Proceedings of the 37th IEEE
Conference on Decision & Control
Tampa, Florida USA • December 1998

WA13 10:00

Controllable Regions of LTI Discrete-time Systems with Input Saturation[1]

Tingshu Hu[2 3] Daniel E. Miller[2 4] Li Qiu[5]

Abstract

In this paper, we present a formula to compute the vertices of the (null) controllable regions for general LTI discrete-time systems with bounded inputs. For n^{th} order systems with only real poles (not necessarily distinct), the formula is simplified to an elementary matrix function, which can be used to show that the set of vertices coincides with a class of time responses of the time-reversed system to bang-bang controls with $n-2$ or less switches. For second-order systems with a pair of complex conjugate poles, a closed form formula to compute the vertices is provided; the set of vertices can also be obtained from the steady state response of the time-reversed system to a periodic or near periodic bang-bang control. The influence of the sampling period on the controllable regions is clearly demonstrated with some examples. A preliminary investigation is made on the existence of nonlinear controllers and the non-existence of linear controllers to achieve certain stabilization tasks.

1 Introduction

The problem to be studied in this paper was formulated several decades ago. The definition of (null) controllable region in this paper is similar to those in [1, 6, 5, 17]. In the 50's and earlier 60's, constrained control was a widely studied topic. It is closely related to time optimal control, e.g., see [5]. This study was continued in the 70's and 80's, e.g., see [6, 12, 14]. The most common type of control constraint is input saturation, i.e., the input is bounded by the ∞-norm. The well known bang-bang control principle was developed for this kind of constrained control.

Recently there is a renewed interest in studying the control of systems with constrained inputs. Great progress has been made in the past few years, e.g., see [15, 16, 7, 11].

There are important differences between the earlier control strategies and the recent developments. Earlier work was mostly aimed at time optimal control and there was typically a heavy on-line computational burden. Recent studies try to achieve global or semi-global asympototic stability, disturbance attenuation, tracking, robustness, etc, with simple feedback control, such as the saturated linear feedback, which is very easy to implement. However, it should be noted that a common assumption in most recent papers is that the open-loop system is *semi-stable*[1]. This assumption was made to guarantee the existence of a global or a semi-global stabilizer, since a semi-stable LTI system controllable in the usual sense is globally controllable with bounded inputs, e.g., see [14].

For strictly unstable systems that have poles outside the unit circle (or in the open right half of the complex plane for continuous-time systems), however, the existing results are quite limited. Just as the controllability result of [8, 12, 14] paved the way for the development of stabilization theory for semi-stable systems with bounded inputs, to achieve easily implementable and nice control for strictly unstable systems, simple and exact descriptions of the controllable regions are required. Along this direction, some nice results have been recently established for continuous-time systems in [3] and [4]. In these papers, we first gave simple exact descriptions for the controllable regions of certain classes of unstable continuous-time systems, and then we showed that for a system with only two anti-stable modes, a saturated linear state feedback can be designed so that any given region in the interior of the controllable region is in the domain of attraction.

As usual, one might anticipate that the results in the continuous-time setting have their counterparts in the discrete-time setting. Indeed, we will show in this paper that through some interesting links, some of the controllability results in [3] have natural discrete-time counterparts, though the development is more technically involved.

Our ultimate goal is to use the newly developed controllable results to design a practical and simple feedback controller to achieve a desired stability region and performance. In Section 4 of this paper, some prelimi-

[1]This work was supported in part by the Natural Sciences and Engineering Research Council of Canada via a research grant.

[2]Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1.
[3]Email: thu@kingcong.uwaterloo.ca
[4]Email: miller@hobbes.uwaterloo.ca
[5]Department of Electrical & Electronic Engineering, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. Email: eeqiu@ee.ust.hk

[1]Here we say that a discrete-time system is *semi-stable* if it has no poles outside the unit circle and a continuous-time system is *semi-stable* if it has no poles in the open right half of the complex plane.

nary investigation is made on this subject.

2 Preliminaries and Notation

Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in \mathbf{R}^n$ is the state and $u(k) \in \mathbf{R}^m$ is the control. A control signal u is said to be *admissible* if $\|u(k)\|_\infty \leq 1$ for all integers $k \geq 0$. In this paper, we are interested in the control of system (1) by using admissible controls.

Definition 1 *A state x_0 is said to be (null) controllable at a given step K if there exists an admissible control u such that the time response x of the system satisfies $x(0) = x_0$ and $x(K) = 0$; a state x_0 is said to be (null) controllable if it is (null) controllable at some $K < \infty$.*

For simplicity, we will drop the word "null" before "controllable" in the rest part of this paper.

Definition 2 *The set of all states controllable at K is called the controllable region of the system at K and is denoted by $\mathcal{C}(K)$; the set of all controllable states is called the controllable region of the system and is denoted by $\mathcal{C}$.*

In this paper, we say that a matrix A is *semi-stable* if it has no eigenvalues outside of the unit circle and A is *anti-stable* if all of its eigenvalues are outside of the unit circle.

Proposition 1 *Assume that (A, B) is controllable.*

(a) *If A is semi-stable, then $\mathcal{C} = \mathbf{R}^n$.*

(b) *If A is anti-stable, then $\mathcal{C}$ is a bounded convex open set containing the origin.*

(c) *If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with $A_1 \in \mathbf{R}^{n_1 \times n_1}$ being anti-stable and $A_2 \in \mathbf{R}^{n_2 \times n_2}$ being semi-stable, and B is partitioned as $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ accordingly, then $\mathcal{C} = \mathcal{C}_1 \times \mathbf{R}^{n_2}$ where $\mathcal{C}_1$ is the controllable region of the anti-stable subsystem $x_1(k+1) = A_1 x_1(k) + B_1 u(k)$.*

Statements (a) and (b) are well-known [14, 5]. Statement (c) are proved in [2]. Because of this proposition, we can concentrate on the study of controllable regions of anti-stable systems. For such systems $\mathcal{C}$ can be approximated by $\mathcal{C}(K)$ for sufficiently large K.

If $B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}$ and the controllable region of the system $x(k+1) = Ax(k) + b_i u_i(k)$, $i = 1, \cdots, m$, is $\mathcal{C}_i$, then

$$\mathcal{C} = \sum_{i=1}^{m} \mathcal{C}_i = \{\sum_{i=1}^{m} x_i : x_i \in \mathcal{C}_i, i \leq m\}.$$

Hence we can begin our study of the controllable regions with single-input systems.

Therefore, in the rest of the paper we will assume that (A, B) is controllable, A is anti-stable, and $m = 1$.

In many situations, it may be more convenient to study the controllability of a system through the reachability of its time-reversed system. The time reversed system of (1) is

$$z(k+1) = A^{-1} z(k) - A^{-1} B v(k). \tag{2}$$

Note that we have assumed that A is anti-stable, so A is invertible. We see that $x(k)$ satisfies (1) with $x(0) = x_0$, $x(k_1) = x_1$, and a given u if and only if $z(k) := x(k_1 - k)$ satisfies (2) with $z(0) = x_1$, $z(k_1) = x_0$, and $v(k) = u(k_1 - k - 1)$. So the two systems have the same set of points as trajectories, but traverse in opposite directions.

Definition 3 *For the system (2), a state z_f is said to be reachable at a given step K if there exists an admissible control v such that the time response z of the system (2) satisfies $z(0) = 0$ and $z(K) = z_f$; a state z_f is said to be reachable if it is reachable at some $K < \infty$.*

Definition 4 *For the system (2), the set of all states reachable at K is called the reachable region of the system (2) at K and is denoted by $\mathcal{R}(K)$; the set of all reachable states is called the reachable region of the system (2) and is denoted by $\mathcal{R}$.*

It is known that $\mathcal{C}(K)$ and $\mathcal{C}$ of (1) are the same as $\mathcal{R}(K)$ and $\mathcal{R}$ of (2), e.g., see [8]. To avoid confusion, we will reserve the notation x, u, $\mathcal{C}(K)$, and $\mathcal{C}$ for the original system (1), and reserve z, v, $\mathcal{R}(K)$, and $\mathcal{R}$ for the time-reversed system (2).

To proceed we need more notation. With X a polytope in $\mathbf{R}^n$, we use $\mathrm{Vert}(X)$ to denote the set of vertices of X. In this paper, the notion of polytope will be extended to include the convex hull of a countable number of vertices in a bounded region. With K_1 and K_2 integers, unless otherwise noted we let $[K_1, K_2]$ denote the set of integers $\{K_1, K_1 + 1, ..., K_2\}$.

3 Controllable Regions

3.1 Description of the controllable region via vertex control

We have assumed in the last section that A is anti-stable, (A, B) is controllable, and $m = 1$. Since B is now a column vector, we rename it as b for clarity. For technical reasons, we first consider the reachable region $\mathcal{R}(K)$. From Definition 4,

$$\mathcal{R}(K) = \left\{ \sum_{\ell=0}^{K-1} A^{-(K-\ell)} b v(\ell) : |v(\ell)| \leq 1, \ell \in [0, K-1] \right\}.$$

It can be shown from the above equation that $\mathcal{R}(K)$ and $\overline{\mathcal{R}}$ depend on A and b continuously in the Hausdorff metric, even if (A, b) is not controllable in the usual sense.

Definition 5 *An admissible control v is said to be a vertex control on $[0, K]$ if the response $z(k)$ of the system (2) is on $\mathrm{Vert}[\mathcal{R}(k)]$ for all $k \in [0, K]$.*

Lemma 1 *If $z_K \in \mathrm{Vert}[\mathcal{R}(K)]$ and v is an admissible control that steers the state from the origin to z_K at step K, then v is a vertex control on $[0, K]$.*

Denote the set of vertex controls on $[0, K]$ as $\mathcal{V}(K)$. It follows that

$$\mathrm{Vert}[\mathcal{R}(K)] = \left\{ \sum_{\ell=0}^{K-1} A^{-(K-\ell)} b v(\ell) : \ v \in \mathcal{V}(K) \right\}.$$

Lemma 2 *[10] An admissible control v^* is a vertex control on $[0, K]$ for the system (2) if and only if there is a nonzero vector $c \in \mathbf{R}^n$ such that $c'A^k b \neq 0$ for all $k \in [0, K-1]$ and $v^*(k) = \mathrm{sgn}(c'A^k b)$ for $k \in [0, K-1]$.*

Therefore,

$$\mathcal{V}(K) = \{v(k) = \mathrm{sgn}(c'A^k b), k \in [0, K-1] : c'A^k b \neq 0 \ \ \forall k \in [0, K-1]\}. \quad (3)$$

So a vertex control is a bang-bang control, i.e., a control that only takes value in $\{1, -1\}$. Using some algebraic manipulations, we can prove

Theorem 1

$$\mathrm{Vert}[\mathcal{R}(K)] = \left\{ \sum_{\ell=1}^{K} A^{-\ell} b \, \mathrm{sgn}(c'A^{-\ell} b) : c'A^{-\ell} b \neq 0 \ \ \forall \ell \in [1, K] \right\}$$

$$\mathrm{Vert}[\overline{\mathcal{R}}] = \left\{ \sum_{\ell=1}^{\infty} A^{-\ell} b \, \mathrm{sgn}(c'A^{-\ell} b) : c'A^{-\ell} b \neq 0 \ \ \forall \ell \geq 1 \right\}.$$

Since $\mathrm{sgn}(c'A^{-\ell}b) = \mathrm{sgn}(\gamma c'A^{-\ell}b)$ for any positive number γ, this formula shows that $\mathrm{Vert}[\overline{\mathcal{R}}]$ can be determined from the surface of a unit ball. It should be noted that each vertex corresponds to a region in the surface of this unit ball rather than just one point. This formula provides a straightforward method to compute the vertices of the controllable region and no optimization is involved. In the following, we will give more attractive formulae to compute the vertices of the controllable regions for some classes of systems.

3.2 Systems with only real eigenvalues

For this kind of discrete-time system, more technical consideration is necessary as compared with a continuous-time system. This difference can be illustrated through a simple example. If $A = -2$, then $c'A^k b$ changes sign at each k. Hence, if A has some negative real eigenvalues, a vertex control can have infinitely many switches. This complexity can be avoided through a technical manipulation. Suppose that A has only real eigenvalues, including some negative ones, and consider

$$y(k+1) = A^2 y(k) + \begin{bmatrix} Ab & b \end{bmatrix} w(k) \quad (4)$$

where $y(k) = x(2k)$ and $w(k) = \begin{bmatrix} u(2k) \\ u(2k+1) \end{bmatrix}$. Then the controllable region of (1) is the same as that of (4), which is the sum of the controllable regions of the following two subsystems:

$$y(k+1) = A^2 y(k) + Abw_1(k)$$

and

$$y(k+1) = A^2 y(k) + bw_2(k).$$

Therefore, without loss of generality, in this section we further assume that A has only positive real eigenvalues. Under this assumption, it is known that any vertex control can have at most $n-1$ switches[10]. It can be shown that the converse is also true. That is, any bang-bang control with $n-1$ or less switches is a vertex control; the proof is based on [3].

Lemma 3 *: For the system (2), suppose that A has only positive real eigenvalues. Then*

(a) *a vertex control has at most $n-1$ switches;*

(b) *any bang-bang control with $n-1$ or less switches is a vertex control.*

It follows from the above lemma that the set of vertex controls on $[0, K]$ can be described as follows:

$$\mathcal{V}(K) = \left\{ \pm v \ : \ v(k) = \begin{cases} 1 & , \quad 0 \leq k < k_1 \\ (-1)^i & , \quad k_i \leq k < k_{i+1} \\ (-1)^{n-1} & , \ k_{n-1} \leq k \leq K-1 \end{cases} , \quad 0 \leq k_1 \leq \cdots \leq k_{n-1} \leq K-1 \right\}.$$

Notice that we allow $k_i = k_{i+1}$ here, so $\mathcal{V}(K)$ includes all the bang-bang controls with $n-1$ or less switches. From the equality $\sum_{k=k_1}^{k_2-1} X^k = (X^{k_1} - X^{k_2})(I - X)^{-1}$, we have

$$\begin{aligned} &\mathrm{Vert}[\mathcal{R}(K)] \\ &= \left\{ \sum_{\ell=0}^{K-1} A^{-(K-\ell)} b v(\ell) : \ v \in \mathcal{V}(K) \right\} \\ &= \left\{ \pm \left[A^{-K} + 2 \sum_{i=1}^{n-1} (-1)^i A^{-\ell_i} + (-1)^n I \right] (I - A)^{-1} b : K \geq \ell_1 \geq \cdots \geq \ell_{n-1} \geq 1 \right\}. \end{aligned}$$

By letting K go to infinity, we get the following theorem.

Theorem 2 *: If A has only real positive eigenvalues, then*

$$\text{Vert}(\overline{\mathcal{R}}) = \left\{ \pm \left[2\sum_{i=1}^{n-1}(-1)^i A^{-\ell_i} + (-1)^n I \right] (I-A)^{-1}b : \right. \\ \left. \infty \geq \ell_1 \geq \cdots \geq \ell_{n-1} \geq 1 \right\}.$$

In particular, for second-order systems, we have,

$$\text{Vert}[\mathcal{R}(K)] = \left\{ \pm [A^{-K} - 2A^{-\ell} + I](I-A)^{-1}b : 1 \leq \ell \leq K \right\}.$$

Hence, there are exactly $2K$ vertices, versus the upperbound of 2^K vertices which emerges from a superficial analysis of $\mathcal{R}(K)$. Furthermore,

$$\text{Vert}(\overline{\mathcal{R}}) = \left\{ \pm(2A^{-\ell} - I)(I-A)^{-1}b : \ 1 \leq \ell \leq \infty \right\}.$$

Similarly, for third-order systems

$$\text{Vert}[\mathcal{R}(K)] = \{\pm[A^{-K} - 2A^{-\ell_1} + 2A^{-\ell_2} - I](I-A)^{-1}b : \\ 1 \leq \ell_2 \leq \ell_1 \leq K\}$$

which has $K(K+1)$ vertices, and

$$\text{Vert}(\overline{\mathcal{R}}) = \{\pm(2A^{-\ell_1} - 2A^{-\ell_2} + I)(I-A)^{-1}b : \\ 1 \leq \ell_2 \leq \ell_1 \leq \infty\}.$$

We can interpret the expressions for $\text{Vert}(\overline{\mathcal{R}})$ as follows. Let $z_e^+ := (I-A)^{-1}b$ be the equilibrium point of the system (2) under the constant control $v(k) \equiv 1$. Then for a second-order system, it can be verified that

$$\text{Vert}(\overline{\mathcal{R}}) = \left\{ \pm \left[A^{-k} z_e^+ + \sum_{\ell=0}^{k-1} A^{-(k-\ell-1)}(-A^{-1}b)(-1) \right] : \right. \\ \left. 1 \leq k \leq \infty \right\}$$

which is exactly the set of points formed by the time-responses of (2) starting from z_e^+ or $-z_e^+$ under the constant control of -1 or $+1$, respectively.

Similarly, for higher-order systems with only positive real eigenvalues, $\text{Vert}(\overline{\mathcal{C}})$ and $\text{Vert}(\overline{\mathcal{R}})$ are *the set of points formed by the time responses of (2) starting from z_e^+ or $-z_e^+$ under any bang-bang control with $n-2$ or less switches.*

3.3 Second-order systems with complex eigenvalues

Assume that $A \in \mathbf{R}^{2\times 2}$ has a pair of complex eigenvalues of the form $r[\cos(\beta) \pm j\sin(\beta)]$, with $r > 1$ and $0 < \beta < \pi$. Then, similar to [3], it can be shown that

$$\left\{ \text{sgn}(c'A^k b) : \ c \neq 0 \right\} = \left\{ \text{sgn}[\sin(\beta k + \theta)] : \ \theta \in [0, 2\pi) \right\}.$$

It follows from (3) that the set of vertex controls for the time reversed system is

$$\mathcal{V}(K) = \{v(k) = \text{sgn}[\sin(\beta k + \theta)], k \in [0, K-1] : \\ \theta \in [0, 2\pi), \sin(\beta k + \theta) \neq 0 \ \ \forall k \in [0, K-1]\}$$

and

$$\text{Vert}[\mathcal{R}(K)] = \left\{ -\sum_{\ell=0}^{K-1} A^{-(K-\ell)} b \ \text{sgn}[\sin(\beta\ell + \theta)] : \right. \\ \left. \theta \in [0, 2\pi), \sin(\beta\ell + \theta) \neq 0 \ \ \forall \ell \in [0, K-1] \right\} \quad (5)$$

First, we consider the case when $\frac{\pi}{\beta}$ is a rational number.

Theorem 3 *Suppose $\beta = \frac{p}{q}\pi$, where p and q are coprime positive integers and $p < q$. Then*

$$\text{Vert}[\mathcal{R}(2q)] = \left\{ \sum_{\ell=0}^{2q-1} A^{-(2q-\ell)} b \ \text{sgn}[\sin(\beta\ell + \frac{\pi}{2q} + \frac{i}{q}\pi)] : \right. \\ \left. i \in [0, 2q-1] \right\} \quad (6)$$

and

$$\text{Vert}(\overline{\mathcal{R}}) = \frac{r^{2q}}{r^{2q}-1} \text{Vert}[\mathcal{R}(2q)].$$

Hence, we can compute the vertices of $\mathcal{R}(2q)$ using (6), and then scale them by $\frac{r^{2q}}{r^{2q}-1}$ to obtain the vertices of $\overline{\mathcal{R}}$.

In parallel to the continuous-time case, $\text{Vert}(\overline{\mathcal{R}})$ also coincides with the time responses of the time-reversed system (2) to some periodic bang-bang control. Let $v^*(k) = \text{sgn}[\sin(\beta k + \frac{\pi}{2q})]$ and $z^*(k)$ be the corresponding time response:

$$z^*(k) = -\sum_{\ell=0}^{k-1} A^{-(k-\ell)} b \ \text{sgn}[\sin(\beta\ell + \frac{\pi}{2q})].$$

Denote $\Gamma(K)$ as

$$\Gamma(K) := \{\pm z^*(K+k) : k \in [0, 2q-1]\}; \quad (7)$$

then $\lim_{K\to\infty} \Gamma(K)$ exists, and this limit is the union of the steady state responses of (2) to $v^*(k)$ and $-v^*(k)$. With $\Gamma = \lim_{K\to\infty} \Gamma(K)$, we have

Proposition 2 *: Suppose that $\beta = \frac{p}{q}\pi$, where p and q are coprime positive integers and $p < q$. Then*

$$\text{Vert}(\overline{\mathcal{C}}) = \text{Vert}(\overline{\mathcal{R}}) = \Gamma.$$

If $\frac{\pi}{\beta}$ is irrational, then $\overline{\mathcal{R}}$ and $\overline{\mathcal{C}}$ will have infinite many vertices. Since $\overline{\mathcal{R}}$ and $\overline{\mathcal{C}}$ depend continuously on the state matrix A, specifically on β, they can be arbitrarily approximated with those having rational $\frac{\pi}{\beta}$. On the other hand, although $v^*(k)$ and $z^*(k)$ are not exactly periodic, $z^*(k)$ has a set of limit points on the state trajectories, which also form the vertex points of $\overline{\mathcal{R}}$ and $\overline{\mathcal{C}}$.

3.4 Examples

The controllable regions of some second-order systems are plotted in the following two examples. By comparing the controllable region of a continuous-time system with those of the discretized systems with different sampling periods, the influence of the sampling period on the controllable regions is clearly demonstrated.

Example 1: The original continuous time system is

$$\dot{x}(t) = A_c x(t) + b_c u(t) = \begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{8}$$

Let h be the sampling period, $A = e^{A_c h}$ and $b = A_c^{-1}(e^{A_c h} - I)b_c$. Consider the discrete-time system

$$x(k+1) = Ax(k) + bu(k) \tag{9}$$

under different h. From Theorem 2 we have

$$\mathrm{Vert}[\overline{\mathcal{C}}] = \{\pm(I - 2A^{-k})(I - A)^{-1}b : \ k = 0, 1, 2, \cdots\}.$$

The boundaries of $\overline{\mathcal{C}}$ corresponding to different sampling periods $h = 0.1, 0.2, 1, 2, 4, 8$ are plotted in Figure 1. When $h = 0.1$, $\mathcal{C}$ is very close to $\mathcal{C}_c$ of the continuous-time system; when $h = 8$, $\mathcal{C}$ is diminished to a narrow strip.

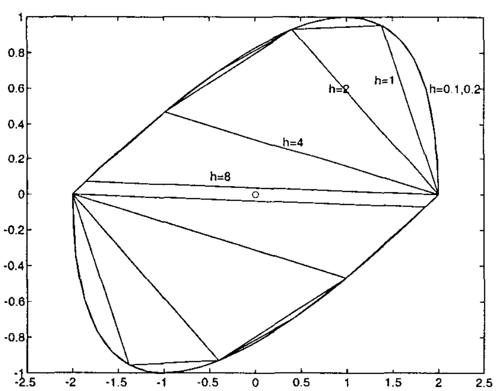

Figure 1: Controllable regions under different sampling periods

Example 2: The original continuous-time system is

$$\dot{x}(t) = A_c x(t) + b_c u(t) = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u(t).$$

With sampling period h, the discrete-time system is

$$x(k+1) = Ax(k) + bu(k) \tag{10}$$

where $A = e^{0.6h} \begin{bmatrix} \cos(0.8h) & -\sin(0.8h) \\ \sin(0.8h) & \cos(0.8h) \end{bmatrix}$ and $b = (e^{A_c h} - I)A_c^{-1} b_c$. With q chosen as in Proposition 2, the controllable regions of (10) under different sampling periods are obtained from the steady state responses of its time-reversed system under a control of $\mathrm{sgn}[\sin(0.8hk + \frac{\pi}{2q})]$.

For $h = \frac{\pi}{0.8}(\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9})$, the controllable regions are plotted in Figure 2 from the outermost to the innermost.

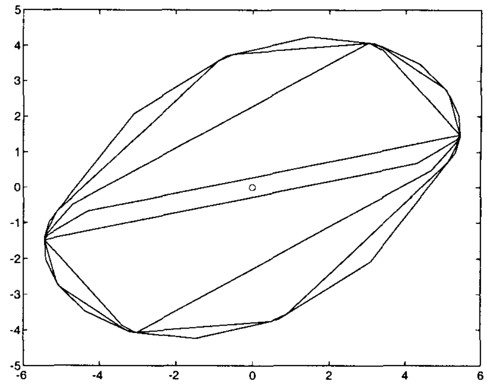

Figure 2: Controllable regions under different sampling periods

4 Controller Synthesis

In this section we briefly discuss some preliminary results on the existence of nonlinear controllers and the non-existence of linear controllers to achieve certain tasks. We will consider two control objectives: (i) keeping x bounded and (ii) forcing x to zero, in each case while using an admissible control. Clearly the former can be achieved only if the initial condition $x_0 \in \overline{\mathcal{C}}$, while the latter can be achieved only if $x_0 \in \mathcal{C}$. In this section, we use $[-1, 1]$ to denote the closed real interval.

4.1 Nonlinear Controllers

Using a standard approach from recent work on nonlinear l_1-optimal control of LTI systems, e.g., see [9, 13], we can prove

Theorem 4 *There exists a nonlinear state feedback controller*

$$u(k) = F[x(k)] \tag{11}$$

which has the property that for every initial condition $x_0 \in \overline{\mathcal{C}}$, the closed loop system consisting of (1) and (11) has the property that u is admissible and $x(k) \in \overline{\mathcal{C}}$ for all $k \geq 0$.

It would typically be desirable that the state go to zero. To this end, with $\lambda \in (0, 1)$, let us define the following subset of $\mathcal{C}(K)$:

$$\mathcal{C}_\lambda(K) := \{\sum_{i=1}^{K} \lambda^i A^{-i} b v(i) : |v(i)| \leq 1\};$$

it is easy to see that $\mathcal{C}_\lambda(K)$ is the controllable region of

$$x(k+1) = \lambda^{-1}Ax(k) + bu(k)$$

at step K and clearly $\lim_{\lambda\to 1}\mathcal{C}_\lambda(K) = \mathcal{C}(K)$. This brings us to

Theorem 5 *Fix $\lambda \in (0,1)$ and $K \in \mathbf{N}$. Then there exists a continuous nonlinear state feedback controller*

$$u(k) = F[x(k)] \tag{12}$$

which has the property that for every initial condition $x_0 \in \mathcal{C}_\lambda(K)$, the closed loop system consisting of (1) and (12) has the property that u is admissible and $x(k) \to 0$ as $k \to 0$.

The above controllers tend to be quite complex, so it would be convenient if the same type of results could be obtained using linear control laws.

4.2 Linear Controllers

Our results to date are restricted to the problem of keeping x bounded. Suppose that we have a linear controller of the form

$$\begin{aligned} w(k+1) &= Jw(k) + Hx(k), \;\; w(0)=0 \\ u(k) &= Gw(k) + Fx(k), \end{aligned}$$

and the goal is to ensure that if x_0 lies in some subset $\mathcal{S} \subset \overline{\mathcal{C}}$, then u is admissible and $x(k) \in \mathcal{S}$ for all $k \geq 0$. Notice that in closed loop

$$x(1) = (A + bF)x_0,$$

so clearly we need

$$(A+bF)\mathcal{S} \subset \mathcal{S}, \;\; F\mathcal{S} \subset [-1,1].$$

Hence, we may as well restrict ourselves to linear state feedback, in which case our goal is to find an F so that

$$(A+bF)\mathcal{S} \subset \mathcal{S}, \;\; F\mathcal{S} \subset [-1,1]. \tag{13}$$

Theorem 6 *Suppose that $n \geq 2$ and that A has only positive real eigenvalues greater than one. With $S = \overline{\mathcal{C}}$, there does not exist an $F \in \mathbf{R}^{1\times n}$ so that (13) hold.*

5 Concluding Remarks

In this paper, we presented some simple formulae to compute the controllable regions for LTI discrete-time systems. A preliminary investigation into controller design was carried out. We showed that there exists a nonlinear controller to keep the state within $\overline{\mathcal{C}}$, the closure of the controllable region, although this controller is quite complicated. We also showed that for a certain class of systems there exists no linear feedback controller to keep the state within $\overline{\mathcal{C}}$. Our next objective is to seek a controller with a simple nonlinearity, such as saturated linear feedback in the continuous-time case[4], to make the stability region of the closed-loop system close to $\overline{\mathcal{C}}$.

References

[1] C. A. Desoer and J. Wing, "An optimal strategy for a saturating sampled data system", *IRE Trans. Automat. Contr.*, AC-6, pp. 5-15, 1961.

[2] O. Hájek, *Control Theory in the Plane*, Springer-Verlag, 1991.

[3] T. Hu, L. Qiu and Z. Lin,"Controllability and stabilization of unstable LTI systems with input saturation", Proc. of the 36th IEEE CDC, pp. 4498-4503, 1997.

[4] T. Hu, L. Qiu and Z. Lin,"Stabilization of LTI systems with planar antistable dynamics using saturated linear feedback", Proc. of the 37th IEEE CDC, 1998.

[5] E. Lee and L. Markus, *Foundations of Optimal Control*, New York, Wiley, 1967.

[6] J. N. Lin, "Determination of reachable set for a linear discrete system,"*IEEE Trans. Automat. Contr.*, AC-15, pp. 339-342, 1970.

[7] Z. Lin and A. Saberi, "A semi-global low-and-high design for linear systems with input saturation: stabilization and disturbance rejection", *Int. J. of Robust and Nonlinear Control*, 5, pp. 381-398, 1995.

[8] J. Macki and M. Strauss, *Introduction to Optimal Control*, Springer-Verlag, 1982.

[9] D. E. Miller and V. M. Lucic, "Low order nonlinear controllers for the rejection of persistent bounded disturbances", submitted to *Automatica*.

[10] X. Qian and J. Song, *Engineering Cybernetics*, Science Academics, Beijing, 1980. (in Chinese).

[11] A. Saberi, Z. Lin and A. R. Teel, "Control of linear systems with saturating actuators", *IEEE Trans. Automat. Contr.* AC-41, pp. 368-378, 1996.

[12] W. E. Schmitendorf and B. R. Barmish, "Null controllability of linear systems with constrained controls", *SIAM J. Control and Optimization*, Vol.18, pp. 327-345,1980.

[13] J. S. Shamma, "Nonlinear state feedback for l_1 optimal control", *Systems & Control Letters*, Vol. 21, pp. 265-270, 1993.

[14] E. D. Sontag, "An algebraic approach to bounded controllability of linear systems", *Int. J. Control*, Vol.39, pp. 181-188, 1984

[15] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls", *IEEE Trans. Automat. Contr.* AC-39, pp. 2411-2425, 1994.

[16] A. R. Teel, "Global stabilization and restricted tracking for multiple integrators with bounded controls", *Systems & Control Letters*, Vol.18, pp. 165-171, 1992.

[17] J. Wing and C. A. Desoer, "The multiple-input minimal time regulator problem (general theory)", *IEEE Trans. Automat. Contr.* AC-8, pp. 125-136, 1963.