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Brief Paper

Connection of Multiplicative/Relative Perturbation in Coprime Factors and Gap Metric Uncertainty*

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Key Words—Robust stability; uncertain linear systems; coprime factorization; gap metric.

Abstract—In this paper, it is shown that a linear uncertain system described by a certain $\mathscr{L}_\infty$ multiplicative or relative perturbation in its coprime factors that are not necessarily normalized, is the same as the one described by a gap or ν-gap metric ball. Hence all the stability robustness results for gap or ν-gap metric uncertainty carry over to this type of coprime factor perturbation. Uncertain systems described by $\mathscr{H}_\infty$ multiplicative or relative perturbations in coprime factors are also studied in this paper. Necessary and sufficient conditions for robust stability of a feedback system with coprime factors of both the plant and the controller subject to simultaneous $\mathscr{H}_\infty$ multiplicative or relative perturbations are obtained. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

In studying controller reduction with multiplicative or relative error bound in coprime factors, a robust stability condition was derived in Gu (1995) for a feedback system whose plant is subject to $\mathscr{H}_\infty$ norm bounded multiplicative or relative perturbation in the coprime factors that are not necessarily normalized. The condition obtained is exactly the same as that for the gap metric or ν-gap metric unceertainty studied in Georgiou and Smith (1990) and Vinnicombe (1993). There thus appears to be an inherent connection between the two different types of uncertainties that is missed in Gu (1995). This paper aims to clarify the missing connection between the multiplicative or relative perturbation in coprime factors and the gap metric or ν-gap metric uncertainty. This is made possible by extending the $\mathscr{H}_\infty$ perturbation studied in (Gu, 1995) to certain $\mathscr{L}_\infty$ perturbations. With the connection established, it becomes easy to analyse the robust stablity of feedback systems with both the plant and the controller subject to simultaneous but independent multiplicative or relative perturbations in coprime factors.

The gap metric was introduced to control literature in Zames and El-Sakkary (1980). Its power and elegance have been demonstrated in subsequent studies, see, e.g., Georgiou (1988), Georgiou and Smith (1990), Qiu and Davison (1992) and Sefton and Ober (1993). A new metric, called ν-gap metric, was invented in Vinnicombe (1993) and was shown to be advantageous over the gap metric. The optimal robust stabilizing controller with respect to gap or ν-gap plant uncertainty has been shown to have some nice properties and has formed the basis of the loop shaping design method in McFarlane and Golver (1990). In the gap or ν-gap based robust control theory, normalized coprime factorizations have been playing a crucial role. In particular, one main result in this theory states that a set of systems in a gap metric ball is equal to a set of systems formed by $\mathscr{H}_\infty$ norm bounded additive perturbations on normalized coprime factors (Georgiou and Smith, 1990). In this paper, by connecting the gap or the ν-gap with perturbations on coprime factors that are not necessarily normalized, we provide more insight into this theory, make the theory more convenient and versatile, and pave the way for the extension of the theory to the cases when normalized coprime factorizations are not desirable, such as infinite dimensional systems (Georgiou and Smith, 1992; Treil, 1994), or to cases when normalized coprime factorizations are not possible, such as systems with Banach input output spaces (Qiu, 1995).

The notation used in this paper is standard. The symbol $\mathscr{L}_2^m$ denotes $\mathscr{C}^m$ valued Lebesgue 2-space defined on the imaginary axis. $\mathscr{H}_2^m$ denotes the $\mathscr{C}^m$ valued Hardy 2-space defined on the right half of the complex plane. $\mathscr{L}_\infty^{p\times m}$ and $\mathscr{H}_\infty^{p\times m}$ denote the $\mathscr{C}^{p\times m}$ valued Lebesgue and Hardy ∞-spaces respectively. $\mathscr{R}\mathscr{L}_\infty^{p\times m}$ and $\mathscr{R}\mathscr{H}_\infty^{p\times m}$ consist of real rational members of $\mathscr{L}_\infty^{p\times m}$ and $\mathscr{H}_\infty^{p\times m}$, respectively. Sometimes we simply write $\mathscr{L}_2$, $\mathscr{L}_\infty$, $\mathscr{R}\mathscr{L}_\infty$, etc. if the dimensions are irrelevant or can be deduced from the context. For $G\in\mathscr{L}_\infty$, we write $G^\sim(s)=\overline{G(-\bar{s})}^{\mathrm{T}}$. For $G\in\mathscr{R}\mathscr{L}_\infty^{m\times m}$ with $G^{-1}\in\mathscr{R}\mathscr{L}_\infty^{m\times m}$, the winding number of G, denoted by wno$G$, is defined to be the excess of the number of zeros over the number of poles of G in the open right half of the complex plane. For a matrix $A\in\mathscr{C}^{p\times m}$, the largest and the smallest singular values are denoted by $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$, respectively. In the following, a diagonal matrix $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ is not necessarily square and its dimensions are deducible from the context.

2. *Uncertainty descriptions*

The systems considered in this paper are assumed to be linear time-invariant and finite dimensional. Thus they can be identified with real rational transfer matrices. The set of such transfer matrices of size $p\times m$ is denoted by $\mathscr{P}^{p\times m}$. A system is said to be stable if its transfer matrix belongs to $\mathscr{H}_\infty$. The feedback system shown in Fig. 1, or simply a pair (P, C), is said to be stable if the transfer matrix from $\left[\begin{smallmatrix}u_1\\u_2\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}e_1\\e_2\end{smallmatrix}\right]$, which is given by $\left[\begin{smallmatrix}I & C\\P & I\end{smallmatrix}\right]^{-1}$, is stable.

Often in practical situations, the exact transfer matrix P of a physical plant is unknown but belongs to a neighborhood of a known nominal transfer matrix P_0. In this case, a feedback controller C_0 is designed based on the nominal plant P_0. However, the implemented controller C may not be exactly C_0 due to the need for controller reduction, finite wordlength effect, etc. but belongs to a neighborhood of C_0. Hence an important problem is whether or not the feedback system in Fig. 1 remains stable when only (P_0, C_0) is known to be stable. This is referred to as robust stability. There are many ways to define neighborhoods of systems. In general, different definitions lead to different conditions for robust stability. Some of the most elegant results on robust control were obtained by using the gap metric and the ν-gap metric to describe uncertainty (Georgiou, 1988; Georgiou and Smith, 1990; Qiu and Davison, 1992; Sefton and Ober, 1993; Vinnicombe, 1993).

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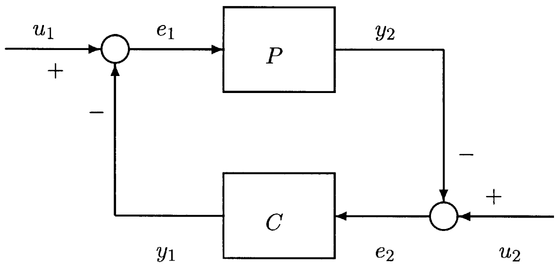

Fig. 1. Feedback control systems.

The gap metric and the ν-gap metric can be defined using Hilbert space geometric language. The definitions adopted below, which are actually computation formulas derived in Sefton and Ober (1993) and Vinnicombe (1993), respectively, appear to be more elementary for control researchers. It is well-known that each member of $\mathscr{P}^{p\times m}$ admits right and left coprime factorizations:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad \tilde{M},\tilde{N},M,N \in \mathscr{RH}_\infty.$$

The coprime factorizations can be made normalized, i.e., satisfying

$$M^{\sim}M + N^{\sim}N = I \quad \text{and} \quad \tilde{M}\tilde{M}^{\sim} + \tilde{N}\tilde{N}^{\sim} = I.$$

Let $P_1,P_2 \in \mathscr{P}^{p\times m}$ and $P_1 = N_1M_1^{-1}$, $P_2 = N_1M_1^{-1}$ be normalized coprime factorizations. The gap metric between P_1 and P_2 is defined as

$$\delta(P_1,P_2) = \inf_{Q,Q^{-1}\in\mathscr{RH}_\infty}\left\|\begin{bmatrix}M_1\\N_1\end{bmatrix} - \begin{bmatrix}M_2\\N_2\end{bmatrix}Q\right\|_\infty. \tag{1}$$

The ν-gap metric between P_1 and P_2 is defined as

$$\delta_\nu(P_1,P_2) = \inf_{\substack{Q,Q^{-1}\in\mathscr{RL}_\infty\\ \mathrm{wno}Q=0}}\left\|\begin{bmatrix}M_1\\N_1\end{bmatrix} - \begin{bmatrix}M_2\\N_2\end{bmatrix}Q\right\|_\infty. \tag{2}$$

A gap ball and a ν-gap ball are then given by

$$\mathscr{B}(P_0,r) = \{P : \delta(P,P_0) < r\}, \tag{3}$$

$$\mathscr{B}_\nu(P_0,r) = \{P : \delta_\nu(P,P_0) < r\}, \tag{4}$$

which give open neighborhoods of P_0 and can be used as uncertain system descriptions.

Assume that $P_0 \in \mathscr{P}^{m\times p}$ and $P_0 = N_0M_0^{-1}$ is a right coprime factorization that may not be normalized. The following neighborhoods of P_0 are introduced in (Gu, 1995):

$$\mathscr{C}_{\rm mul}(P_0,r) = \left\{P = NM^{-1}: \begin{bmatrix}M\\N\end{bmatrix} = (I+\Delta)\begin{bmatrix}M_0\\N_0\end{bmatrix}, \Delta\in\mathscr{RH}_\infty, \|\Delta\|_\infty < r\right\}, \tag{5}$$

$$\mathscr{C}_{\rm rel}(P_0,r) = \left\{P = NM^{-1}: \begin{bmatrix}M\\N\end{bmatrix} = (I+\Delta)^{-1}\begin{bmatrix}M_0\\N_0\end{bmatrix}, \Delta\in\mathscr{RH}_\infty, \|\Delta\|_\infty < r\right\}. \tag{6}$$

It is shown in Gu (1995) that if these neighborhoods are used to describe the uncertainty of the plant for a feedback system, the necessary and sufficient conditions for the robust stability of the feedback system are exactly the same as in the case when the uncertainty is described by gap metric or the ν-gap metric. This hints a connection between gap metric ball or ν-gap metric ball and the sets given in equations (5) and (6). In this paper, we will show that the gap ball is actually more closely related to the following enlarged sets:

$$\mathscr{C}'_{\rm mul}(P_0,r) = \left\{P = NM^{-1}: \begin{bmatrix}M\\N\end{bmatrix} = (I+\Delta)\begin{bmatrix}M_0\\N_0\end{bmatrix} \in \mathscr{RH}_\infty,\right.$$
$$\left. M \text{ and } N \text{ are coprime}, \Delta\in\mathscr{RL}_\infty, \|\Delta\|_\infty < r\right\}, \tag{7}$$

$$\mathscr{C}'_{\rm rel}(P_0,r) = \left\{P = NM^{-1}: \begin{bmatrix}M\\N\end{bmatrix} = (I+\Delta)^{-1}\begin{bmatrix}M_0\\N_0\end{bmatrix} \in \mathscr{RH}_\infty,\right.$$
$$\left. M \text{ and } N \text{ are coprime}, \Delta\in\mathscr{RL}_\infty, \|\Delta\|_\infty < r\right\}. \tag{8}$$

To connect to the ν-gap metric, we need to enlarge the sets further. First, we need to extend the concept of winding number to nonsquare transfer matrices. Let $G \in \mathscr{RL}_\infty^{n\times m}$ have Smith–McMillan form

$$\mathrm{diag}(\gamma_1,\gamma_2,\ \dots\ ,\gamma_{\min\{n,m\}}).$$

If all $\gamma_1,\gamma_2,\ \dots\ ,\gamma_{\min\{n,m\}}$ are nonzero and $\mathscr{L}_\infty$ invertible, then the winding number of G is defined as

$$\mathrm{wno}G = \mathrm{wno}\,\gamma_1\gamma_2\cdots\gamma_{\min\{n,m\}}.$$

The sets that are connected to ν-gap ball are now defined by

$$\mathscr{C}''_{\rm mul}(P_0,r) = \left\{P = NM^{-1}: \begin{bmatrix}M\\N\end{bmatrix} = (I+\Delta)\begin{bmatrix}M_0\\N_0\end{bmatrix},\right.$$
$$\left. \mathrm{wno}\begin{bmatrix}M\\N\end{bmatrix} = 0, \Delta\in\mathscr{RL}_\infty, \|\Delta\|_\infty < r\right\}, \tag{9}$$

$$\mathscr{C}''_{\rm rel}(P_0,r) = \left\{P = NM^{-1}: \begin{bmatrix}M\\N\end{bmatrix} = (I+\Delta)^{-1}\begin{bmatrix}M_0\\N_0\end{bmatrix},\right.$$
$$\left. \mathrm{wno}\begin{bmatrix}M\\N\end{bmatrix} = 0, \Delta\in\mathscr{RL}_\infty, \|\Delta\|_\infty < r\right\}. \tag{10}$$

Notice that the definitions (5)–(10) do not depend on the particular coprime factorization used in their definitions. To be absolute rigorous, we need to require in the sets (5)–(10) that M^{-1} exists. Also notice that the perturbation matrices Δ in equations (7)–(10) are not required to be stable. It is clear that

$$\mathscr{C}_{\rm mul}(P,r) \subset \mathscr{C}'_{\rm mul}(P,r) \subset \mathscr{C}''_{\rm mul}(P,r), \tag{11}$$

$$\mathscr{C}_{\rm rel}(P,r) \subset \mathscr{C}'_{\rm rel}(P,r) \subset \mathscr{C}''_{\rm rel}(P,r). \tag{12}$$

This is because matrices M and N in (5)–(6) are always right coprime, whereas matrices M and N in (9)–(10) are required to satisfy $\mathrm{wno}\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right] = 0$.

3. *Connections*

In this section, two theorems are stated which completely establish the connections between sets $\mathscr{B}(P_0,r)$, $\mathscr{C}'_{\rm mul}(P,r)$, $\mathscr{C}'_{\rm rel}(P,r)$, and between $\mathscr{B}_\nu(P_0,r)$, $\mathscr{C}''_{\rm mul}(P_0,r)$, $\mathscr{C}''_{\rm rel}(P_0,r)$.

Theorem 1. $\mathscr{B}(P_0,r) = \mathscr{C}'_{\rm mul}(P,r) = \mathscr{C}'_{\rm rel}(P,r)$.

Proof. The theorem is trivially true when $r > 1$. Thus only the case $r \le 1$ will be considered in the following. We first prove $\mathscr{B}(P_0,r) = \mathscr{C}'_{\rm mul}(P_0,r)$. Since $P_0 \in \mathscr{P}^{m\times p}$, normalized coprime factoriztaion $P_0 = N_0M_0^{-1}$ can be assumed in the definition of $\mathscr{C}'_{\rm mul}(P,r)$. Suppose that $P \in \mathscr{C}'_{\rm mul}(P_0,r)$. Then there exists a right coprime factorization $P = NM^{-1}$ such that

$$\begin{bmatrix}M\\N\end{bmatrix} = \begin{bmatrix}M_0\\N_0\end{bmatrix} + \begin{bmatrix}\Delta M\\\Delta N\end{bmatrix},$$

$$\begin{bmatrix}\Delta M\\\Delta N\end{bmatrix} = \Delta\begin{bmatrix}M_0\\N_0\end{bmatrix} \in \mathscr{RH}_\infty,$$

$$\left\|\begin{bmatrix}\Delta M\\\Delta N\end{bmatrix}\right\|_\infty \le \|\Delta\|_\infty < r.$$

Since M and N are right coprime, there exists $Q \in \mathscr{RH}_\infty$ with $Q^{-1} \in \mathscr{RH}_\infty$ such that $\left[\begin{smallmatrix} MQ^{-1} \\ NQ^{-1} \end{smallmatrix}\right]$ is an isometry. Hence

$$\left\| \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} MQ^{-1} \\ NQ^{-1} \end{bmatrix} Q \right\|_\infty = \left\| \Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \right\|_\infty < r.$$

It follows from the definition (1) that $P \in \mathscr{B}(P_0, r)$ that concludes $\mathscr{C}'_{\text{mul}}(P_0,r) \subset \mathscr{B}(P_0,r)$. Now assume $P \in \mathscr{B}(P_0,r)$. Then there exist normalized coprime factorization $P = NM^{-1}$ and some Q such that

$$\left\| \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q \right\|_\infty < r, \quad Q, Q^{-1} \in \mathscr{RH}_\infty.$$

Define

$$\Delta = -\left(\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q \right) [M_0^\sim \quad N_0^\sim].$$

Then

$$\begin{bmatrix} MQ \\ NQ \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \quad \|\Delta\|_\infty = \left\| \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q \right\|_\infty < r.$$

It follows from the definition (7) that $P \in \mathscr{C}'_{\text{mul}}(P_0, r)$. This proves $\mathscr{B}(P_0,r) \subset P \in \mathscr{C}'_{\text{mul}}(P_0,r)$.
By the definitions (7) and (8), it is obvious that

$$P \in \mathscr{C}'_{\text{rel}}(P_0,r) \Leftrightarrow P_0 \in \mathscr{C}'_{\text{mul}}(P,r).$$

Since

$$P \in \mathscr{B}(P_0,r) \Leftrightarrow P_0 \in \mathscr{B}(P,r) \Leftrightarrow P_0 \in \mathscr{C}'_{\text{mul}}(P,r),$$

it follows that $\mathscr{B}(P_0,r) = \mathscr{C}'_{\text{rel}}(P_0,r)$. □

Theorem 2. $\mathscr{B}_v(P_0,r) = \mathscr{C}''_{\text{mul}}(P_0,r) = \mathscr{C}''_{\text{rel}}(P_0,r)$.

Proof. Again the case $r > 1$ is trivial. Thus $r \leq 1$ is assumed. We first prove $\mathscr{B}_v(P_0,r) = \mathscr{C}''_{\text{mul}}(P_0,r)$. Suppose that $P \in \mathscr{C}''_{\text{mul}}(P_0,r)$. Using a normalized coprime factorization $P_0 = N_0 M_0^{-1}$ in the definition of $\mathscr{C}''_{\text{mul}}(P,r)$ yields

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} + \Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \in \mathscr{RH}_\infty,$$

$$\text{wno} \begin{bmatrix} M \\ N \end{bmatrix} = 0, \qquad \left\| \Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \right\|_\infty \leq \|\Delta\|_\infty < r,$$

for some factorization $P = NM^{-1}$ that may not be coprime. Hence there exists some Q with $Q, Q^{-1} \in \mathscr{RL}_\infty$ and $\text{wno}Q = 0$ such that MQ^{-1} and NQ^{-1} belong to $\mathscr{RH}_\infty$ and are right coprime. Hence

$$\left\| \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} MQ^{-1} \\ NQ^{-1} \end{bmatrix} Q \right\|_\infty = \left\| \Delta \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} \right\|_\infty < r.$$

It follows from equation (2) that $P \in \mathscr{B}_v(P_0,r)$ that concludes $\mathscr{C}''_{\text{mul}}(P_0,r) \subset \mathscr{B}_v(P_0,r)$. Now assume that $P \in \mathscr{B}_v(P_0,r)$. Then there exist normalized coprime factorization $P = NM^{-1}$ and some Q such that

$$\left\| \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q \right\|_\infty < r, \quad Q, Q^{-1} \in \mathscr{RL}_\infty, \quad \text{wno}Q = 0.$$

Define

$$\Delta = -\left(\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q \right) [M_0^\sim \quad N_0^\sim].$$

Then

$$\begin{bmatrix} MQ \\ NQ \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix},$$

$$\text{wno} \begin{bmatrix} MQ \\ NQ \end{bmatrix} = 0,$$

$$\|\Delta\|_\infty = \left\| \begin{bmatrix} M_0 \\ N_0 \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} Q \right\|_\infty < r.$$

It follows from definition (9) that $P \in \mathscr{C}'_{\text{mul}}(P_0,r)$. This proves $\mathscr{B}(P_0,r) \subset \mathscr{C}'_{\text{mul}}(P_0,r)$.

Note that if

$$\begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \qquad \text{wno} \begin{bmatrix} M \\ N \end{bmatrix} = 0,$$

then there exists some $Q \in \mathscr{RL}_\infty$ with $Q^{-1} \in \mathscr{RL}_\infty$ and $\text{wno}Q = 0$ such that MQ and NQ are right coprime. Hence

$$\begin{bmatrix} M_0 Q \\ N_0 Q \end{bmatrix} = (I + \Delta) \begin{bmatrix} MQ \\ NQ \end{bmatrix}, \qquad \text{wno} \begin{bmatrix} M_0 Q \\ N_0 Q \end{bmatrix} = 0.$$

This shows that

$$P \in \mathscr{C}''_{\text{rel}}(P_0,r) \Leftrightarrow P_0 \in \mathscr{C}''_{\text{mul}}(P,r) .$$

Since

$$P \in \mathscr{B}_v(P_0,r) \Leftrightarrow P_0 \in \mathscr{B}_v(P,r) \Leftrightarrow P_0 \in \mathscr{C}''_{\text{mul}}(P,r) ,$$

it follows that $\mathscr{B}_v(P_0,r) = \mathscr{C}''_{\text{rel}}(P_0,r)$. □

In Georgiou and Smith (1990), T-gap and T-gap ball were introduced that can be defined as

$$\delta_T(P_1,P_2) = \delta(P_1^{\text{T}}, P_2^{\text{T}}), \mathscr{B}_T(P_0,r) = \mathscr{B}^{\text{T}}(P_0^{\text{T}},r).$$

Let $P_0 = \tilde{M}_0^{-1} \tilde{N}_0$ be a left coprime factorization P_0 that may not be normalized. Define

$$\begin{aligned} \tilde{\mathscr{C}}'_{\text{mul}}(P_0,r) &= \{P = \tilde{M}^{-1}\tilde{N} : [\tilde{M} \quad \tilde{N}] \\ &= [\tilde{M}_0 \quad \tilde{N}_0]\,(I + \tilde{\Delta}) \in \mathscr{RH}_\infty, \\ &\tilde{M} \text{ and } \tilde{N} \text{ are coprime}, \Delta \in \mathscr{RL}_\infty, \|\tilde{\Delta}\|_\infty < r\}, \\ \tilde{\mathscr{C}}'_{\text{rel}}(P_0,r) &= \{P = \tilde{M}^{-1}\tilde{N} : [\tilde{M} \quad \tilde{N}] \\ &= [\tilde{M}_0 \quad \tilde{N}_0]\,(I + \tilde{\Delta})^{-1} \in \mathscr{RH}_\infty, \\ &\tilde{M} \text{ and } \tilde{N} \text{ are coprime}, \Delta \in \mathscr{RL}_\infty, \|\tilde{\Delta}\|_\infty < r\}. \end{aligned}$$

Then the following result is true.

Corollary 3. $\mathscr{B}_T(P_0,r) = \tilde{\mathscr{C}}'_{\text{mul}}(P_0,r) = \tilde{\mathscr{C}}'_{\text{rel}}(P_0,r)$.

Proof. The results follow from those of Theorem 1 by noting that $\tilde{\mathscr{C}}'_{\text{mul}}(P_0,r) = \mathscr{C}'^{\text{T}}_{\text{mul}}(P_0^{\text{T}},r)$ and $\tilde{\mathscr{C}}'_{\text{rel}}(P_0,r) = \mathscr{C}'^{\text{T}}_{\text{rel}}(P_0^{\text{T}},r)$. □

It is shown in Georgiou and Smith (1990) that $\delta_T(P_0,r) \neq \delta(P_0^{\text{T}},r)$ in general. Thus in general,

$$\mathscr{C}'_{\text{mul}}(P,r) \neq \tilde{\mathscr{C}}'_{\text{rel}}(P,r), \quad \mathscr{C}'_{\text{rel}}(P,r) \neq \tilde{\mathscr{C}}'_{\text{mul}}(P,r).$$

However if the perturbation is restricted to be stable as those for $\mathscr{C}_{\text{mul}}(P_0,r)$ and $\mathscr{C}_{\text{rel}}(P_0,r)$, then the equality holds. In the remaining part of this section, we take a close look at $\mathscr{C}_{\text{mul}}(P_0,r)$ and $\mathscr{C}_{\text{rel}}(P_0,r)$.

For $P_0 \in \mathscr{P}^{p \times m}$, let $P_0 = \tilde{M}_0^{-1}\tilde{N}_0$ be a left coprime factorization. Define

$$\begin{aligned} \tilde{\mathscr{C}}_{\text{mul}}(P_0,r) &= \{P = \tilde{M}^{-1}\tilde{N} : [\tilde{M} \quad \tilde{N}] \\ &= [\tilde{M}_0 \quad \tilde{N}_0]\,(I + \Delta), \Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty < r\}, \\ \tilde{\mathscr{C}}_{\text{rel}}(P_0,r) &= \{P = \tilde{M}^{-1}\tilde{N} : [\tilde{M} \quad \tilde{N}] \\ &= [\tilde{M}_0 \quad \tilde{N}_0]\,(I + \Delta)^{-1}, \Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty < r\}. \end{aligned}$$

Proposition 4. $\mathscr{C}_{\text{mul}}(P_0,r) = \tilde{\mathscr{C}}_{\text{rel}}(P_0,r)$, and $\mathscr{C}_{\text{rel}}(P_0,r) = \tilde{\mathscr{C}}_{\text{mul}}(P_0,r)$.

Proof. Again, we only need consider the case for $r \leq 1$. Let $P \in \mathscr{C}_{\text{mul}}(P_0,r)$. Then there exists a right coprime factorization $P = NM^{-1}$ such that

$$\begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \qquad \Delta \in \mathscr{RH}_\infty, \qquad \|\Delta\|_\infty < r,$$

where $P_0 = N_0 M_0^{-1}$ is a right coprime factorization. Let $P_0 = \tilde{M}_0^{-1}\tilde{N}_0$ be a left coprime factorization.

Define

$$[\tilde{M} \quad \tilde{N}] = [\tilde{M}_0 \quad \tilde{N}_0]\,(I + \tilde{\Delta})^{-1},$$

$$\tilde{\Delta} = T\Delta T^{-1}, \qquad T = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{13}$$

Then $\tilde{M}$ and $\tilde{N}$ are left coprime, since $(I + \tilde{\Delta})^{-1}$ is a unit $\mathscr{RH}_\infty$ due to $\|\tilde{\Delta}\|_\infty = \|\Delta\|_\infty < r \le 1$. Furthermore,

$$\begin{aligned}[-\tilde{N} \;\; \tilde{M}]\begin{bmatrix} M \\ N \end{bmatrix} &= [\tilde{M} \;\; \tilde{N}]\,T\begin{bmatrix} M \\ N \end{bmatrix} \\ &= [\tilde{M}_0 \quad \tilde{N}_0]\,(\mathrm{I} + \tilde{\Delta})^{-1}\,T\begin{bmatrix} M \\ N \end{bmatrix} \\ &= [\tilde{M}_0 \quad \tilde{N}_0]\,T(I + \Delta)^{-1}\begin{bmatrix} M \\ N \end{bmatrix} \\ &= [-\tilde{N}_0 \quad \tilde{M}_0]\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} = 0.\end{aligned}$$

Consequently, $\tilde{M}^{-1}\tilde{N} = NM^{-1} = P$, which implies that $P \in \tilde{\mathscr{C}}_{\mathrm{rel}}(P_0, r)$ by (13). Therefore, $\mathscr{C}_{\mathrm{mul}}(P_0, r) \subset \tilde{\mathscr{C}}_{\mathrm{rel}}(P_0, r)$. Reversing the procedure above shows $\tilde{\mathscr{C}}_{\mathrm{rel}}(P_0, r) \subset \mathscr{C}_{\mathrm{mul}}(P_0, r)$, leading to the conclusion that $\mathscr{C}_{\mathrm{mul}}(P_0, r) = \tilde{\mathscr{C}}_{\mathrm{rel}}(P_0, r)$. The equality $\mathscr{C}_{\mathrm{rel}}(P_0, r) = \tilde{\mathscr{C}}_{\mathrm{mul}}(P_0, r)$ can be similarly shown. □

It is not clear if $\mathscr{C}_{\mathrm{mul}}(P_0, r) = \mathscr{C}_{\mathrm{rel}}(P_0, r)$ holds in general. However, it holds trivially when P_0 is a scalar (SISO) system.

It has been shown in Vinnicombe (1993) that the containment $\mathscr{B}(P_0, r) \subset \mathscr{B}_\nu(P_0, r)$ is in general strict. This, together with the following example, shows that the containments in equations (11) and (12) are all strict in general.

Example: Let $P_0(s) = (s-1)/(s+1)$ and $P(s) = (2s-1)/(s+1)$. Then a right coprime factorization of P_0 is $P_0 = N_0M_0^{-1}$ where

$$\begin{bmatrix} M_0 \\ N_0 \end{bmatrix} = \begin{bmatrix} 1 \\ (s-1)/(s+1) \end{bmatrix}$$

and all right coprime factorizations of P is given by $P = NM^{-1}$ where

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} 1 \\ (2s-1)/(s+1) \end{bmatrix} Q$$

and Q is a unit in $\mathscr{RH}_\infty$. Let $\Delta \in \mathscr{RH}_\infty$ satisfies

$$\begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)\begin{bmatrix} M_0 \\ N_0 \end{bmatrix}.$$

Evaluating this equation at $s = 1$, we get

$$\Delta(1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} Q(1) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence,

$$\|\Delta(1)\| \ge \min_{Q(1)} \left\| \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} Q(1) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = \frac{1}{\sqrt{5}}.$$

This shows that $P \in \mathscr{C}_{\mathrm{mul}}(P_0, r) = \mathscr{C}_{\mathrm{rel}}(P_0, r)$ only if $r > 1/\sqrt{5}$. However, it is computed in Georgiou and Smith (1990) and Vinnicombe (1993) that $\delta(P, P_0) = \frac{1}{3}$ and $\delta_\nu(P, P_0) = 1/\sqrt{10}$.

4. *Robust stability*

We are interested in the robust stability conditions for the feedback system shown in Fig. 1 when the plant and the controller are subject to simultaneous perturbations of the form described in equations (5)–(10). Let

$$b_{P,C} = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \quad C] \right\|_\infty^{-1}.$$

The following theorems are due to Qiu and Davison (1992) and Vinnicombe (1993).

Theorem 5. Let $\mathrm{P}_0 \in \mathscr{P}^{p \times m}$, $C_0 \in \mathscr{P}^{m \times p}$, and (P_0, C_0) be stable. Then (P, C) is stable for all $P \in \mathscr{B}(P_0, r_1) = \mathscr{C}'_{\mathrm{mul}}(P_0, r_1) = \mathscr{C}'_{\mathrm{rel}}(P_0, r_1)$ and $C \in \mathscr{B}(C_0, r_2) = \mathscr{C}'_{\mathrm{mul}}(C_0, r_2) = \mathscr{C}'_{\mathrm{rel}}(C_0, r_2)$ if and only if

$$\arcsin r_1 + \arcsin r_2 \le \arcsin b_{P_0, C_0}.$$

Theorem 6. Let $\mathrm{P}_0 \in \mathscr{P}^{p \times m}$, $C_0 \in \mathscr{P}^{m \times p}$, and (P_0, C_0) be stable. Then (P, C) is stable for all $P \in \mathscr{B}_\nu(P_0, r_1) = \mathscr{C}''_{\mathrm{mul}}(P_0, r_1) = \mathscr{C}''_{\mathrm{rel}}(P_0, r_1)$ and $C \in \mathscr{B}_\nu(C_0, r_2) = \mathscr{C}''_{\mathrm{mul}}(C_0, r_2) = C''_{\mathrm{rel}}(C_0, r_2)$ if and only if

$$\arcsin r_1 + \arcsin r_2 \le \arcsin b_{P_0, C_0}.$$

Since $\mathscr{C}_{\mathrm{mul}}(P_0, r) \subset \mathscr{B}(P_0, r)$ and $\mathscr{C}_{\mathrm{rel}}(P_0, r) \subset \mathscr{B}(P_0, r)$ and the containment is strict in general, one wonders if the condition in Theorem 5 or 6 can be relaxed if P belongs to $\mathscr{C}_{\mathrm{mul}}(P_0, r_1)$ or $\mathscr{C}_{\mathrm{rel}}(P_0, r_1)$, and C belongs to $\mathscr{C}_{\mathrm{mul}}(C_0, r_2)$ or $\mathscr{C}_{\mathrm{rel}}(C_0, r_2)$. The answer is negative.

Theorem 7. Let $\mathrm{P}_0 \in \mathscr{P}^{p \times m}$, $C_0 \in \mathscr{P}^{m \times p}$, and (P_0, C_0) be stable. Then (P, C) is stable for all $P \in \mathscr{C}_{\mathrm{mul}}(P_0, r_1)$ and $C \in \mathscr{C}_{\mathrm{rel}}(C_0, r_2)$ if and only if

$$\arcsin r_1 + \arcsin r_2 \le \arcsin b_{P_0, C_0}.$$

Proof. The sufficiency follows from Theorem 5 or 6. It remains to show the necessity. Assume that

$$\arcsin r_1 + \arcsin r_2 > \arcsin b_{P_0, C_0}.$$

We need to construct $P \in \mathscr{C}_{\mathrm{mul}}(P_0, r_1)$, $C \in \mathscr{C}_{\mathrm{rel}}(P_0, r_2)$ such that (P, C) is unstable. Let $\theta = \arcsin(b_{P_0, C_0})$. Then there exist $\theta_1 < \arcsin(r_1)$ and $\theta_2 < \arcsin(r_2)$ such that $\theta_1 + \theta_2 = \theta$.

Let $P_0 = N_0M_0^{-1}$ be a normalized right coprime factorization and $C_0 = \tilde{V}_0^{-1}\tilde{U}_0$ be a normalized left coprime factorization. Then

$$b_{P_0, C_0} = \inf_{\omega \in [0, \infty)} \underline{\sigma}[\tilde{V}_0(j\omega)M_0(j\omega) - \tilde{U}_0(j\omega)N_0(j\omega)].$$

There must exist $\bar{\omega} \in [0, \infty]$ such that

$$\underline{\sigma}[\tilde{V}_0(j\bar{\omega})M_0(j\bar{\omega}) - \tilde{U}_0(j\bar{\omega})N_0(j\bar{\omega})] = b_{P_0, C_0}.$$

By Stewart and Sun (1990) (Theorem 1.5.2.), there exist unitary matrices X, Y, and Z such that

$$\begin{bmatrix} M_0(j\bar{\omega}) \\ N_0(j\bar{\omega}) \end{bmatrix} = X\begin{bmatrix} I \\ 0 \end{bmatrix} Y^*,$$

$$[\tilde{V}_0(j\bar{\omega}) - \tilde{U}_0(j\bar{\omega})] = Z[C \quad S]X^*,$$

where

$$C = \mathrm{diag}(c_1, c_2, \ldots, c_m) \in \mathscr{R}^{m \times m},$$

$$S = \mathrm{diag}(\sqrt{1 - c_1^2}, \sqrt{1 - c_2^2}, \ldots, \sqrt{1 - c_{\min\{p,m\}}^2}) \in \mathscr{R}^{m \times p},$$

and $0 \le c_1 \le c_2 \le \cdots \le c_m$. This implies $c_1 = b_{P_0, C_0} = \sin\theta$. Define

$$\bar{\Delta}_1 = X^*\begin{bmatrix} \mathrm{diag}(-\sin\theta_1, \sin\theta_1 0, \ldots, 0) & 0 \\ \mathrm{diag}(-\cos\theta_1, \sin\theta_1 0, \ldots, 0) & 0 \end{bmatrix} X$$

and

$$\bar{\Delta}_2 = X^*\begin{bmatrix} 0 & \mathrm{diag}(-\sin\theta\sin\theta_2, 0, \ldots, 0) \\ 0 & \mathrm{diag}(-\cos\theta\sin\theta_2, 0, \ldots, 0) \end{bmatrix}$$

$$\times\begin{bmatrix} \mathrm{diag}(\sin\theta_1, 0, \ldots, 0) & \mathrm{diag}(\cos\theta_1, 0, \ldots, 0) \\ \mathrm{diag}(\cos\theta_1, 0, \ldots, 0) & \mathrm{diag}(-\sin\theta_1, 0, \ldots, 0) \end{bmatrix} X.$$

Then it is straightforward to verify that $\bar{\sigma}(\bar{\Delta}_1) = \sin\theta_1$, $\bar{\sigma}(\bar{\Delta}_2) = \sin\theta_2$, and

$$[\tilde{V}(j\bar{\omega}) - \tilde{U}(j\omega_0)]\,(\mathrm{I} + \bar{\Delta}_2)(I + \bar{\Delta}_1)\begin{bmatrix} M_0(j\omega) \\ N_0(j\omega) \end{bmatrix}$$

in singular. Notice that $\bar{\Delta}_1$ and $\bar{\Delta}_2$ are of rank one. Standard techniques exist to construct $\Delta_1, \Delta_2 \in \mathscr{RH}_\infty$ such that $\Delta_1(j\bar{\omega}) = \bar{\Delta}_1, \bar{\Delta}_2(j\bar{\omega}) = \bar{\Delta}_2$, $\|\Delta_1\|_\infty = \bar{\sigma}(\bar{\Delta}_1)$, and $\|\Delta_2\|_\infty = \bar{\sigma}(\bar{\Delta}_2)$

(Vidyasagar, 1985). Now let

$$\begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta_1) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix},$$

$$[\tilde{V} \quad \tilde{U}] = [\tilde{V}_0 \quad \tilde{U}_0] \left(I + \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Delta_2 \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \right).$$

Then it follows that $P = NM^{-1} \in \mathscr{C}_{\mathrm{mul}}(P_0, r_1)$, $C = \tilde{V}^{-1}\tilde{U} \in \mathscr{C}_{\mathrm{mul}}(C_0, r_2) = \mathscr{C}_{\mathrm{rel}}(C_0, r_2)$, and (P, C) is unstable. □

5. *Conclusion*

The main contribution of this paper is the establishment of the connection between the gap or ν-gap metric uncertainty and the multiplicative/relative perturbation in coprime factors that are not necessarily normalized. Consequently, the robust stability probem raised in Gu (1995) is completely solved for the plants and controllers whose coprime factors involve simultaneous and independent perturbations. Although only finite dimensional systems are studied, the results can be generalized to infinite dimensional systems. In particular, Theorems 5 and 6 are applicable to plants and controllers of infinite dimension that admit coprime factor perturbations of multiplicative type, where the only requirement for the nominal plant or/and ocntroller is the existence of some coprime factorization having continuous frequency response. This is contrast to the gap metric case as studied in Georgiou and Smith (1992) that requires the existence of normalized coprime factorizations that admit continuous frequency response. As indicated in Treil (1994), an infinite-dimensional system having coprime factorization with continuous frequency response may not have normalized coprime factorization with continuous frequency response. Thus the robust stability results in this paper complement those in Georgiou and Smith (1992).

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References

Georgiou, T. T. (1988). On the computation of the gap metric. *System Control Lett.*, **11,** 253–257.

Georgiou, T. T. and M. C. Smith (1990). Optimal robustness in the gap metric. *IEEE Trans. Automat. Control*, **35,** 673–685.

Georgiou, T. T. and M. C. Smith (1992). Robust stabilization in the gap metric: controller design for distributed systems. *IEEE Trans. Automat. Control*, **37,** 1133–1143.

Gu, G. (1995). Model reduction with relative/multiplicative error bounds and relations to controller reduction. *IEEE Trans. Automat. Control*, **40,** 1478–1485.

McFarlane, D. C. and K. Glover (1990). *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*. Springer, Berlin.

Qiu, L. (1995). Robust control for ℓ_p gap perturbations. In *Proc. European Control Conference*, pp. 1020–1025.

Qiu, L. and E. J. Davison (1992). Feedback stability under simultaneous gap metric uncertainties in plant and controller. *Systems Control Lett.*, **18,** 9–22.

Sefton, J. A. and R. J. Ober (1993). On the gap metric and coprime factor perturbations. *Automatica*, **29,** 723–734.

Stewart, G. W. and J. G. Sun (1990). *Matrix Perturbation Theory*. Academic Press, San Diego.

Treil, S. (1994). A counterexample on continuous coprime factors. *IEEE Trans. Automat. Control*, **39,** 1261 –1263.

Vidyasagar, M. (1985). *Control System Synthesis*: *A Factorization Approach*. The MIT Press, Cambridge, MA.

Vinnicombe, G. (1993). Structured uncertainty and the graph topology. *IEEE Trans. Automat. Control*, **38,** 1371–1383.

Zames, G. and A. K. El-Sakkary (1980). Unstable systems and feedback: the gap metric. In *Proc. 16th Allerton Conf.*, pp. 380–385.