned} \tag{13}$$

$$\begin{aligned} &\tilde{\mathcal{C}}_{rel}(P_0, r) \\ &= \{P = \tilde{M}^{-1} \tilde{N} : \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix} (I + \Delta)^{-1} \\ &\qquad \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty < r\}. \end{aligned} \tag{14}$$

Lemma 3

$$\mathcal{C}_{mul}(P_0, r) = \tilde{\mathcal{C}}_{rel}(P_0, r)$$
$$\mathcal{C}_{rel}(P_0, r) = \tilde{\mathcal{C}}_{mul}(P_0, r).$$

Proof: Let $P \in \mathcal{C}_{mul}(P_0, r)$. Then P has a right coprime factorization NM^{-1} such that $\begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}$, where $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty < r$. Define

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix} \left(I + \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \Delta \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\right)^{-1}.$$

Then $\tilde{M}$ and $\tilde{N}$ are coprime since $(I + \Delta)^{-1}$ is a unit in $\mathcal{RH}_\infty$. Furthermore,

$$\begin{bmatrix} -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{M}_0 & \tilde{N}_0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} (I+\Delta)^{-1} \begin{bmatrix} M \\ N \end{bmatrix}$$
$$= \begin{bmatrix} -\tilde{N}_0 & \tilde{M}_0 \end{bmatrix} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}$$
$$= 0.$$

Consequently, $\tilde{M}^{-1}\tilde{N} = NM^{-1} = P$, which implies $P \in \tilde{\mathcal{C}}_{rel}(P_0, r)$. Therefore, $\mathcal{C}_{mul}(P_0, r) \subset \tilde{\mathcal{C}}_{rel}(P_0, r)$. Reversing the procedure above shows $\tilde{\mathcal{C}}_{rel}(P_0, r) \subset \mathcal{C}_{mul}(P_0, r)$

The equality $\mathcal{C}_{rel}(P_0, r) = \tilde{\mathcal{C}}_{mul}(P_0, r)$ can be similarly shown.

It is not clear if $\mathcal{C}_{mul}(P_0, r) = \mathcal{C}_{rel}(P_0, r)$ holds in general. However, it holds trivially when P_0 is a scalar (SISO) system.

It has been shown in (Vinnicombe, 1993) that the containment $\mathcal{B}(P_0, r) \subset \mathcal{B}_\nu(P_0, r)$ is in general strict. This, together with the following example, shows that the containments in (11)-(12) are all strict in general.

Example: Let $P_0(s) = \frac{s-1}{s+1}$ and $P(s) = \frac{2s-1}{s+1}$, then a right coprime factorization of P_0 is $P_0 = N_0 M_0^{-1}$ where $\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{s-1}{s+1} \end{bmatrix}$ and all right coprime factorizations of P is given by $P = NM^{-1}$ where $\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2s-1}{s+1} \end{bmatrix} Q$ and Q is a unit in $\mathcal{RH}_\infty$. Let $\Delta \in \mathcal{RH}_\infty$ satisfies

$$\begin{bmatrix} M \\ N \end{bmatrix} = (I+\Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}.$$

Evaluating this equation at $s = 1$, we get

$$\Delta(1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} Q(1) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence,

$$\|\Delta(1)\| \geq \min_{Q(1)} \left\| \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} Q(1) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = \frac{1}{\sqrt{5}}.$$

This shows that $P \in \mathcal{C}_{mul}(P_0, r) = \mathcal{C}_{rel}(P_0, r)$ only if $r > \frac{1}{\sqrt{5}}$. However, it is computed in (Georgiou and Smith, 1990) and (Vinnicombe, 1993) that $\delta(P, P_0) = \frac{1}{3}$ and $\delta_\nu = \frac{1}{\sqrt{10}}$.

Finally, we would like to remark that there may be a close connection between $\mathcal{C}_{mul}(P_0, r)$ or $\mathcal{C}_{rel}(P_0, r)$ and a pointwise gap metric ball defined in (Qiu and Davison, 1992*b*).

4. ROBUST STABILITY

We are interested in the robust stability conditions for the feedback system shown in Figure 1 when the plant and the controller are subject to simultaneous perturbations of the form described in (5) - (10). Let

$$b_{P,C} = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} I & C \end{bmatrix} \right\|_\infty^{-1}.$$

The following theorems are due to (Qiu and Davison, 1992*a*; Vinnicombe, 1993).

Theorem 4 *Let $P_0 \in \mathcal{P}^{p\times m}$ and $C_0 \in \mathcal{P}^{m\times p}$, and (P_0, C_0) is stable. Then (P, C) is stable for all $P \in \mathcal{B}(P_0, r_1)$ and $C \in \mathcal{B}(C_0, r_2)$ if and only if*

$$\arcsin r_1 + \arcsin r_2 \leq \arcsin b_{P_0, C_0}.$$

Theorem 5 *Let $P_0 \in \mathcal{P}^{p\times m}$ and $C_0 \in \mathcal{P}^{m\times p}$, and (P_0, C_0) is stable. Then (P, C) is stable for all $P \in \mathcal{B}_\nu(P_0, r_1)$ and $C \in \mathcal{B}_\nu(C_0, r_2)$ if and only if*

$$\arcsin r_1 + \arcsin r_2 \leq \arcsin b_{P_0, C_0}.$$

Since $\mathcal{C}_{mul}(P_0, r) \subset \mathcal{B}(P_0, r)$ and $\mathcal{C}_{rel}(P_0, r) \subset \mathcal{B}(P_0, r)$ and the containment is strict in general, one wonders if the condition in Theorem 4 or 5 can be relaxed if P belongs to $\mathcal{C}_{mul}(P_0, r_1)$ or $\mathcal{C}_{rel}(P_0, r_1)$ and C belongs to $\mathcal{C}_{mul}(C_0, r_2)$ or $\mathcal{C}_{rel}(C_0, r_2)$. The answer is negative.

Theorem 6 *Let $P_0 \in \mathcal{P}^{p\times m}$ and $C_0 \in \mathcal{P}^{m\times p}$, and (P_0, C_0) is stable. Then (P, C) is stable for all $P \in \mathcal{C}_{mul}(P_0, r_1)$ and $C \in \mathcal{C}_{rel}(C_0, r_2)$ if and only if*

$$\arcsin r_1 + \arcsin r_2 \leq \arcsin b_{P_0, C_0}.$$

Proof: The sufficiency follows from Theorem 4 or 5. It remains to show the necessity. Assume

$$\arcsin r_1 + \arcsin r_2 > \arcsin b_{P_0, C_0}.$$

We need to construct $P \in \mathcal{C}_{mul}(P_0, r_1)$, $C \in \mathcal{C}_{rel}(P_0, r_2)$ such that (P, C) is unstable. Let $\theta = \arcsin(b_{P_0, C_0})$. Then there exist $\theta_1 < \arcsin(r_1)$ and $\theta_2 < \arcsin(r_2)$ such that $\theta_1 + \theta_2 = \theta$.

Let $P_0 = N_0 M_0^{-1}$ be a normalized right coprime factorization and $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$ be a normalized left coprime factorization. Then

$$b_{P_0, C_0} = \inf_{\omega \in [0,\infty]} \underline{\sigma}[\tilde{V}_0(j\omega) M_0(j\omega) - \tilde{U}_0(j\omega) N_0(j\omega)].$$

There must exist $\bar{\omega} \in [0, \infty]$ such that

$$\underline{\sigma}[\tilde{V}_0(j\bar{\omega}) M_0(j\bar{\omega}) - \tilde{U}_0(j\bar{\omega}) N_0(j\bar{\omega})] = b_{P_0, C_0}.$$

By (Stewart and Sun, 1990, Theorem 1.5.2), there exist unitary matrices X, Y, and Z such that

$$\begin{bmatrix} M_0(j\bar{\omega}) \\ N_0(j\bar{\omega}) \end{bmatrix} = X \begin{bmatrix} I \\ 0 \end{bmatrix} Y^*$$

and

$$\left[\tilde{V}_0(j\omega) \;\; -\tilde{U}_0(j\omega) \right] = Z \left[C \;\; S \right] X^*$$

where

$$C = \mathrm{diag}(c_1, c_2, \ldots, c_m) \in \mathcal{R}^{m\times m},$$
$$S = \mathrm{diag}\left(\sqrt{1-c_1^2}, \sqrt{1-c_2^2}, \ldots, \sqrt{1-c_{\min\{p,m\}}^2}\right) \in \mathcal{R}^{m\times p},$$

and $0 \le c_1 \le c_2 \le \cdots \le c_m$. This implies $c_1 = b_{P_0,C_0} = \sin\theta$. Define

$$\overline{\Delta}_1 = X^* \begin{bmatrix} \mathrm{diag}(-\sin\theta_1 \sin\theta_1, 0, \ldots, 0) & 0 \\ \mathrm{diag}(-\cos\theta_1 \sin\theta_1, 0, \ldots, 0) & 0 \end{bmatrix} X$$

and

$$\overline{\Delta}_2 = X^* \begin{bmatrix} 0 & \mathrm{diag}(-\sin\theta \sin\theta_2, 0, \ldots, 0) \\ 0 & \mathrm{diag}(-\cos\theta \sin\theta_2, 0, \ldots 0) \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\sin\theta_1, 0, \ldots, 0) & \mathrm{diag}(\cos\theta_1, 0, \ldots, 0) \\ \mathrm{diag}(\cos\theta_1, 0, \ldots, 0) & \mathrm{diag}(-\sin\theta_1, 0, \ldots, 0) \end{bmatrix} X.$$

Then it is straightforward to verify that $\overline{\sigma}(\overline{\Delta}_1) = \sin\theta_1$, $\overline{\sigma}(\overline{\Delta}_2) = \sin\theta_2$, and

$$\left[\tilde{V}(j\omega) \;\; -\tilde{U}(j\omega) \right](I + \overline{\Delta}_2)(I + \overline{\Delta}_1) \begin{bmatrix} M_0(j\omega) \\ N_0(j\omega) \end{bmatrix}$$

is singular. Notice that $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are of rank one. Standard techniques exist to contruct $\Delta_1, \Delta_2 \in \mathcal{RH}_\infty$ such that $\Delta_1(j\omega) = \overline{\Delta}_1$, $\Delta_2(j\omega) = \overline{\Delta}_2$, $\|\Delta_1\|_\infty = \overline{\sigma}(\overline{\Delta}_1)$, and $\|\Delta_2\|_\infty = \overline{\sigma}(\overline{\Delta}_2)$ (Vidyasagar, 1985). Now let

$$\begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta_1) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}$$
$$\left[\tilde{V} \;\; \tilde{U} \right] = \left[\tilde{V}_0 \;\; \tilde{U}_0 \right] \left(I + \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Delta_2 \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \right).$$

Then it follows that $P = NM^{-1} \in \mathcal{C}_{mul}(P_0, r_1)$, $C = \tilde{V}^{-1}\tilde{U} \in \tilde{\mathcal{C}}_{mul}(C_0, r_2) = \mathcal{C}_{rel}(C_0, r_2)$, and (P, C) is unstable.

5. CONCLUSION

The main contribution of this paper is the establishment of the connection between the gap or ν-gap metric uncertainty and the perturbation on coprime factors that are not necessarily normalized. In this paper, we have only dealt with right coprime factorizations. A parallel analysis can be carried out for left coprime factorizations. The connection between certain multiplicative or relative perturbation on left coprime factors and the T-gap metric uncertainty (Georgiou and Smith, 1990) can be established.

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