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CONTRACTIVE COMPLETION OF BLOCK MATRICES AND ITS APPLICATION TO $\mathcal{H}_\infty$ CONTROL OF PERIODIC SYSTEMS *

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Design of $\mathcal{H}_\infty$-optimal controllers for discrete-time periodic systems requires proper handling of a causality constraint, which in turn is related to factorization and contractive completion problems associated with block lower-triangular matrices. For a given block upper-triangular matrix, this paper gives a parametrization of all possible contractive completions. The unique contractive completion which minimizes an entropy function is also given. This is then applied to $\mathcal{H}_\infty$ control of periodic systems: We explicitly characterize the set of all periodic, causal controllers which achieve a certain closed-loop $\mathcal{H}_\infty$ norm bound and also give the unique controller which further minimizes a linear-exponential-quadratic-Gaussian cost functional.

1 Introduction

$\mathcal{H}_\infty$-optimal control of linear time-invariant (LTI) systems has been thoroughly studied; its importance in robust control is widely recognized, see the books [8, 12, 24] and the references therein.

$\mathcal{H}_\infty$-optimal control of discrete-time linear *periodic* systems was first studied in [7, 10] in the one-block case and was later extended to the general case in [22]. The common technique used in the study of periodic systems is lifting [17], which amounts to extending input and output spaces of periodic systems and obtaining equivalent LTI systems. This process of lifting has the norm-preserving property and therefore allows an equivalent $\mathcal{H}_\infty$ control problem to be posed for the lifted LTI systems. However, lifting also introduces a

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design constraint due to causality requirement of the controllers; this causality constraint requires that the feedthrough terms in the lifted controllers be block lower-triangular.

In the general case in [22], this constraint was treated using a convex search over a finite-dimensional space. However, it is possible to give explicit solutions to the $\mathcal{H}_\infty$ design problem using factorization involving block triangular matrices, as is discussed in [5] for multirate sampled-data systems. In [22, 5], only *one* solution was computed for the associated $\mathcal{H}_\infty$ problems. In this paper we shall characterize all possible solutions achieving a certain $\mathcal{H}_\infty$ performance and also give the unique solution which further minimizes an auxiliary cost functional. Similar techniques have also been applied to multirate sample-data systems [21].

The new result for $\mathcal{H}_\infty$ periodic control is based on the study of completing partial (block upper-triangular) matrices to contractions, or equivalently, the matrix distance problem stated as follows: Given a full matrix and a certain associated block lower-triangular structure, find all possible block lower-triangular matrices which are within a pre-specified distance, measured by the spectral norm, from the given matrix. Such problems were studied before [3, 4, 23]: In [3, 4] the solutions are derived based on J-spectral factorizations using operator theory and the Krein-space geometry; in [23] the solution may be considered as a finite-dimensional analogue of Schur's algorithm and is derived by using elementary matrix algebra. In this paper, we shall present another solution which keeps the flavor of J-spectral factorization of [3, 4] but uses only elementary linear algebra.

J-spectral factorization roots deeply in $\mathcal{H}_\infty$ control theory, which was studied before primarily via matrix interpolations [8, 11]. For an overview on the role of J-spectral factorization in matrix interpolations, see [2].

We remark that it is possible to develop a complete theory for the $\mathcal{H}_\infty$ control problem with the causality constraint if, as suggested in [2, Section 3] and [13, Chapter 7], one redefines the $\mathcal{H}_\infty$ space to be the set of bounded analytic matrices on the unit disk which are block lower-triangular when evaluated at the origin. However, we feel that our approach in this paper is advantageous because it connects the existing standard discrete time $\mathcal{H}_\infty$ solution in [14] with the solution to the matrix completion problem and the new results are obtained with relatively less effort.

The organization of this paper is as follows. In the next section, we state precisely our $\mathcal{H}_\infty$ control problem for periodic systems, convert the problem via lifting into an equivalent problem for LTI systems with a causality constraint on the controllers, and relate the causality condition to a certain block lower-triangular structure.

In Section 3, we solve the matrix distance problem stated earlier using elementary linear algebra. The proofs of the results are also given.

In Section 4, the results in Section 3 are applied to our $\mathcal{H}_\infty$ control problem for periodic systems. For a given $\mathcal{H}_\infty$ norm bound (normalized to 1), we characterize the set of all periodic controllers satisfying the causality constraint and achieving the $\mathcal{H}_\infty$ norm bound

for the closed-loop system. Furthermore, we give the controller which minimizes a linear-exponential-quadratic-Gaussian functional, or equivalently, an entropy cost.

Section 5 offers some concluding remarks.

Now we introduce some notation. Given an operator K and two operator matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

the linear fractional transformation associated with P and K is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

and the star product of P and Q is

$$P \star Q = \begin{bmatrix} P_{11} + P_{12}Q_{11}(I - P_{22}Q_{11})^{-1}P_{21} & P_{12}(I - Q_{11}P_{22})^{-1}Q_{12} \\ Q_{21}(I - P_{22}Q_{11})^{-1}P_{21} & Q_{21}(I - P_{22}Q_{11})^{-1}P_{22}Q_{12} + Q_{22} \end{bmatrix}.$$

Here, we assume that the domains and co-domains of the operators are compatible and the inverses exist. With these definitions, we have

$$\mathcal{F}(P, \mathcal{F}(Q, K)) = \mathcal{F}(P \star Q, K).$$

2 $\mathcal{H}_\infty$ Periodic Control and Lifting

Let ℓ be the space of discrete-time signals, possibly vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. Let U be the unit delay operator on ℓ and U^* the unit advance operator. For a positive integer l, a linear, causal discrete-time system $\tilde{G}$ is *l-periodic* if $(U^*)^l \tilde{G} U^l = \tilde{G}$. A 1-periodic system is normally known as time-invariant.

A linear l-periodic system can be viewed as an LTI system via lifting [17]. Define the *lifting operator* L_l via $v = L_l \tilde{v}$:

$$\{v(0), v(1), \cdots\} \mapsto \left\{ \begin{bmatrix} \tilde{v}(0) \\ \tilde{v}(1) \\ \vdots \\ \tilde{v}(l-1) \end{bmatrix}, \begin{bmatrix} \tilde{v}(l) \\ \tilde{v}(l+1) \\ \vdots \\ \tilde{v}(2l-1) \end{bmatrix}, \cdots \right\}. \tag{1}$$

L_n maps ℓ to ℓ^l, the external direct sum of l copies of ℓ. The inverse L_l^{-1}, mapping ℓ^l to ℓ, amounts to reversing the operation in (1). The lifted system is defined as

$$G = L_l \tilde{G} L_l^{-1}.$$

It is a fact that G is LTI iff $\tilde{G}$ is linear and l-periodic. Moreover, since the lifting preserves the norm on ℓ_2, G is ℓ_2-bounded iff $\tilde{G}$ is and in this case they have the same ℓ_2-induced norm: $\|G\| = \|\tilde{G}\|$.

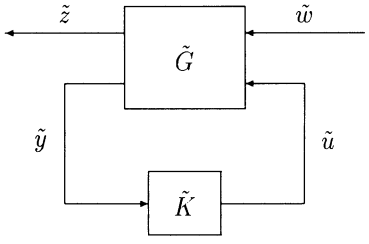

Figure 1: The discrete-time periodic system

The linear periodic control system to be studied is shown in Figure 1. Here $\tilde{G}$, the system to be controlled, is l-periodic and causal with two input vectors, the exogenous input $\tilde{w}$ and the control input $\tilde{u}$, and two output vectors, the output to be controlled $\tilde{z}$ and the measured output $\tilde{y}$; $\tilde{K}$, which processes $\tilde{y}$ and generates $\tilde{u}$, is the controller to be designed. Since $\tilde{G}$ is l-periodic, we shall require that $\tilde{K}$ be l-periodic and causal. (The causality is for implementability of the controller.) We shall be interested in only finite-dimensional $\tilde{G}$ and $\tilde{K}$, i.e., those $\tilde{G}$ and $\tilde{K}$ which have finite-dimensional state space realizations. Let us take any minimal state space realization of $\tilde{G}$ and $\tilde{K}$ in Figure 1; the closed system is said to be internally stable if the state vectors of $\tilde{G}$ and $\tilde{K}$ tend to zero from every initial condition.

The $\mathcal{H}_\infty$ control problem is as follows: Given $\tilde{G}$, design $\tilde{K}$ so that the closed-loop system is internally stable and the map $\tilde{w} \mapsto \tilde{z}$, denoted $T_{\tilde{z}\tilde{w}}$, has ℓ_2-induced norm less than a pre-specified number γ, or, $\|T_{\tilde{z}\tilde{w}}\| < \gamma$. By normalization, we can take $\gamma = 1$. Clearly, the solutions to this $\mathcal{H}_\infty$ control problem, if they exist, are not unique. We first seek a characterization of all solutions, and then find among those solutions the unique one which minimizes the following linear-exponential-quadratic-Gaussian (LEQG) cost:

$$\Omega(T_{\tilde{z}\tilde{w}}) = \lim_{T\to\infty} \frac{2}{T} \ln \mathbf{E}\left\{\exp\left[\frac{1}{2}\sum_{k=0}^{T-1} \tilde{z}'(k)\tilde{z}(k)\right]\right\},$$

where $\tilde{z}$ is the response of the closed-loop system when the input $\tilde{w}$ is a Gaussian white noise with zero mean and unit covariance, and $\mathbf{E}$ is the expectation operator.

Previous work on such an $\mathcal{H}_\infty$ problem are [7, 10], which studied a special case (the one-block problem), and [22]; none of them contained the characterization of all possible solutions or derived the particular solution minimizing the LEQG cost.

Now we lift the system in Figure 1 to get Figure 2, where all signals are lifted, e.g., $w = L_n\tilde{w}$, and the two lifted systems

$$G = L_l\tilde{G}L_l^{-1}, \quad K = L_l\tilde{K}L_l^{-1},$$

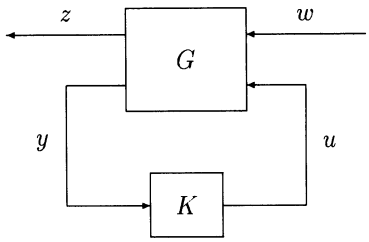

Figure 2: The lifted LTI system

are both LTI and causal with transfer functions

$$\hat{G}(\lambda) = \begin{bmatrix} \hat{G}_{11}(\lambda) & \hat{G}_{12}(\lambda) \\ \hat{G}_{21}(\lambda) & \hat{G}_{22}(\lambda) \end{bmatrix}, \quad \hat{K}(\lambda)$$

respectively. Here $\hat{G}(\lambda)$ is partitioned according to the dimensions of its two inputs and two outputs. (In the transfer functions, we used λ-transforms instead of the more traditional z-transforms, where $\lambda = z^{-1}$.)

To proceed, we need to introduce some notation. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Define the set of block matrices:

$$\mathcal{M}(\mathbb{F}^{m\times n}) := \left\{ \begin{bmatrix} M_{11} & \cdots & M_{1l} \\ \vdots & & \vdots \\ M_{l1} & \cdots & M_{ll} \end{bmatrix} : \; M_{ij} \in \mathbb{F}^{m\times n} \right\}.$$

The integer l is not reflected in the notation since we will assume it is fixed. The block lower-triangular subset of $\mathcal{M}(\mathbb{F}^{m\times n})$, denoted by $\mathcal{T}(\mathbb{F}^{m\times n})$, consists of all matrices with $M_{ij} = 0$, $i < j$, and the strictly block lower-triangular subset, $\mathcal{T}_s(\mathbb{F}^{m\times n})$, consists of matrices with $M_{ij} = 0$, $i \leq j$.

Let the dimensions of $\tilde{w}$, $\tilde{z}$, $\tilde{u}$, $\tilde{y}$ be p, q, m, n, respectively. Due to causality of $\tilde{G}$ and $\tilde{K}$, the transfer functions of the lifted systems satisfy $\hat{G}_{11}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$, $\hat{G}_{12}(0) \in \mathcal{T}(\mathbb{R}^{q\times m})$, $\hat{G}_{21}(0) \in \mathcal{T}(\mathbb{R}^{n\times p})$, $\hat{G}_{22}(0) \in \mathcal{T}(\mathbb{R}^{n\times m})$, and $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})$. It follows from [17] that each LTI causal nl-input ml-output controller K satisfying $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})$ corresponds to an l-periodic, causal, n-input m-output controller.

Letting T_{zw} be the closed-loop map $w \mapsto z$ in Figure 2, we have

$$\|T_{\tilde{z}\tilde{w}}\| = \|T_{zw}\| = \|\hat{T}_{zw}\|_\infty,$$

the last quantity being the $\mathcal{H}_\infty$ norm of the transfer function for T_{zw}. By using the techniques in [9], we can show that when $\|\hat{T}_{zw}\|_\infty < 1$,

$$\Omega(T_{\tilde{z}\tilde{w}}) = \frac{1}{l}\mathcal{I}(\hat{T}_{zw}),$$

where $\mathcal{I}(\hat{T}_{zw})$, the entropy of $\hat{T}_{zw}$, is defined as

$$\mathcal{I}(\hat{T}_{zw}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det\left[I - \hat{T}_{zw}^*(e^{j\omega})\hat{T}_{zw}(e^{j\omega})\right] d\omega.$$

Hence the equivalent LTI $\mathcal{H}_\infty$ problem is: Given the LTI system G from the lifting of $\tilde{G}$, characterize all LTI and causal controllers K satisfying $\hat{K}(0) \in \mathcal{T}(\mathbf{R}^{m\times n})$ such that the closed-loop system is internally stable and $\|\hat{T}_{zw}\|_\infty < 1$; furthermore, find the unique such controller which minimizes $\mathcal{I}(\hat{T}_{zw})$.

One cannot apply standard $\mathcal{H}_\infty$ techniques [14, 18, 15, 16] to this problem directly because the causality constraint on $\hat{K}(0)$: $\hat{K}(0) \in \mathcal{T}(\mathbf{R}^{m\times n})$. How to handle this constraint is the main concern of this paper.

3 Matrix Contractive Completion

In this section we consider the following matrix completion problem: Given $M \in \mathcal{M}(\mathbf{F}^{m\times n})$, characterize all $T \in \mathcal{T}(\mathbf{F}^{m\times n})$ such that $\|M + T\| < 1$; and then find the one, among those T characterized, which minimizes

$$\mathcal{I}(M + T) := -\ln\det[I - (M + T)^*(M + T)].$$

We shall need the following notation: For a block matrix

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1q} \\ \vdots & & \vdots \\ M_{p1} & \cdots & M_{pq} \end{bmatrix}$$

and integers i_1, i_2, j_1, j_2 with $1 \le i_1 \le i_2 \le p$ and $1 \le j_1 \le j_2 \le q$, we write the submatrix

$$[M_{ij}]_{i=i_1,j=j_1}^{i_2,j_2} := \begin{bmatrix} M_{i_1j_1} & \cdots & M_{i_1j_2} \\ \vdots & & \vdots \\ M_{i_2j_1} & \cdots & M_{i_2j_2} \end{bmatrix}.$$

The following theorem gives several equivalent conditions for the solvability of the matrix completion problem.

Theorem 1 *Let $M \in \mathcal{M}(\mathbf{F}^{m\times n})$. The following statements are equivalent:*

(a) For $k = 1, 2, \ldots, l-1$, $\|[M_{ij}]_{i=1,j=k+1}^{k,l}\| < 1$.

(b) There exists $T \in \mathcal{T}(\mathbf{F}^{m\times n})$ such that $\|M + T\| < 1$.

(c) There exists

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

with $W_{11} \in \mathcal{T}(\mathbb{F}^{m\times m})$, $W_{12} \in \mathcal{T}(\mathbb{F}^{m\times n})$, $W_{21} \in \mathcal{T}_s(\mathbb{F}^{n\times m})$, *and* $W_{22} \in \mathcal{T}(\mathbb{F}^{n\times n})$ *such that* $W^*JW = G^*JG$, *where*

$$G = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

(d) There exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with $P_{11} \in \mathcal{T}(\mathbb{F}^{m\times n})$, $P_{12} \in \mathcal{T}(\mathbb{F}^{m\times m})$, $P_{21} \in \mathcal{T}(\mathbb{F}^{n\times n})$, $P_{22} \in \mathcal{T}_s(\mathbb{F}^{n\times m})$, *and* P_{12}, P_{21} *both invertible such that*

$$\begin{bmatrix} M + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary.

About the proof of this theorem, (a) $\Leftrightarrow$ (b) follows from the Arveson's distance formula [6]; (d) $\Rightarrow$ (b) is obvious since $\|M + P_{11}\| < 1$ and $P_{11} \in \mathcal{T}(\mathbb{F}^{m\times n})$; the rest is rather involved and hence is left in the appendix. Matrices W and P in conditions (c) and (d) are essential to the solution of the matrix problem; their existence is proven constructively and hence W and P can be computed if condition (a) in Theorem 1 holds, which is easily verifiable.

If condition (c) in Theorem 1 holds, we have $W^*JW = G^*JG$. Because G and J are invertible, so is W. Furthermore, the $(1,1)$ block of the equation $W^*JW = G^*JG$ reads

$$W_{11}^*W_{11} - W_{21}^*W_{21} = I,$$

which implies that W_{11} is invertible too. Using this condition, we can parametrize all solutions to our matrix problem.

Theorem 2 *Let* $M \in \mathcal{M}(\mathbb{F}^{m\times n})$ *and assume condition (c) in Theorem 1 is satisfied. Then the set of all* $T \in \mathcal{T}(\mathbb{F}^{m\times n})$ *such that* $\|M + T\| < 1$ *is given by*

$$\left\{ T = Q_1Q_2^{-1} : \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = W^{-1} \begin{bmatrix} U \\ I \end{bmatrix},\ U \in \mathcal{T}(\mathbb{F}^{m\times n}),\ \textit{and}\ \|U\| < 1 \right\}. \tag{2}$$

Proof: Let $V = W^{-1}$ and partition V compatibly. Then $V_{11} \in \mathcal{T}(\mathbb{F}^{m\times m})$, $V_{12} \in \mathcal{T}(\mathbb{F}^{m\times n})$, $V_{21} \in \mathcal{T}_s(\mathbb{F}^{n\times m})$, and $V_{22} \in \mathcal{T}(\mathbb{F}^{n\times n})$. Since W_{11} is invertible, so is V_{22} by some calculation. Now inverting both sides of $W^*JW = G^*JG$ and noting $J^{-1} = J$, we get $G^{-1}JG^{*-1} = VJV^*$. The $(2,2)$ block of the latter equation gives

$$V_{21}V_{21}^* - V_{22}V_{22}^* = -I,$$

which implies $\|V_{22}^{-1}V_{21}\| < 1$. Since

$$Q_2 = V_{22} + V_{21}U = V_{22}(I + V_{22}^{-1}V_{21}U),$$

it follows that for every $U \in \mathcal{T}(\mathbf{F}^{m\times n})$ with $\|U\| < 1$, Q_2^{-1} exists and belongs to $\mathcal{T}(\mathbf{F}^{n\times n})$. Letting T be given by (2), we have

$$\begin{aligned}
(M+T)^*(M+T) - I &= \begin{bmatrix} M+T \\ I \end{bmatrix}^* J \begin{bmatrix} M+T \\ I \end{bmatrix} \\
&= \begin{bmatrix} T \\ I \end{bmatrix}^* G^*JG \begin{bmatrix} T \\ I \end{bmatrix} \\
&= Q_2^{*-1} \begin{bmatrix} U \\ I \end{bmatrix}^* V^*G^*JGV \begin{bmatrix} U \\ I \end{bmatrix} Q_2^{-1} \\
&= Q_2^{*-1} \begin{bmatrix} U \\ I \end{bmatrix}^* J \begin{bmatrix} U \\ I \end{bmatrix} Q_2^{-1} \\
&= Q_2^{*-1}(U^*U - I)Q_2^{-1} \\
&< 0.
\end{aligned}$$

This implies $\|M+T\| < 1$.

Conversely, suppose $T \in \mathcal{T}(\mathbf{F}^{m\times n})$ with $\|M+T\| < 1$. Define

$$\begin{aligned}
\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} &:= W \begin{bmatrix} T \\ I \end{bmatrix} \\
&= WG^{-1} \begin{bmatrix} M+T \\ I \end{bmatrix}.
\end{aligned} \tag{3}$$

Then we have $U_1 \in \mathcal{T}(\mathbf{F}^{m\times n})$ and $U_2 \in \mathcal{T}(\mathbf{F}^{n\times n})$. Since

$$\begin{aligned}
U_1^*U_1 - U_2^*U_2 &= \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^* J \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\
&= \begin{bmatrix} M+T \\ I \end{bmatrix}^* G^{*-1}W^*JWG^{-1} \begin{bmatrix} M+T \\ I \end{bmatrix} \\
&= \begin{bmatrix} M+T \\ I \end{bmatrix}^* J \begin{bmatrix} M+T \\ I \end{bmatrix} \\
&= (M+T)^*(M+T) - I \\
&< 0,
\end{aligned}$$

it follows that U_2 is invertible and $\|U_1U_2^{-1}\| < 1$. Define $U := U_1U_2^{-1}$ and

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} := W^{-1} \begin{bmatrix} U \\ I \end{bmatrix},$$

it follows from (3) that $Q_1 = TU_2^{-1}$ and $Q_2 = U_2^{-1}$. Hence $T = Q_1Q_2^{-1}$, which satisfies the parametrization in (2). □

Alternatively, we can use condition (d) in Theorem 1 to find all solutions to the same matrix problem.

Theorem 3 *Let $M \in \mathcal{M}(\mathbb{F}^{m\times n})$ and assume condition (d) in Theorem 1 is satisfied. Then the set of all $M \in \mathcal{T}(\mathbb{F}^{m\times n})$ such that $\|M+T\| < 1$ is given by*

$$\{T = \mathcal{F}(P,U) : \ U \in \mathcal{T}(\mathbb{F}^{m\times n}) \text{ and } \|U\| < 1\}. \tag{4}$$

Proof: Since the matrix

$$\begin{bmatrix} M+P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary and P_{12}, P_{21} are invertible, it follows from [20] that the map

$$U \mapsto \mathcal{F}\left(\begin{bmatrix} M+P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, U\right) = M + \mathcal{F}(P,U)$$

is a bijection from the open unit ball of $\mathcal{M}(\mathbb{F}^{m\times n})$ onto itself. What is left to show is that $\mathcal{F}(P,U) \in \mathcal{T}(\mathbb{F}^{m\times n})$ iff $U \in \mathcal{T}(\mathbb{F}^{m\times n})$. The "if" part follows from simple matrix manipulation. For the "only if" part, assume $T := \mathcal{F}(P,U) \in \mathcal{T}(\mathbb{F}^{m\times n})$ for some $U \in \mathcal{M}(\mathbb{F}^{m\times n})$; we need to show that U too belongs to $\mathcal{T}(\mathbb{F}^{m\times n})$. From

$$T = P_{11} + P_{12}U(I - P_{22}U)^{-1}P_{21},$$

we obtain after some algebra

$$P_{12}^{-1}(T - P_{11})P_{21}^{-1} = [I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22}]U. \tag{5}$$

Since

$$\begin{aligned} I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22} &= I + P_{12}^{-1}P_{12}U(I - P_{22}U)^{-1}P_{21}P_{21}^{-1}P_{22}] \\ &= I + U(I - P_{22}U)^{-1}P_{22} \\ &= (I - UP_{22})^{-1}, \end{aligned}$$

it follows that $I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22}$ is invertible. Hence from (5)

$$U = [I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22}]^{-1}P_{12}^{-1}(T - P_{11})P_{21}^{-1}$$

Therefore U belongs to $\mathcal{T}(\mathbb{F}^{m\times n})$. □

The characterizations in Theorems 2 and 3 also give easy expression to the T which minimizes $\mathcal{I}(M+T)$.

Theorem 4 *Let $M \in \mathcal{M}(\mathbb{F}^{m\times n})$ and assume condition (c) or (d) in Theorem 1 is satisfied. Then the unique T satisfying $\|M+T\| < 1$ which minimizes $\mathcal{I}(M+T)$ is given by $T = P_{11}$ or $T = -W_{11}^{-1}W_{12}$.*

Proof: We will show that $T = P_{11}$ minimizes the entropy, the proof that $T = -W_{11}^{-1}W_{12}$ also minimizes the entropy is omitted. According to Theorem 3, all T satisfying $\|M+T\| < 1$ are characterized by (4), Consequently, all resulting $M + T$ are given by

$$\left\{\mathcal{F}\left(\begin{bmatrix} M + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, U\right) : \ U \in \mathcal{T}(\mathbf{F}^{m\times n}) \text{ and } \|U\| < 1\right\}.$$

By Lemma 2 i) in [16], we obtain

$$\mathcal{I}(M + T) = \mathcal{I}(U) + \mathcal{I}(M + P_{11}) + 2\ln|\det(I - P_{22}U)|.$$

Notice that the second term is independent of U and $P_{22}U \in \mathcal{T}_s(\mathbf{F}^{n\times m})$, which implies that the third term is zero. Therefore the U which minimizes $\mathcal{I}(M + T)$ is given by $U = 0$. □

One implication of Theorem 4 is that although W given in condition (c) and P in condition (d) of Theorem 1 are not unique, P_{11} and $-W_{11}^{-1}W_{12}$ are uniquely determined and they are equal.

4 All $\mathcal{H}_\infty$ Suboptimal Periodic Controllers

Now we return to the $\mathcal{H}_\infty$ periodic control problem stated in Section 2: Given LTI G resulted from the lifting of $\tilde{G}$, characterize all LTI, causal K with $\hat{K}(0) \in \mathcal{T}(\mathbf{R}^{m\times n})$ that stabilize G and achieve $\||\mathcal{F}(\hat{G}, \hat{K})\|_\infty < 1$. This problem is called the constrained $\mathcal{H}_\infty$ problem. To make use of the standard results in the literature and to simplify our solution, we make the following assumptions:

1. $\hat{G}_{12}(0)$ has full column rank, $\hat{G}_{21}(0)$ has full row rank.
2. $\hat{G}_{12}$ and $\hat{G}_{21}$ have no transmission zeros on the unit circle.
3. $\hat{G}_{22}(0) \in \mathcal{T}_s(\mathbf{R}^{n\times m})$ or, equivalently, $\tilde{G}$ is strictly causal.

For this constrained $\mathcal{H}_\infty$ problem to be solvable, it is necessary that the unconstrained problem be solvable. Hence first we drop the causality constraint temporarily and consider the corresponding unconstrained $\mathcal{H}_\infty$ problem: Find all LTI, causal K to stabilize G and achieve $\||\mathcal{F}(\hat{G}, \hat{K})\|_\infty < 1$. This is a standard problem and has been extensively studied in, e.g., [14]. Several solutions to the standard $\mathcal{H}_\infty$ problem exist in the literature. Here we adopt the solution in [14]. Assume the solvability conditions are satisfied, then all stabilizing controllers K satisfyinging $\|\mathcal{F}(\hat{G}, \hat{K})\|_\infty < 1$ are characterized by

$$\left\{\hat{K} = \mathcal{F}\left(\begin{bmatrix} 0 & I \\ I & -\hat{G}_{22}(0) \end{bmatrix} \star \hat{M}, \hat{\Phi}\right) : \quad \hat{\Phi} \in \mathcal{RH}_\infty, \ \|\hat{\Phi}\|_\infty < 1, \right.$$
$$\left. \text{and } I + \hat{G}_{22}(0)\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] \text{ is invertible.}\right\} \quad (6)$$

Here the $\mathcal{RH}_\infty$ matrix

$$\hat{M} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix}$$

is not uniquely given in [14] and by using Cholesky factorizations we can always choose $\hat{M}$ so that

$$\begin{aligned} \hat{M}_{12}(0) &\in \mathcal{T}(\mathbf{R}^{m\times m}) \\ \hat{M}_{21}(0) &\in \mathcal{T}(\mathbf{R}^{n\times n}) \\ \hat{M}_{22}(0) &= 0. \end{aligned}$$

Furthermore, $\hat{M}_{12}(0)$ and $\hat{M}_{21}(0)$ are invertible.

Now let us return to the constrained $\mathcal{H}_\infty$ problem.

Theorem 5 *The constrained $\mathcal{H}_\infty$ problem is solvable iff the corresponding unconstrained problem is solvable and there exists $T \in \mathcal{T}(\mathbf{R}^{m\times n})$ such that*

$$\| - \hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + T\| < 1. \tag{7}$$

Proof: Obviously, the corresponding unconstrained problem has to be solvable in order for the constrained problem to be solvable. Assume that the unconstrained problem is solvable. Since $\hat{G}_{22}(0) \in \mathcal{T}_s(\mathbf{R}^{n\times m})$, it follows that $\hat{K}(0) \in \mathcal{T}(\mathbf{R}^{m\times n})$ iff

$$\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] = \hat{M}_{11}(0) + \hat{M}_{12}(0)\hat{\Phi}(0)\hat{M}_{21}(0) \in \mathcal{T}(\mathbf{F}^{m\times n}).$$

Pre- and post-multiply this by $\hat{M}_{12}(0)^{-1}$ and $\hat{M}_{21}(0)^{-1}$ respectively to get

$$\hat{M}_{12}(0)^{-1}\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)]\hat{M}_{21}(0)^{-1} = \hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \hat{\Phi}(0).$$

If there exists $\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] \in \mathcal{T}(\mathbf{F}^{m\times n})$ such that $\|\hat{\Phi}(0)\| < 1$, then $\hat{M}_{12}(0)^{-1}\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)]$ $\hat{M}_{21}(0)^{-1} \in \mathcal{T}(\mathbf{R}^{m\times n})$ and

$$\| - \hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \hat{M}_{12}(0)^{-1}\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)]\hat{M}_{21}(0)^{-1}\| < 1.$$

Conversely, if $T \in \mathcal{T}(\mathbf{R}^{m\times n})$ such that (7) is true, let

$$\Phi = -\hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + T$$

Then

$$\hat{K} = \mathcal{F}\left(\begin{bmatrix} 0 & I \\ I & -\hat{G}_{22}(0) \end{bmatrix} \star \hat{M}, \Phi\right)$$

achieves $\hat{K}(0) \in \mathcal{T}(\mathbf{R}^{m\times n})$. □

The solvability condition for the corresponding unconstrained problem is given in [14]; the existence of $T \in \mathcal{T}(\mathbb{R}^{m\times n})$ such that (7) is satisfied can be checked easily by using Theorem 1. If the conditions in Theorem 5 are satisfied, then there exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with $P_{11} \in \mathcal{T}(\mathbb{R}^{m\times n})$, $P_{12} \in \mathcal{T}(\mathbb{R}^{m\times m})$, $P_{21} \in \mathcal{T}(\mathbb{R}^{n\times n})$, $P_{22} \in \mathcal{T}_s(\mathbb{R}^{n\times m})$, and P_{12}, P_{21} both invertible such that

$$U = \begin{bmatrix} -\hat{M}_{12}^{-1}(0)\hat{M}_{11}(0)\hat{M}_{21}^{-1}(0) + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is orthogonal (Theorem 1). Define

$$\hat{N} = \begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \star \hat{M} \star U.$$

It is easy to check that $\hat{N}_{11}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})$, $\hat{N}_{12}(0) \in \mathcal{T}(\mathbb{R}^{m\times m})$, $\hat{N}_{21}(0) \in \mathcal{T}(\mathbb{R}^{n\times n})$, $\hat{N}_{22}(0) \in \mathcal{T}_s(\mathbb{R}^{n\times m})$, and $\hat{N}_{12}(0)$, $\hat{N}_{21}(0)$ are both invertible. The set in (6) can be rewritten as

$$\{\hat{K} = \mathcal{F}(\hat{N}, \hat{\Phi}) : \hat{\Phi} \in \mathcal{RH}_\infty, \|\hat{\Phi}\|_\infty < 1, I - \hat{N}_{22}(0)\hat{\Phi}(0) \text{ is invertible}\}.$$

Theorem 6 *Assume solvability of the constrained $\mathcal{H}_\infty$ problem. Then the set of all controllers solving the problem is given by*

$$\{\hat{K} = \mathcal{F}(\hat{N}, \hat{\Phi}) : \hat{\Phi} \in \mathcal{RH}_\infty, \|\hat{\Phi}\|_\infty < 1, \hat{\Phi}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})\ \}.$$

Proof: First notice that $I - \hat{N}_{22}(0)\hat{\Phi}(0)$ is always invertible if $\hat{\Phi}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})$. the special properties of $\hat{N}(0)$ guarantees that $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})$ iff $\hat{\Phi}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})$. Then the result follows immediately. □

It follows from Theorem 6 that all $\mathcal{H}_\infty$ suboptimal closed-loop transfer functions are

$$\{\mathcal{F}(\hat{G}, \hat{K}) = \mathcal{F}(\hat{J}, \hat{\Phi}) :\ \hat{\Phi} \in \mathcal{RH}_\infty,\ \|\hat{\Phi}\|_\infty < 1,\ \hat{\Phi}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})\}$$

where

$$\hat{J} = \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \end{bmatrix} = \hat{G} \star \hat{N}.$$

It follows from the internal stability requirement that $\hat{J} \in \mathcal{RH}_\infty$. Also we have $\hat{J}_{11}(0) \in \mathcal{T}(\mathbb{R}^{m\times n})$, $\hat{J}_{12}(0) \in \mathcal{T}(\mathbb{R}^{m\times m})$, $\hat{J}_{21}(0) \in \mathcal{T}(\mathbb{R}^{n\times n})$, and $\hat{J}_{22}(0) \in \mathcal{T}_s(\mathbb{R}^{n\times m})$. Since $\|\mathcal{F}(\hat{J}, \hat{\Phi})\| < 1$ for all $\hat{\Phi} \in \mathcal{RH}_\infty$ with $\|\hat{\Phi}\|_\infty < 1$, it can be shown by using an idea in [19] that there exists a scalar transfer function $\hat{d} \in \mathcal{RH}_\infty$ with $\hat{d}^{-1} \in \mathcal{RH}_\infty$ such that

$$\left\| \begin{bmatrix} \hat{J}_{11} & \hat{d}\hat{J}_{12} \\ \hat{d}^{-1}\hat{J}_{21} & \hat{J}_{22} \end{bmatrix} \right\|_\infty < 1.$$

Consequently, we can find $\hat{J}_{13}$, $\hat{J}_{23}$, $\hat{J}_{31}$, $\hat{J}_{32}$, $\hat{J}_{33}$, all belonging to $\mathcal{RH}_\infty$, such that

$$\hat{J}_{aug} = \begin{bmatrix} \hat{J}_{11} & \hat{d}\hat{J}_{12} & \hat{J}_{13} \\ \hat{d}^{-1}\hat{J}_{21} & \hat{J}_{22} & \hat{J}_{23} \\ \hat{J}_{31} & \hat{J}_{32} & \hat{J}_{33} \end{bmatrix}$$

is para-unitary. Then another way to characterize the $\mathcal{H}_\infty$ suboptimal closed-loop transfer functions is

$$\left\{ \mathcal{F}\left(\hat{J}_{aug}, \begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix} \right) : \ \hat{\Phi} \in \mathcal{RH}_\infty, \ \ \|\hat{\Phi}\|_\infty < 1, \ \ \hat{\Phi}(0) \in \mathcal{T}(\mathbb{R}^{m\times n}) \right\}.$$

By Lemma 2 i) in [16],

$$\begin{aligned} \mathcal{I}[\mathcal{F}(\hat{G},\hat{K})] &= \mathcal{I}\left(\begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix} \right) + \mathcal{I}(\hat{J}_{11}) + 2\ln \left| \det\left(I - \begin{bmatrix} \hat{J}_{22}(0) & \hat{J}_{23}(0) \\ \hat{J}_{32}(0) & \hat{J}_{33}(0) \end{bmatrix} \begin{bmatrix} \hat{\Phi}(0) & 0 \\ 0 & 0 \end{bmatrix} \right) \right| \\ &= \mathcal{I}(\hat{\Phi}) + \mathcal{I}(\hat{J}_{11}) + 2\ln|\det[I - \hat{J}_{22}(0)\hat{\Phi}(0)]| \\ &= \mathcal{I}(\hat{\Phi}) + \mathcal{I}(\hat{J}_{11}). \end{aligned}$$

The last equality is due to $\hat{J}_{22}(0)\hat{\Phi}(0) \in \mathcal{T}_s(\mathbb{R}^{m\times m})$. Therefore, the minimum of $\mathcal{I}[\mathcal{F}(\hat{G},\hat{K})]$ is achieved at $\hat{\Phi} = 0$. The following theorem is thus obtained.

Theorem 7 *The minimum entropy controller is given by $\hat{K}_{ME} = \hat{N}_{11}$. The minimum entropy value of the closed-loop transfer function is $\mathcal{I}(\hat{J}_{11})$.*

5 Concluding Remarks

Using a parametrization of all solutions for a certain matrix completion problem, we obtained a simple characterization of all $\mathcal{H}_\infty$ suboptimal periodic controllers satisfying a causality constraint. This characterization also gives a simple expression of the unique $\mathcal{H}_\infty$ suboptimal periodic controller which further minimizes an LEQG cost. The results obtained are explicit and require no numerical optimization. The ideas used in the paper can be applied to study similar problems involving multirate sampled-data controllers as in [5, 21].

Appendix: Proof of Theorem 1

To prove (c) $\Rightarrow$ (d), let

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

with $W_{11} \in \mathcal{T}(\mathbb{F}^{m\times m})$, $W_{12} \in \mathcal{T}(\mathbb{F}^{m\times n})$, $W_{21} \in \mathcal{T}_s(\mathbb{F}^{n\times m})$, and $W_{22} \in \mathcal{T}(\mathbb{F}^{n\times n})$ satisfy $W^*JW = G^*JG$. Define

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} := \begin{bmatrix} -W_{11}^{-1}W_{12} & W_{11}^{-1} \\ W_{22} - W_{21}W_{11}^{-1}W_{12} & W_{21}W_{11}^{-1} \end{bmatrix}.$$

It is easy to see that $P_{11} \in \mathcal{T}(\mathbb{F}^{m\times n})$, $P_{12} \in \mathcal{T}(\mathbb{F}^{m\times m})$, $P_{21} \in \mathcal{T}(\mathbb{F}^{n\times n})$, $P_{22} \in \mathcal{T}_s(\mathbb{F}^{n\times m})$. Since W_{11} and W are invertible (see the remarks below Theorem 1), so are P_{12} and P_{21}. It can be verified then that

$$\begin{bmatrix} M + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is a unitary matrix.

It remains to show (a) $\Rightarrow$ (c). As it is well-known, the inertia of a Hermitian matrix H is an ordered triple $\{\pi_+(H), \pi_-(H), \pi_0(H)\}$ of positive numbers, where $\pi_+(H), \pi_-(H), \pi_0(H)$ are numbers of positive, negative, and zero eigenvalues of H respectively, all counting multiplicities. In the following, we denote G^*JG by H and prove two claims related to H.

Claim 1 *Matrices*

$$\begin{bmatrix} [H_{ij}]_{i=k,j=k}^{l,l} & [H_{ij}]_{i=k,j=l+k}^{l,2l} \\ [H_{ij}]_{i=l+k,j=k}^{2l,l} & [H_{ij}]_{i=l+k,j=l+k}^{2l,2l} \end{bmatrix}, \quad k = 1, 2, \ldots, l,$$

are invertible and their inertias are $\{(l-k+1)m, (l-k+1)n, 0\}$ *respectively.*

Proof: The claim is obviously true when $k = 1$. Now assume $2 \le k \le l$. Since

$$H = G^*JG = \begin{bmatrix} I & M \\ M^* & M^*M - I \end{bmatrix},$$

we have

$$\begin{aligned}
&\begin{bmatrix} [H_{ij}]_{i=k,j=k}^{l,l} & [H_{ij}]_{i=k,j=l+k}^{l,2l} \\ [H_{ij}]_{i=l+k,j=k}^{2l,l} & [H_{ij}]_{i=l+k,j=l+k}^{2l,2l} \end{bmatrix} \\
&= \begin{bmatrix} I & [M_{ij}]_{i=k,j=k}^{l,l} \\ ([M_{ij}]_{i=k,j=k}^{l,l})^* & ([M_{ij}]_{i=1,j=k}^{l,l})^*[M_{ij}]_{i=1,j=k}^{l,l} - I \end{bmatrix} \\
&= \begin{bmatrix} I & [M_{ij}]_{i=k,j=k}^{l,l} \\ ([M_{ij}]_{i=k,j=k}^{l,l})^* & ([M_{ij}]_{i=1,j=k}^{k-1,l})^*[M_{ij}]_{i=1,j=k}^{k-1,l} + ([M_{ij}]_{i=k,j=k}^{l,l})^*[M_{ij}]_{i=k,j=k}^{l,l} - I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ ([M_{ij}]_{i=k,j=k}^{l,l})^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & ([M_{ij}]_{i=1,j=k}^{k-1,l})^*[M_{ij}]_{i=1,j=k}^{k-1,l} - I \end{bmatrix} \begin{bmatrix} I & [M_{ij}]_{i=k,j=k}^{l,l} \\ 0 & I \end{bmatrix}.
\end{aligned}$$

Since $\|[M_{ij}]_{i=1,j=k}^{k-1,l}\| < 1$, the claim follows immediately. □

Claim 2 *Matrices*

$$\begin{bmatrix} [H_{ij}]_{i=k,j=k}^{l,l} & [H_{ij}]_{i=k,j=l+k+1}^{l,2l} \\ [H_{ij}]_{i=l+k+1,j=k}^{2l,l} & [H_{ij}]_{i=l+k+1,j=l+k+1}^{2l,2l} \end{bmatrix}, \quad k = 1, 2, \ldots, l-1,$$

are invertible and their inertias are $\{(l-k+1)m, (l-k)n, 0\}$.

Proof: Following the same argument as in the proof of Claim 1, we can show that

$$\begin{bmatrix} [H_{ij}]_{i=k,j=k}^{l,l} & [H_{ij}]_{i=k,j=l+k+1}^{l,2l} \\ [H_{ij}]_{i=l+k+1,j=k}^{2l,l} & [H_{ij}]_{i=l+k+1,j=l+k+1}^{2l,2l} \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ ([M_{ij}]_{i=k,j=k+1}^{l,l})^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & ([M_{ij}]_{i=1,j=k+1}^{k-1,l})^*[M_{ij}]_{i=1,j=k+1}^{k-1,l} - I \end{bmatrix} \begin{bmatrix} I & [M_{ij}]_{i=k,j=k+1}^{l,l} \\ 0 & I \end{bmatrix}.$$

Since $[M_{ij}]_{i=1,j=k+1}^{k-1,l}$ is a submatrix of $[M_{ij}]_{i=1,j=k+1}^{k,l}$, we also have $\|[T_{ij}]_{i=1,j=k+1}^{k-1,l}\| < 1$. The claim thus follows. □

Now let us permute the rows and columns of H and J to form

$$\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{21}^* & \cdots & \tilde{H}_{l1}^* \\ \tilde{H}_{21} & \tilde{H}_{22} & \cdots & \tilde{H}_{l2}^* \\ \vdots & \vdots & & \vdots \\ \tilde{H}_{l1} & \tilde{H}_{l2} & \cdots & \tilde{H}_{ll} \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} \tilde{J}_1 & 0 & \cdots & 0 \\ 0 & \tilde{J}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{J}_l \end{bmatrix}$$

where

$$\tilde{H}_{ij} = \begin{bmatrix} H_{ij} & H_{i(l+j)} \\ H_{(l+i)j} & H_{(l+i)(l+j)} \end{bmatrix}, \quad \tilde{J}_i = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

With this permutation, the desired factorization becomes $\tilde{W}^*\tilde{J}\tilde{W} = \tilde{H}$ where $\tilde{W}$ belongs to $\mathcal{T}(\mathsf{F}^{(m+n)\times(m+n)})$:

$$\tilde{W} = \begin{bmatrix} \tilde{W}_{11} & 0 & \cdots & 0 \\ \tilde{W}_{21} & \tilde{W}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{W}_{l1} & \tilde{W}_{l2} & \cdots & \tilde{W}_{ll} \end{bmatrix}.$$

Further partition gives

$$\tilde{W}_{ij} = \begin{bmatrix} (\tilde{W}_{ij})_{11} & (\tilde{W}_{ij})_{12} \\ (\tilde{W}_{ij})_{21} & (\tilde{W}_{ij})_{22} \end{bmatrix}.$$

Hence $W_{21} \in \mathcal{T}_s(\mathsf{F}^{n\times m})$ implies $(\tilde{W}_{ii})_{21} = 0$.

Claim 3 *Matrices*

$$\tilde{H}_{kk} - [\tilde{H}_{kj}]_{j=k+1}^{l}([\tilde{H}_{ij}]_{i=k+1,j=k+1}^{l,l})^{-1}[\tilde{H}_{ik}]_{i=k+1}^{l}, \quad k = 1, 2, \ldots, l-1,$$

are invertible and their inertias are $\{m, n, 0\}$ *respectively. If these matrices are further partitioned into* 2×2 *block matrices with* $m \times m$ *(1,1) blocks, then their (1,1) blocks are positive definite.*

Proof: By Claim 1, $[\tilde{H}_{ij}]_{i=k,j=k}^{l,l}$ and $[\tilde{H}_{ij}]_{i=k+1,j=k+1}^{l,l}$ are invertible and their inertias are $\{(l-k+1)m, (l-k+1)n, 0\}$ and $\{(l-k)m, (l-k)n, 0\}$ respectively. Write

$$[\tilde{H}_{ij}]_{i=k,j=k}^{l,l} = \begin{bmatrix} \tilde{H}_{kk} & [\tilde{H}_{kj}]_{j=k+1}^{l} \\ [\tilde{H}_{ik}]_{i=k+1}^{l} & [\tilde{H}_{ij}]_{i=k+1,j=k+1}^{l,l} \end{bmatrix} =: \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}.$$

Note that

$$\begin{bmatrix} A & B^* \\ B & C \end{bmatrix} = \begin{bmatrix} I & B^*C^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - B^*C^{-1}B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & 0 \\ C^{-1}B & I \end{bmatrix}.$$

Hence $\tilde{H}_{kk} - [\tilde{H}_{kj}]_{j=k+1}^{l}([\tilde{H}_{ij}]_{i=k+1,j=k+1}^{l,l})^{-1}[\tilde{H}_{ik}]_{i=k+1}^{l}$ is invertible and its inertia is $\{m, n, 0\}$.

If we apply the same argument to matrix

$$\begin{bmatrix} H_{kk} & [H_{kj}]_{j=k+1}^{l} \\ [H_{ik}]_{i=k+1}^{l} & [\tilde{H}_{ij}]_{i=k+1,j=k+1}^{l,l} \end{bmatrix}$$

and notice that its inertia is $\{(l-k+1)m, (l-k)n, 0\}$ (Claim 2), then we see that the inertia of $H_{kk} - [H_{kj}]_{j=k+1}^{l}([\tilde{H}_{ij}]_{i=k+1,j=k+1}^{l,l})^{-1}[H_{ik}]_{i=k+1}^{l}$ is $\{m, 0, 0\}$. This matrix is exactly the (1,1) block of $\tilde{H}_{kk} - [\tilde{H}_{kj}]_{j=k+1}^{l}([\tilde{H}_{ij}]_{i=k+1,j=k+1}^{l,l})^{-1}[\tilde{H}_{ik}]_{i=k+1}^{l}$.

Suppose now that we can carry out the following computation:

For i from l to 1,

find $\tilde{W}_{ii}$ with $(W_{ii})_{21} = 0$ such that

$$\tilde{W}_{ii}^*\tilde{J}_i\tilde{W}_{ii} = \begin{cases} \tilde{H}_{ll} & \text{if } i = l \\ \tilde{H}_{ii} - \begin{bmatrix} \tilde{W}_{(i+1)i}^* & \cdots & \tilde{W}_{li}^* \end{bmatrix} \begin{bmatrix} \tilde{J}_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{J}_l \end{bmatrix} \begin{bmatrix} \tilde{W}_{(i+1)i} \\ \vdots \\ \tilde{W}_{li} \end{bmatrix} & \text{if } i < l; \end{cases} \tag{8}$$

for $j = 1, \ldots, i-1$, let

$$\tilde{W}_{ij} = \tilde{J}_i\tilde{W}_{ii}^{*-1}\left(\tilde{H}_{ij} - \sum_{k=i+1}^{l} \tilde{W}_{ki}^*\tilde{J}_k\tilde{W}_{kj}\right). \tag{9}$$

end

end

Then we obtain all $\tilde{W}_{ij}$ for $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, i$, and it is straightforward to check that we have $\tilde{W}^*\tilde{J}\tilde{W} = \tilde{H}$. In order to show that the above computation can be carried out, we need to show that the factorization in (8) can be done and $\tilde{W}_{ii}$, $i = 1, 2, \ldots, l$, are invertible. For this purpose, a technical lemma is needed.

Lemma 1 *Given a nonsingular Hermitian matrix* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \in \mathbb{F}^{(m+n)\times(m+n)}$ *and* $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathbb{F}^{(m+n)\times(m+n)}$, *there exists* $B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \in \mathbb{F}^{(m+n)\times(m+n)}$ *such that* $B^*JB = A$ *if and only if* $A_{11} > 0$ *and the inertia of* A *is* $\{m, n, 0\}$.

Proof: The necessity is obvious since $B_{11}^* B_{11} = A_{11}$ and the inertia is invariant under congruence. To see the sufficiency, notice that if $A_{11} > 0$, then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{12}^* A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{12}^* A_{11}^{-1} A_{12} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} \\ 0 & I \end{bmatrix}.$$

The inertia of A being $\{m, n, 0\}$ implies that $A_{22} - A_{12}^* A_{11}^{-1} A_{12} < 0$. Hence we can define

$$B_{11} = A_{11}^{\frac{1}{2}}, \quad B_{12} = A_{11}^{-\frac{1}{2}} A_{12}, \quad B_{22} = (A_{12}^* A_{11}^{-1} A_{12} - A_{22})^{\frac{1}{2}}.$$

With this definition, $B^* J B = A$ is satisfied. □

By this lemma, it becomes obvious that when $i = l$ the factorization in (8) can be done and $\tilde{W}_{ll}$ is invertible. When $i < l$, we have

$$\begin{bmatrix} \tilde{W}_{(i+1)(i+1)}^* & \cdots & \tilde{W}_{l(i+1)}^* \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{W}_{ll}^* \end{bmatrix} \begin{bmatrix} \tilde{J}_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{J}_l \end{bmatrix} \begin{bmatrix} \tilde{W}_{(i+1)i} \\ \vdots \\ \tilde{W}_{li} \end{bmatrix} = \begin{bmatrix} \tilde{H}_{(i+1)i} \\ \vdots \\ \tilde{H}_{li} \end{bmatrix}.$$

Then

$$\begin{aligned}
&\tilde{H}_{ii} - \begin{bmatrix} \tilde{W}_{(i+1)i}^* & \cdots & \tilde{W}_{li}^* \end{bmatrix} \begin{bmatrix} \tilde{J}_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{J}_l \end{bmatrix} \begin{bmatrix} \tilde{W}_{(i+1)i} \\ \vdots \\ \tilde{W}_{li} \end{bmatrix} \\
&= \tilde{H}_{ii} - \begin{bmatrix} \tilde{H}_{(i+1)i}^* & \cdots & \tilde{H}_{li}^* \end{bmatrix} \begin{bmatrix} \tilde{W}_{(i+1)(i+1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \tilde{W}_{l(i+1)} & \cdots & \tilde{W}_{ll} \end{bmatrix}^{-1} \\
&\qquad \begin{bmatrix} \tilde{J}_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{J}_l \end{bmatrix} \begin{bmatrix} \tilde{W}_{(i+1)(i+1)}^* & \cdots & \tilde{W}_{l(i+1)}^* \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{W}_{ll}^* \end{bmatrix}^{-1} \begin{bmatrix} \tilde{H}_{(i+1)i} \\ \vdots \\ \tilde{H}_{li} \end{bmatrix} \\
&= \tilde{H}_{ii} - \begin{bmatrix} \tilde{H}_{(i+1)i}^* & \cdots & \tilde{H}_{li}^* \end{bmatrix} \begin{bmatrix} \tilde{H}_{(i+1)(i+1)} & \cdots & \tilde{H}_{l(i+1)}^* \\ \vdots & & \vdots \\ \tilde{H}_{l(i+1)} & \cdots & \tilde{H}_{ll} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{H}_{(i+1)i} \\ \vdots \\ \tilde{H}_{li} \end{bmatrix}.
\end{aligned}$$

It then follows from Claim 3 that the factorization in (8) can be carried out and the resulting $\tilde{W}_{ii}$ are invertible.

After we get $\tilde{W}$, certain row and column permutations will give us W which satisfies $W^* J W = G^* J G$ in the standard matrix representation.

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