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Controllable regions of linear systems with bounded inputs [1]

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Abstract

It is known that the controllable region of a general unstable system with bounded control is the Cartesian product of the controllable region of its subsystem with antistable modes and that of its subsystem with stable and marginally stable modes. While the controllable region of a system with only stable and marginally stable modes is well known to be the whole state space, that of an antistable system is studied in this paper. A necessary and sufficient condition for a state of an antistable system to be controllable is given. The boundary of the controllable region is characterized.

1. Introduction

There has been a surge of activity recently on the control of linear systems whose inputs have a priori bounds, see, e.g., [10–12, 2, 6]. This problem is well motivated from a practical point of view since there is hardly any actuator which does not saturate when applied with excessive command. Notice that such systems are not linear systems since the input space is not a linear space although their dynamics is linear. A fundamental issue for such systems is their controllability, namely, the characterization of those states which can be controlled to the origin by using control signals which are within the prespecified bound. Such states are said to be controllable. Clearly, an uncontrollable state cannot be made to belong to the domain of contraction of the closed loop system no matter what feedback controller is used. It is then of interest to investigate the set of all controllable states. This is the theme of this paper.

Consider a discrete-time system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

or a continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control. Here and in the sequel, the time variable t or T may take values in $\mathbb{Z}_+$ or $\mathbb{R}_+$, depending on the context. We assume that (A, B) is controllable. Let Ω be a compact convex set in $\mathbb{R}^m$ containing 0 in its interior. A control signal u is said to be *admissible* if $u(t) \in \Omega$ for all $t \geqslant 0$.

Definition 1. (a) A state x_0 is said to be controllable at a given time (or step) T if there exists an admissible control u such that the state trajectory x of the system satisfies $x(0) = x_0$ and $x(T) = 0$. (b) A state x_0 is said to be controllable if it is controllable at some $T < \infty$.

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Definition 2. (a) The set of all controllable states x_0 at T is called the controllable region at T and is denoted by $\mathscr{C}_T$. (b) The set of all controllable states is called the controllable region and is denoted by $\mathscr{C}$.

Depending on whether discrete-time systems or continuous-time systems are of concern, a square matrix will be said to be *semistable* if all its eigenvalues are contained in the closed unit disk or if all its eigenvalues have nonpositive real parts; it will be said to be *antistable* if all its eigenvalues have modulus greater than one or if all its eigenvalues have positive real parts.

For a general unstable discrete-time system, we can assume, without loss of generality, that the system is of the following form:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t),$$
$$u(t) \in \Omega,$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ is antistable and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ is semistable. Similarly, for a general unstable continuous-time system, we can assume that the system is of the following form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t),$$
$$u(t) \in \Omega,$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ is antistable and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ is semistable.

Let $\mathscr{C}_1$ denote the controllable region of the following subsystem:

$$x_1(t+1) = A_1 x(t) + B_1 u(t), \quad u(t) \in \Omega$$

or

$$\dot{x}_1(t) = A_1 x(t) + B_1 u(t), \quad u(t) \in \Omega$$

depending on the circumstances.

Lemma 1 (Hájek [1]). *$\mathscr{C} = \mathscr{C}_1 \times \mathbb{R}^{n_2}$ and $\mathscr{C}_1$ is a bounded, convex, and open set in $\mathbb{R}^{n_1}$ which contains the origin.*

The author of [1] credited Lemma 1 to a thesis of Hsu in 1974. Its specialization to semistable systems was also proved in [8, 9].

Lemma 1 says that the controllable region of a general controllable system is a cylinder whose cross section is a bounded convex open set containing the origin. It is desirable to have an explicit characterization of $\mathscr{C}$, or of its cross section. A simpler problem is to find a necessary and sufficient condition for a given state x_0 to belong to $\mathscr{C}$. In the next two sections, we solve the problem of characterizing $\mathscr{C}$. Because of Lemma 1, we only deal with antistable systems.

2. Background

We will need some tools on convex analysis. The material in this section is mainly from [7, 4, 5]. Let $\mathscr{V}$ be a topological vector space over $\mathbb{R}$ and S be a subset of $\mathscr{V}$. We use αS to denote the set $\{\alpha v : v \in S\}$. The convex hull of S, denoted by $\mathrm{co}(S)$, is the smallest convex set containing S. The closure of S, denoted by $\mathrm{cl}(S)$, is the smallest closed set containing S. The interior of S, denoted by $\mathrm{int}\, S$, is the largest open set it contains.

Let $S \subset \mathscr{V}$ be an arbitrary set.

Definition 3. The gauge (or Minkowski functional) of S is the function $\mu_S : \mathscr{V} \to [0, \infty]$ defined by

$$\mu_S(v) = \inf\{\alpha \geqslant 0 : v \in \alpha S\}.$$

The gauge can be considered as a generalization of the norm. If S is bounded, balanced (in the sense that $\alpha S \subset S$ for every $\alpha \in [-1, 1]$), convex, and contains 0 in its interior, then μ_S is a norm.

The dual space of $\mathscr{V}$ is denoted by $\mathscr{V}^*$. For $v \in \mathscr{V}$ and $u \in \mathscr{V}^*$, we write $u(v)$ as $\langle v, u\rangle$. Again let $S \subset \mathscr{V}$ be an arbitrary set.

Definition 4. The polar of S, denoted by S°, is a set in $\mathscr{V}^*$ defined by

$$S^\circ = \{u \in \mathscr{V}^* : \langle v, u\rangle \leqslant 1, \ \forall v \in S\}.$$

Proposition 1. *The polar S° of any set $S \subset \mathscr{V}$ is a closed convex set and contains the origin. If $0 \in \mathrm{int}\, S$, then S° is bounded. If S is bounded, then $0 \in \mathrm{int}\, S^\circ$. If S is balanced, so is S°.*

Definition 5. The gauge μ_{S° of S° is called the dual gauge of μ_S.

The dual gauge μ_{S° of μ_S can also be related to μ_S through the following proposition.

Proposition 2. *Let μ_S be the gauge of a set $S \subset \mathscr{V}$. Then its dual gauge μ_{S° satisfies*

$$\mu_{S^\circ}(u) = \sup_{v \in S} \langle v, u \rangle .$$

If μ_Ω is a norm, then μ_{Ω° is its dual norm.

Since S° is a set in $\mathscr{V}^*$, it has a polar in $\mathscr{V}^{**}$ which is written as $S^{\circ\circ}$ instead of $(S^\circ)^\circ$. The set $S^{\circ\circ}$ is called the *bipolar* of S.

Proposition 3. *If $\mathscr{V}$ is reflexive, then* $S^{\circ\circ} = \mathrm{cl}[\mathrm{co}(\{0\} \cup S)]$.

For the compact convex set $\Omega \in \mathbb{R}^m$ with $0 \in \mathrm{int}\,\Omega$ which is associated with our bounded control problem, it follows from Propositions 1 and 2 that $\Omega^\circ \subset \mathbb{R}^m$ is also a compact convex set containing 0 in its interior and $\Omega^{\circ\circ} = \Omega$. Hence, gauges μ_Ω and μ_{Ω° are dual to each other. In our application, we need to evaluate $\mu_{\Omega^\circ}(v)$. In most interesting cases, this is easy. If Ω is the unit ball of a weighted Hölder p-norm, then μ_{Ω° is the inversely weighted Hölder q-norm, where $1/p + 1/q = 1$. If Ω is a polytope given by

$$\Omega = \mathrm{co}\{u_1, u_2, \dots, u_l\},$$

then

$$\mu_{\Omega^\circ}(v) = \max\{v' u_i \colon i = 1, \dots, l\}.$$

Next let us consider the space of $\mathbb{R}^m$-valued sequences from 0 to $T-1$:

$$\ell_T^m = \{\{u(t)\}_0^{T-1} \colon u(t) \in \mathbb{R}^m\}.$$

This space can be identified with $\mathbb{R}^{mT}$. The dual of ℓ_T^m is itself with the linear functional defined as

$$\langle v, u \rangle = \sum_{t=0}^{T-1} v(t)' u(t)$$

for $v, u \in \ell_T^m$. Let us define

$$R(\infty, \Omega, T) = \left\{ u \in \ell_T^m \colon \max_{0 \leqslant t \leqslant T-1} \mu_\Omega[u(t)] \leqslant 1 \right\},$$

$$R(1, \Omega^\circ, T) = \left\{ v \in \ell_T^m \colon \sum_{t=0}^{T-1} \mu_{\Omega^\circ}[v(t)] \leqslant 1 \right\}.$$

Clearly $R(\infty, \Omega, T)$ and $R(1, \Omega^\circ, T)$ are compact convex sets containing 0 in their interior. Hence they have gauges

$$\mu_{R(\infty,\Omega,T)}(v) = \max_{0 \leqslant k \leqslant T-1} \mu_\Omega[u(k)],$$

$$\mu_{R(1,\Omega^\circ,T)}(v) = \sum_{k=0}^{T-1} \mu_{\Omega^\circ}[v(k)].$$

Proposition 4. $R(\infty, \Omega, T)^\circ = R(1, \Omega^{\mathrm{c}}, T)$ *and* $R(1, \Omega^\circ, T)^\circ = R(\infty, \Omega, T)$.

Proof. We prove the second equality. The first one, which actually will not be used in the sequel, can be proved in a similar way. If $u \in R(\infty, \Omega, T)$, then it follows from Proposition 2 that for each $v \in R(1, \Omega^\circ, T)$,

$$\langle v, u \rangle = \sum_{t=0}^{T-1} v(t)' u(t) \leqslant \sum_{t=0}^{T-1} \mu_{\Omega^\circ}[v(t)] \leqslant 1.$$

This shows $u \in R(1, \Omega^\circ, T)^\circ$. If $u \notin R(\infty, \Omega, T)$, then there exists a t_0 satisfying $0 \leqslant t_0 \leqslant T-1$ such that $\mu_\Omega[u(t_0)] > 1$. This means that we can choose $v \in \ell_T^m$ such that $\mu_{\Omega^\circ}[v(t_0)] \leqslant 1$ but $v(t_0)' u(t_0) > 1$ and $v(t) = 0$ for all $t \neq t_0$. Then

$$\sum_{t=0}^{T-1} \mu_{\Omega^\circ}[v(t)] = \mu_{\Omega^\circ}[v(t_0)] < 1,$$

which says that $v \in R(1, \Omega^\circ, T)$, but

$$\langle v, u \rangle = \sum_{t=0}^{T-1} v(t)' u(t) > 1.$$

This shows $u \notin R(1, \Omega^\circ, T)^\circ$. □

Therefore, $\mu_{R(\infty,\Omega,T)}$ and $\mu_{R(1,\Omega^\circ,T)}$ are dual to each other.

Now let us consider the $\mathbb{R}^m$-valued Lebesgue spaces $\mathscr{L}_\infty^m[0,T]$ and $\mathscr{L}_1^m[0,T]$. It is well known that the dual space of $\mathscr{L}_1^m[0,T]$ is $\mathscr{L}_\infty^m[0,T]$ with the linear functional defined as

$$\langle v, u \rangle = \int_0^T v(t)' u(t)\, \mathrm{d}t$$

for $v \in \mathscr{L}_1^m[0,T]$ and $u \in \mathscr{L}_\infty^m[0,T]$. Define

$$S(\infty, \Omega, T) = \left\{ u \in \mathscr{L}_\infty^m[0,T] \colon \operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \mu_\Omega[u(t)] \leqslant 1 \right\},$$

$$S(1, \Omega^\circ, T) = \left\{ v \in \mathscr{L}_1^m[0,T] \colon \int_0^T \mu_{\Omega^\circ}[v(t)]\, \mathrm{d}t \leqslant 1 \right\}.$$

Clearly $S(\infty,\Omega,T)$ and $S(1,\Omega^\circ,T)$ are bounded closed convex sets containing 0 in their interior. Hence they have gauges

$$\mu_{S(\infty,\Omega,T)}(u) = \operatorname*{ess\ sup}_{0\leqslant t\leqslant T} \mu_\Omega[u(t)],$$

$$\mu_{S(1,\Omega^\circ,T)}(v) = \int_0^T \mu_{\Omega^\circ}[v(t)]\,\mathrm{d}t.$$

Proposition 5. $S(1,\Omega^\circ,T)^\circ = S(\infty,\Omega,T)$.

Proof. If $u\in S(\infty,\Omega,T)$, then it follows from Proposition 2 that for each $v\in S(1,\Omega^\circ,T)$,

$$\langle v,u\rangle = \int_0^T v(t)'u(t)\,\mathrm{d}t \leqslant \int_0^T \mu_{\Omega^\circ}[v(t)]\,\mathrm{d}t\leqslant 1.$$

This shows $u\in S(1,\Omega^\circ,T)^\circ$. If $u\notin S(\infty,\Omega,T)$, then there exists a set $W\subset[0,T]$ with nonzero measure $v(W)$ such that $\mu_\Omega[u(t)]>1$ for all $t\in W$. This means that we can choose $v_0\in\mathscr{L}_1^m[0,T]$ such that $\mu_{\Omega^\circ}[v_0(t)]\leqslant 1$ but $v_0(t)'u(t)>1$ for all $t\in W$ and $v_0(t)=0$ for all $t\in[0,T]\backslash W$. Let $v=v_0/v(W)$. Then

$$\int_0^T \mu_{\Omega^\circ}[v(t)]\,\mathrm{d}t\leqslant \int_W \frac{1}{v(W)}\mu_{\Omega^\circ}[v_0(t)]\,\mathrm{d}t<1,$$

which says that $v\in S(1,\Omega^\circ,T)$, but

$$\langle v,u\rangle = \int_W \frac{1}{v(W)} v_0(t)'u(t)\,\mathrm{d}t>1.$$

This shows $u\notin S(1,\Omega^\circ,T)^\circ$. □

Therefore, $\mu_{S(\infty,\Omega,T)}$ is the dual of $\mu_{S(1,\Omega^\circ,T)}$. Notice that $S(\infty,\Omega,T)^\circ\neq S(1,\Omega^\circ,T)$ for the same reason why $\mathscr{L}_\infty[0,T)^*\neq\mathscr{L}_1[0,T)$.

Finally, we state the Hahn–Banach theorem here for easy reference.

Proposition 6. *Suppose that $\mathscr{W}$ is a subspace of a vector space $\mathscr{V}$, $p:\mathscr{V}\to\mathbb{R}$ is a function satisfying $p(v+w)\leqslant p(v)+p(w)$ and $p(\alpha v)=\alpha p(v)$ for all $v,w\in\mathscr{V}$ and $\alpha\geqslant 0$, and f is a linear functional on $\mathscr{W}$ and $f(v)\leqslant p(v)$ for all $v\in\mathscr{W}$. Then there exists a linear functional F on $\mathscr{V}$ such that $F(v)=f(v)$ for all $v\in\mathscr{W}$ and $F(v)\leqslant p(v)$ for all $v\in\mathscr{V}$.*

3. Main results

Let us first characterize those $x_0\in\mathbb{R}^n$ that are contained in $\mathscr{C}$. As we have seen in Section 2, we can assume, without loss of generality, that the system is antistable. Define functions for nonzero x_0

$$\gamma(x_0)=\min_{x\in x_0^\perp}\sum_{t=0}^{\infty}\mu_{\Omega^\circ}\left[-B'A'^{-t-1}\left(\frac{x_0}{x_0'x_0}+x\right)\right]$$

for discrete-time systems and

$$\gamma(x_0)=\min_{x\in x_0^\perp}\int_0^{\infty}\mu_{\Omega^\circ}\left[-B'\mathrm{e}^{-A't}\left(\frac{x_0}{x_0'x_0}+x\right)\right]\mathrm{d}t$$

for continuous-time systems. Here we have used $x_0^\perp$ to denote the set of annihilators (or the orthogonal complement if the usual inner product is defined on $\mathbb{R}^n$) of x_0. Note that the series and the integral above converge because A is assumed to be antistable. Since μ_{Ω° is a continuous convex function, the minimization problems on the right-hand sides are well defined and can be computed easily if μ_{Ω° can be evaluated easily.

Theorem 1. *Assume the system is antistable. Let $x_0\in\mathbb{R}^n$ and $x_0\neq 0$. Then $x_0\in\mathscr{C}$ if and only if $\gamma(x_0)>1$.*

Proof. Let us first prove for the discrete-time case. Define

$$\gamma_T(x_0)=\min_{x\in x_0^\perp}\sum_{t=0}^{T-1}\mu_{\Omega^\circ}\left[-B'A'^{-t-1}\left(\frac{x_0}{x_0'x_0}+x\right)\right].$$

We now show that $x_0\in\mathscr{C}_T$ if and only if $\gamma_T(x_0)\geqslant 1$. We know that $x\in\mathscr{C}_T$ if and only if there exists control sequence $u_T = \ell_T^m$ with $\mu_{R(\infty,\Omega,T)}(u_T)\leqslant 1$ such that

$$-\sum_{t=0}^{T-1}A^{-t-1}Bu(t)=x_0.$$

This equality is true if and only if

$$-\sum_{t=0}^{T-1}\frac{x_0'}{x_0'x_0}A^{-t-1}Bu(t)=1, \tag{3}$$

and

$$-\sum_{t=0}^{T-1}x'A^{-t-1}Bu(t)=0 \tag{4}$$

for all $x\in x_0^\perp$.

The control sequence u_T can be identified with a linear functional on ℓ_T^m. Eqs. (3) and (4) define this linear functional on the n-dimensional subspace $\mathscr{V}_T$ of ℓ_T^m spanned by $\{-B'A'^{-t-1}x_0/x_0'x_0\}_0^{T-1}$ and $\{-B'A'^{-t-1}x\}_0^{T-1}$ for $x\in x_0^\perp$. By Propositions 2 and

4, the existence of u_T with $\mu_{R(\infty,\Omega,T)}(u_T)\leqslant 1$ satisfying Eqs. (3) and (4) implies that

$$\langle v_T,u_T\rangle \leqslant \mu_{R(1,\Omega^\circ,T)}(v_T) \tag{5}$$

for all $v_T\in\mathscr{V}_T$.

On the other hand, a u_T specified by Eqs. (3) and (4) can be initially considered as a linear functional on $\mathscr{V}_T$. Suppose Eq. (5) holds for all $v_T\in\mathscr{V}_T$. The function $\mu_{R(1,\Omega^\circ,T)}:\ell_T^m\to\mathbb{R}$ satisfies the requirement for the function p in Proposition 6. By Proposition 6, u_T can be extended to a linear functional on ℓ_T^m, i.e., u_T can be made to become an element of ℓ_T^m, satisfying Eq. (5) for all $v_T\in\ell_T^m$. By Propositions 2 and 4 again, we have $\mu_{R(\infty,\Omega,T)}(u_T)\leqslant 1$.

So far we have proved that $x_0\in\mathscr{C}_T$ if and only if Eq. (5) is satisfied for all $v_T\in\mathscr{V}_T$, which is equivalent to

$$\alpha\leqslant\sum_{t=0}^{T-1}\mu_{\Omega^\circ}\left(-\alpha B'A'^{-t-1}\frac{x_0}{x_0'x_0}-B'A'^{-t-1}x\right)$$

for all $\alpha\in\mathbb{R}$ and $x\in x_0^\perp$. This last condition is equivalent to

$$\begin{aligned}&\gamma_T(x_0)\\&=\min_{x\in x_0^\perp}\sum_{t=0}^{T-1}\mu_{\Omega^\circ}\left(-B'A'^{-t-1}\frac{x_0}{x_0'x_0}-B'A'^{-t-1}x\right)\\&\geqslant 1.\end{aligned}$$

Now the theorem follows from the fact that $\gamma(x_0)>\gamma_T(x_0)$ for all $x_0\in\mathbb{R}^n$ and $T>0$.

The proof for the continuous-time case goes similarly. Define

$$\gamma_T(x_0)=\min_{x\in x_0^\perp}\int_0^T\mu_{\Omega^\circ}\left[-B'e^{-A't}\left(\frac{x_0}{x_0'x_0}+x\right)\right]dt.$$

We now show that $x_0\in\mathscr{C}_T$ if and only if $\gamma_T(x_0)\geqslant 1$. We know that $x\in\mathscr{C}_T$ if and only if there exists control signal $u_T\in\mathscr{L}_\infty^m[0,T]$ with $\mu_{S(\infty,\Omega,T)}(u_T)\leqslant 1$ such that

$$-\int_0^T e^{-At}Bu(t)\,dt=x_0.$$

This is true if and only if

$$-\int_0^T\frac{x_0'}{x_0'x_0}e^{-At}Bu(t)\,dt=1, \tag{6}$$

and

$$-\int_0^T x'e^{-At}Bu(t)\,dt=0 \tag{7}$$

for all $x\in x_0^\perp$.

The control signal u_T can be identified with a linear functional on $\mathscr{L}_1^m[0,T]$. Eqs. (6) and (7) define this linear functional on the n-dimensional subspace $\mathscr{V}_T$ of $\mathscr{L}_1^m[0,T]$ spanned by $B'e^{-A'(\cdot)}x_0/x_0'x_0$ and $B'e^{-A'(\cdot)}x$ for $x\in x_0^\perp$. By Propositions 2 and 5, the existence of u_T with $\mu_{S(\infty,\Omega,T)}(u_T)\leqslant 1$ satisfying Eqs. (6) and (7) implies that

$$\langle v_T,u_T\rangle\leqslant\mu_{S(1,\Omega^\circ,T)}(v_T) \tag{8}$$

for all $v_T\in\mathscr{V}_T$.

On the other hand, a u_T specified by Eqs. (6) and (7) can be initially considered as a linear functional on $\mathscr{V}_T$. Suppose Eq. (8) holds for all $v_T\in\mathscr{V}_T$. The function $\mu_{S(1,\Omega^\circ,T)}:\mathscr{L}_1^m[0,T]\to\mathbb{R}$ satisfies the requirement for the function p in Proposition 6. By Proposition 6, u_T can be extended to a linear functional on $v_T\in\mathscr{L}_1^m[0,T]$, i.e., u_T can be made to become an element of $\mathscr{L}_\infty^m[0,T]$, satisfying Eq. (8) for all $v_T\in\mathscr{L}_1^m[0,T]$. By Propositions 2 and 5 again, we have $\mu_{S(\infty,\Omega,T)}(u_T)\leqslant 1$.

So far we have proved that $x_0\in\mathscr{C}_T$ if and only if Eq. (8) is satisfied for all $v_T\in\ell_T^m$, which is equivalent to

$$\alpha\leqslant\int_0^T\mu_{\Omega^\circ}\left(-\alpha B'e^{-A't}\frac{x_0}{x_0'x_0}-B'e^{-A't}x\right)dt$$

for all $\alpha\in\mathbb{R}$ and $x\in x_0^\perp$. This last condition is equivalent to

$$\begin{aligned}&\gamma_T(x_0)\\&=\min_{x\in x_0^\perp}\int_0^T\mu_{\Omega^\circ}\left(-B'e^{-A't}\frac{x_0}{x_0'x_0}-B'e^{-A't}x\right)dt\\&\geqslant 1.\end{aligned}$$

Now the theorem follows from the fact that $\gamma(x_0)>\gamma_T(x_0)$ for all $x_0\in\mathbb{R}^n$ and $T>0$. □

For the continuous-time case, a condition with the same spirit was obtained in [8]. However, our derivation is completely different and provides extra insight. Our condition also takes a simpler form. Also for the continuous time case, a related problem was studied in [3] in the content of time optimal control. The technique used was similar to ours but the result took different form.

Denote the boundary of $\mathscr{C}$ by $\partial\mathscr{C}$. The following theorem gives an explicit description of $\partial\mathscr{C}$ and follows from Theorem 1 immediately.

Theorem 2. $\partial\mathscr{C}=\{\gamma(w)w: w\in\mathbb{R}^n$ *and* $\|w\|_2=1\}$.

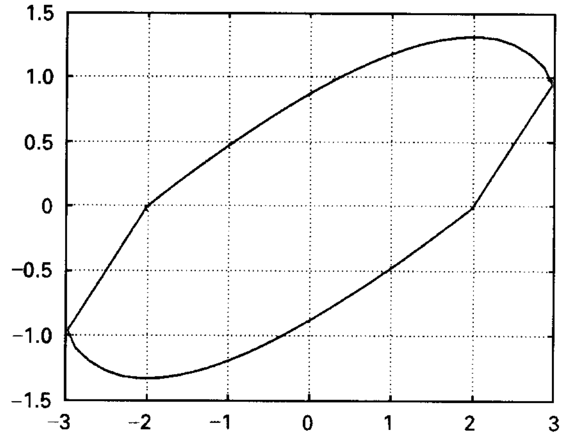

Fig. 1. Controllable region of Example 1.

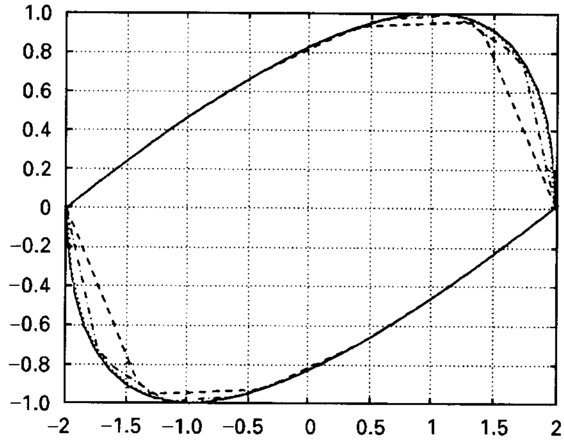

Fig. 2. Controllable regions of different discretized systems in Example 2.

4. Examples

Example 1. A second-order antistable continuous-time system is described by Eq. (2) with

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Ω is the unit ball of the Hölder 1-norm in $\mathbb{R}^2$. Its controllable region is computed using Theorem 4 and is the enclosed region shown in Fig. 1.

Example 2. In this example, we investigate the effect of sampling-period on the controllable region of discretized systems. The original antistable continuous-time system is described by (2) with

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Omega = [-1, 1].$$

This system is discretized using four different sampling period to get four different antistable discrete-time systems. In Fig. 2, the boundaries of the controllable regions of these discrete-time systems are drawn. The innermost dashed one corresponds to the sampling period $h = 1$. The other three from the inner to the outermost correspond to $h = 0.6$, 0.2, and 0.1, respectively. Actually, it can be proved that as h tends to zero, the limit of $\mathscr{C}$ is the controllable region of the continuous-time system.

5. Conclusions

This paper gives the characterization of the controllable region of a general unstable system with bounded control. The next stage of investigation is to study the relationship between controllability and stabilizability. In particular, we wish to design feedback controllers to stabilize such systems so that the domain of attraction is close to the controllable region. For a semistable system with bounded inputs, recent results in the literature shows that a nonlinear state feedback controller can be designed to achieve global stabilization [11, 12] and a linear saturated state feedback can be designed to accomplish semiglobal stabilization, i.e., to make the domain of attraction arbitrarily large [2, 6]. In our continuing study, we investigate the possibility of designing feedback control laws for a general unstable system so that the domain of attraction of the closed loop system is the same as or arbitrarily close to the controllable region.

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References

[1] O. Hájek, Control Theory in the Plane, Springer, Berlin, 1991.
[2] Z. Lin, A. Saberi, Semi-global exponential stabilization of linear discrete-time systems subject to input saturation via linear feedbacks, Systems Control Lett. 24 (1995) 125–132.
[3] X. Li, H. Xie, Time optimal control of linear systems, Acta Sci. Nat. Univ. Fudan 9 (1964) 501–512 (in Chinese).
[4] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.
[5] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1973.

[6] A. Saberi, Z. Lin, A.R. Teel, Control of linear systems with saturating actuators, IEEE Trans. Automat. Control 41 (1996) 368–378.

[7] H.H. Schaefer, Topological Vector Space, Springer, Berlin, 1980.

[8] W.E. Schmitendorf, B.R. Barmish, Null controllability of linear systems with constrained controls, SIAM J. Control Optim. 18 (1980) 327–345.

[9] E.D. Sontag, An algebraic approach to bounded controllability of linear systems, Int. J. Control 39 (1984) 181–188.

[10] E.D. Sontag, H.J. Sussmann, Nonlinear output feedback design for linear systems with saturating controls, in: Proc. IEEE Conf. Decision and Control, 1990, pp. 3414–3416.

[11] H.J. Sussmann, E.D. Sontag, Y. Yang, A general result on the stabilization of linear systems using bounded controls, IEEE Trans. Automat. Control 39 (1994) 2411–2425.

[12] A.R. Teel, Global stabilization and restricted tracking for multiple integrators with bounded controls, Systems Control Lett. 18 (1992) 165–171.