Direct State Space Solution of Multirate Sampled-Data $\mathcal{H}_2$ Optimal Control

LI QIU† and KAN TAN†

Key Words—Multirate; sampled-data systems; lifting technique; $\mathcal{H}_2$ optimization; causality constraint; nested algebra.

Abstract—In solving the multirate sampled-data $\mathcal{H}_2$ control problem using the lifting approach, one needs to solve a constrained discrete-time $\mathcal{H}_2$ optimal control problem for a generalized plant with infinite dimensional input/output spaces. To solve this problem, the existing sampled-data $\mathcal{H}_2$ design technique computes an equivalent finite dimensional discrete-time system and then designs the optimal $\mathcal{H}_2$ controller for the equivalent system. In this paper, we will show that this problem can be solved using state space formulas by dealing with operators directly. The operator compositions are computed explicitly using discrete multirate lifting and matrix exponentials. The advantages of the direct method are: it is straightforward, it has clear physical meanings, and it is more efficient computationally. A sufficient condition for the existence and uniqueness of multirate sampled-data $\mathcal{H}_2$ optimal controller is given in terms of the continuous-time plant. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

For a sampled-data control system, the plant is in general a continuous-time LTI system, the controller is composed of A/D converters (samplers), a digital computer, and D/A converters (holds). Hence, a sampled-data control system is a hybrid system involving both continuous-time and discrete-time signals. In many applications, the samplers and the holds do not necessarily operate in the same rate. In such cases, the system is called a multirate sampled-data control system. Since the plant evolves in the continuous-time, performance criteria are most readily formulated in the continuous-time domain. The studies of the single-rate sampled-data $\mathcal{H}_2$ design include (Bamieh and Pearson, 1992; Chen and Francis, 1991, 1995; Khargonekar and Sivashankar, 1991; Trentelman and Stoovevogel, 1995), where the sampled-data $\mathcal{H}_2$ design problem is converted to a pure discrete-time $\mathcal{H}_2$ design for an equivalent discrete-time system. For the multirate case, after lifting, causality constraint arises (Colaneri and Nicolao, 1995; Chen and Qiu, 1994; Qiu and Chen, 1994; Al-Rahmani and Franklin, 1992; Shu and Chen, 1996; Voulgaris and Bamieh, 1993; Voulgaris et al., 1994). Addressing $\mathcal{H}_2$ optimal control in particular, references (Colaneri and Nicolao, 1995; Voulgaris and Bamieh, 1993) find an equivalent pure discrete-time system and then design a controller with the causality constraint for the equivalent discrete-time system, references (Chen and Qiu, 1994; Qiu and Chen, 1994) give a direct method based on the frequency-domain technique and the nest algebra. Reference (Shu and Chen, 1996) treats a multirate pure discrete-time system by a state-space method. This paper gives a direct multirate sampled-data $\mathcal{H}_2$ design: when a system arises with infinite-dimensional input/output spaces by continuous-time lifting, we treat it directly instead of converting it to another equivalent pure discrete-time system with finite dimensional input/output spaces. It is shown that two Riccati equations are to be solved which contain matrix-valued operator compositions, and these compositions can be computed explicitly in state space formulas. We will be interested in the conditions guaranteeing the existence and uniqueness of the optimal sampled-data controller. Results in the single-rate case were given by (Khargonekar and Sivashankar, 1991; Trentelman and Stoovevogel, 1995). We generalize them to get a sufficient condition for the multirate case. The advantages of the direct state-space solution are: physical meanings are preserved so it is conceptionally more clear, and less computation is needed because the conversion to the pure discrete-time system is no longer necessary.

The results in this paper have been implemented in a MATLAB toolbox for multirate systems and control currently under development (Qiu et al., 1996).

The organization of this paper is as follows: Section 2 presents the multirate sampled-data configuration and the continuous-time lifting. Section 3 derives the direct state space solution of multirate sampled-data $\mathcal{H}_2$ optimal control. Section 4 addresses the computational issues of the operator compositions involved in Section 3. Section 5 gives a sufficient condition in terms of the continuous-time plant for the existence and uniqueness of the $\mathcal{H}_2$ optimal sampled-data control.

The notation used in this paper will be standard. For operator $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ from space $\mathcal{W} \oplus \mathcal{W}$ to space $\mathcal{Z} \oplus \mathcal{Z}$ and operator $Q$ from $\mathcal{W}$ to $\mathcal{Z}$, the linear fractional transformation $P_{11} + P_{12}(I - P_{22}Q)^{-1}P_{21}$ is denoted by $\mathcal{W}(P, Q)$. For discrete-time signals and systems, the $z$-transform, which is obtained from the $z$-transform by replacing $z$ with $1/\omega$, is used.

2. Setup

The setup of a multirate sampled-data control system is shown in Fig. 1. Here $G_s$ is an analog generalized plant with two (vector) inputs, the exogenous input $w$ and the control input $u$, and two (vector) outputs, the signal $z$ to be regulated and the measured signal $y$. We assume that $G_s$ is LTI with a state-space model

$$\hat{G}_s(z) = \begin{bmatrix} A_s & B_{12} \\ C_{11} & D_{11} \end{bmatrix}$$

Three blocks in the direct feedthrough matrix of $G_s$ are assumed to be zero with $D_{11} = 0$ for the finiteness of the $\mathcal{H}_2$ norm which

* Received 6 May 1997; revised 16 December 1997; received in final form 28 April 1998. A preliminary version of this paper was presented at the 2nd Asian Control Conference, which was held in Seoul, Korea, during July, 1997. This paper was recommended for publication in revised form by Associate Editor André L. Tits under the direction of Editor Tamer Başar. Corresponding author Professor Li Qiu. Tel. + 852 2538 7067; Fax + 852 2538 1485; E-mail eequi@ee.ust.hk.
†Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.
will be introduced later, $D_{211} = 0$ for the proper functioning of the samplers when the exogenous input is an impulsive function, and $D_{212} = 0$ for simplicity. We furthermore assume that all matrices in the state space model (1) are real. Symbols $\mathcal{F}$ and $\mathcal{M}$ represent multirate sampling $(A,D)$ and hold $(D,A)$ operations and are defined as follows:

$$\mathcal{M} = \begin{bmatrix} L_{n_1} & & \\ & \ddots & \\ & & L_{n_p} \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} S_{m_1} & & \\ & \ddots & \\ & & S_{m_p} \end{bmatrix}. $$

These correspond to performing the $A/D$ conversions for the $p$ channels of $y$ periodically with periods $m_i h_i$, respectively, and the $D/A$ conversions for the $q$ channels of $u$ with periods $n_i h_i$, respectively. Here $m_i$ and $n_i$ are integers and $h_i$ is a real number referred to as the base period. The linear multirate controller $K_m$ is assumed to satisfy three properties: periodicity, causality, and finite dimensionality; then they can be implemented in the form of some difference equations (Chen and Qiu, 1994).

The closed-loop system in Fig. 1 can be converted to an LTI discrete-time system with infinite dimensional input/output spaces by the lifting technique. Let $K_m$ be a strictly causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, finite-dimensional, and causal, 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The solutions of $X$ and $Y$ of the two equations (5) and (6) are said to be stabilizing if

$$A - B_2YD_1^{-1}(B_2XA + DT_2C_1)$$

and

$$A - (AYC_1 + B_2YD_1^{-1}(C_2YC_1 + B_2YD_2)C_2)$$

are stable, respectively.

**Proposition 1.** Suppose the lifted plant $G$ satisfies the assumptions (A1)-(A3). Then there exist unique stabilizing solutions to Riccati equations (5) and (6).

The proof of this proposition is the same as that for the case when $B_1$, $C_1$, $D_{12}$, and $D_2$ are matrices, with the simple modification of replacing matrix transposes by operator adjoints wherever appropriate.

Let $X$ and $Y$ be the stabilizing solutions of Riccati equations (5) and (6). Then as usual it can be shown that $B_2YD_1^{-1}(B_2XA + DT_2C_1)$, $C_2YC_1 + B_2YD_2$ and $B_2YD_2^{-1}$, $C_2YC_1 + B_2YD_2$ are stable.

$$F = - (B_2YD_1^{-1}(B_2XA + DT_2C_1),$$

$$L = - (AYC_1 + B_2YD_1^{-1}(C_2YC_1 + B_2YD_2)C_2)$$

where $H = - R^*P(IY_1)^{-1} \mathcal{P}^{-1}R(IY_1)^{-1}$ is the orthogonal projection from $\mathcal{P}(\theta, \nu)$ onto $\mathcal{P}(\theta, \nu)$ and $R \in \mathcal{F}(\theta, \nu)$ and $S \in \mathcal{F}(\theta, \nu)$.

$$\mathcal{P} = B_2YD_1^{-1}(B_2XA + DT_2C_1),$$

$$\mathcal{S} = C_2YC_1 + B_2YD_2$$

The factorizations in equations (7) and (8) are always possible (Chen and Qiu, 1994). A simple choice might be the Cholesky factorization.

**Theorem 1.** Assume the plant $G$ in the form of equation (4) satisfies assumptions (A1)-(A3). Then the lifted multirate sampled-data $\mathcal{H}_\infty$ optimal controller is given by

$$\hat{K}_{opt}(\lambda) = \left[ \frac{A_k}{B_k} \frac{C_k}{D_k} \right],$$

where

$$A_k = A + B_2F + LC_2 - B_2HC_2,$$

$$L = - L + B_2HD_2^{-1}(I + HD_2)F - HC_2,$$

$$B_k = - L + B_2H(I + D_2)H^{-1},$$

$$C_k = (I + HD_2)^{-1}F - HC_2,$$

$$D_k = (I + HD_2)^{-1}H.$$

The optimal $\mathcal{H}_\infty$ norm is

$$\| \mathcal{F}(\hat{G}, \hat{K}_{opt}) \|^2 = \text{tr}(A^*XAY + XB_1B_1^* + C_1C_2Y - XY) + \|D_1\|^2_{\infty} - \|R\|S\|^2.$$

**Proof.** The proof will be sketchy since it involves many standard materials as given in Chen and Francis (1995). The emphasis will be on the handling of the causality constraint. Let $X$ and $Y$ be the stabilizing solutions of equations (5) and (6). Then all stabilizing controllers without causality constraint are characterized by a linear fractional transformation

$$\hat{K} = \mathcal{F}(\hat{G}, \hat{Q}), \hat{Q} \in \mathcal{H}_\infty,$$

where

$$\hat{J}(\lambda) = \left[ \begin{array}{ccc} A + B_2F + LC_2 - LD_2F & - L & B_2 + LD_2 \\ F & 0 & I \\ - C_2 - D_2F & I & - D_2 \end{array} \right].$$

Note that

$$\hat{K}(0) = \mathcal{F}(\hat{G}(0), \hat{Q}(0)) = \hat{Q}(0)(I + D_2\hat{Q}(0))^{-1}.$$

Since $D_2 \in \mathcal{L}(\theta, \nu)$, it is easy to show (Chen and Qiu, 1994) that $\hat{K}(0) \in \mathcal{L}(\theta, \nu)$ if and only if $\hat{Q}(0) \in \mathcal{L}(\theta, \nu)$. Now it follows from the standard analysis, see e.g. Chen and Francis (1995), that under the controllers characterized in equation (9), the closed-loop transfer function is

$$\mathcal{F}(\hat{G}, \hat{K}) = \mathcal{F}(\hat{G}, \hat{Q} \hat{J}(\lambda)) = \hat{T}_{11} + \hat{T}_{12} \hat{Q} \hat{T}_{21},$$

where

$$\hat{T}_{11}(\lambda) = \begin{bmatrix} A + B_2F & B_2F & B_1 \\ 0 & A + LC_2 & - B_1 - LD_2 \end{bmatrix},$$

$$\hat{T}_{12}(\lambda) = \begin{bmatrix} A + B_2F & B_2 \\ C_1 + D_2F & D_2 \end{bmatrix},$$

$$\hat{T}_{21}(\lambda) = \begin{bmatrix} A + LC_2 & B_1 + LD_2 \\ C_2 & D_2 \end{bmatrix}.$$

Let $\hat{T}_{ij}(\lambda)$ be the adjoint of $\hat{T}(\lambda)$. Then for $\hat{T}_{11}, \hat{T}_{12}, \hat{T}_{21}$ defined by above equations, we have (Chen and Francis, 1995)

$$\hat{T}_{12}, \hat{T}_{12} = B_2YD_1^{-1}(B_2XA + DT_2C_1),$$

$$\hat{T}_{21}, \hat{T}_{21} = C_2YC_1 + B_2YD_2,$$

$$\hat{T}_{21}, \hat{T}_{21} = B_2YD_1^{-1}(B_2XA + DT_2C_1),$$

$$\hat{T}_{21}, \hat{T}_{21} = C_2YC_1 + B_2YD_2.$$

and ($\hat{T}_{12}, \hat{T}_{12}, \hat{T}_{21}, \hat{T}_{21}$) $\in \mathcal{H}_\infty$ with

$$\Pi_{\mathcal{H}_\infty}(\hat{T}_{12}, \hat{T}_{12}, \hat{T}_{21}, \hat{T}_{21}) = B_2YD_1^{-1}(B_2XA + DT_2C_1),$$

$$\hat{T}_{21}, \hat{T}_{21} = C_2YC_1 + B_2YD_2.$$

Now carry out matrix factorizations in equations (7) and (8). Define

$$\hat{U} = \begin{bmatrix} R^* \hat{T}_{12} \\ I - \hat{T}_{11}^{-1}R^* \hat{T}_{12} \end{bmatrix},$$

$$\hat{V} = \begin{bmatrix} \hat{T}_{11} \hat{T}_{21}S^{-1} \\ I - \hat{T}_{11} \hat{T}_{21}S^{-1} \hat{T}_{12} \end{bmatrix}.$$

Then equations (10) and (11) imply $\hat{U} \hat{U} = I$ and $\hat{V} \hat{V} = I$. Hence

$$\| \mathcal{F}(\hat{G}, \hat{K}) \|^2 = \| \hat{T}_{11} + \hat{T}_{12} \hat{Q} \hat{T}_{21} \|^2 = \| \hat{U} \hat{T}_{11} \hat{Q} \hat{T}_{21} \|^2 = \| R^* \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1} + R \hat{Q} \hat{S} \|^2 + \| \hat{W}_{11} \|^2 + \| \hat{W}_{21} \|^2 + \| \hat{W}_{22} \|^2.$$

where

$$\| R^* \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1} + R \hat{Q} \hat{S} \|^2 = \left[ \begin{array}{cc} \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1} + R \hat{Q} \hat{S} \\ \hat{W}_{12} \hat{W}_{21} \hat{W}_{22} \end{array} \right].$$

Note that $\hat{W}_{12}, \hat{W}_{21}, \hat{W}_{22}$ are independent of $\hat{Q}$. So minimizing $\| \mathcal{F}(\hat{G}, \hat{K}) \|^2$ is equivalent to minimizing $\| R^* \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1} + R \hat{Q} \hat{S} \|^2$ for $\hat{Q} \in \mathcal{H}_\infty$ satisfying $\hat{Q}(0) \in \mathcal{L}(\theta, \nu)$. From equation (12), we know that

$$\| R^* \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1} + R \hat{Q} \hat{S} \|^2 = \| R^* \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1} + R \hat{Q} \hat{S} \|^2.$$

Since $R \hat{Q} \hat{S} \in \mathcal{H}_\infty$, it follows that $R \hat{Q} \hat{S}$ can only be used to cancel part of the constant term of $R^* \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1}$. Consider the causality constraint $\hat{Q}(0) \in \mathcal{L}(\theta, \nu)$ and that $R \in \mathcal{L}(\theta, \nu)$ and $S \in \mathcal{L}(\theta, \nu)$ is invertible. The optimal $\hat{Q}$ is given by

$$\hat{Q}_{opt} = - \left( \Pi_{\mathcal{H}_\infty}(\theta, \nu) \right) - \left( R^* \hat{T}_{12} \hat{T}_{11} \hat{T}_{21}S^{-1} + R \hat{Q} \hat{S} \right).$$
Hence $\hat{Q}_H = H$. The optimal $\| \cdot \|$ norm is given by
\[
\| \mathcal{F}(G, K) \| = \| \hat{T}_1(t) \| = \| R\hat{Q}_H \| = \| \hat{T}_1(t) \| = \| \text{RHS} \|.
\]

It remains to show that
\[
\| \hat{T}_1(t) \| = \text{tr}(A^*XAY + XB_1B_1^* + C^*C_1Y - XY) + \| D_{11} \|_{\infty, \text{F}}.
\]

It is easy to verify that
\[
\hat{T}_1(t) = \hat{T}_p(t) + \hat{T}_{12}(t)\hat{T}_L(t),
\]
where
\[
\hat{T}_p(t) = \begin{bmatrix}
    A + B_1F & B_21 & \cdots & B_{12} \\
    C_1 + D_{12}F & D_{11} & \cdots & D_{12}
\end{bmatrix}
\]
\[
\hat{T}_L(t) = \begin{bmatrix}
    A + LC_1 & -B_1 - LD_{12} & \cdots & -D_{12}
\end{bmatrix}
\]

Straightforward computation shows that $\hat{T}_p$ and $\hat{T}_{12}\hat{T}_L$ are orthogonal to each other and equation (10) implies $\| \hat{T}_1(t) \| = |R\hat{T}_L(t)|$. Hence,
\[
\| \hat{T}_1(t) \| = \| \hat{T}_p(t) \| + |R\hat{T}_L(t)|
\]
\[
= \| D_{11} \|_{\infty, \text{F}} + \text{tr}(A^*XAY + XB_1B_1^* + C^*C_1Y - XY)
\]
\[
+ \text{tr}(A^*XAY + C^*C_1Y)Y
\]
\[
= \| D_{11} \|_{\infty, \text{F}} + \text{tr}(A^*XAY + C^*C_1Y).
\]

The optimal control formula in Theorem 1 first appeared in Qu et al. (1996). A slightly less general (complete version of this theorem is also independently obtained in Mirkin and Palmor (1997). The proof here is different from that in Mirkin and Palmor (1997).

4. Computation of the operator compositions

From the development in the last section, it is seen that to compute the multirate sampled-data $\mathcal{F}_n$ optimal controller and the optimal $\| \cdot \|$ norm using the direct state space solution we need from the lifted system $G$ matrices $A, B_1, C_2, D_{22}$, operator compositions:
\[
\begin{bmatrix}
    B_1 & B_1^*D_{12}^* & C_1^* & D_{11}^*D_{12}^*
\end{bmatrix}
\]
\[
D_{12}D_{11}D_{12}, \text{ and norm } \| D_{11} \|_{\infty, \text{F}} \text{ as the input data.}
\]

The matrices $A, B_1, C_2, D_{22}$ are easy to obtain. A way to compute $\| D_{11} \|_{\infty, \text{F}}$ using matrix exponentials is given in Ramineh and Pearson (1992). The computation of the required operator compositions, however, is rather nontrivial. For a special case when all $m_i$, $i = 1, \ldots, p$, are the same and all $n_{i,j}$, $i = 1, \ldots, q$, are the same (the dual rate case), integral formulas for these operator compositions are obtained in Qu and Chen (1994). There are characteristic functions involved in integral formulas, which make the computation quite complicated. The characteristic functions arise due to the multirate nature of the controller. To avoid this complication, we will show that these operator compositions can be obtained through a two-step lifting: first lift the plant $G_0$ in the base period $h$ and then lift $1$-fold in discrete-time. The resulted system can be related to $G_0$ easily and because of this the data on the lifted system $G_0$ can be obtained from some data associated with the intermediate system obtained after lifting $G_0$ in the base period. In the end it is shown that the required data of $G$ can also be obtained using matrix exponentials.

Recall equation (2) in Section 2:
\[
G = \begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix} = \begin{bmatrix}
    L_{0} & \mathcal{L}_{N_{\mathcal{F}}} \mathcal{L}_{\mathcal{F}} \mathcal{L}_{N_{\mathcal{F}}} \\
    G_{21} & G_{22}
\end{bmatrix} \begin{bmatrix}
    L_{1}^{-1} \mathcal{F} \mathcal{F} \mathcal{F}^{-1} \\
    L_{0}^{-1} \mathcal{F} \mathcal{F} \mathcal{F}^{-1}
\end{bmatrix}.
\]

It is possible to find matrices $L_{0}$ and $L_{N_{\mathcal{F}}}$ such that
\[
\mathcal{L}_{\mathcal{F}} = L_{0}L_{S_{0}} \text{ and } \mathcal{F} \mathcal{F}^{-1} = H_{L_{0}}L_{S_{0}}^{-1}L_{0}.
\]

Actually, in the first system period $[0, M]$, $L_{1}$ and $L_{N_{\mathcal{F}}}$ are required to satisfy
\[
\begin{bmatrix}
    y_1(0) \\
    y_2(0) \\
    \vdots \\
    y_1(h) \\
    \vdots \\
    y_p(h)
\end{bmatrix} = \begin{bmatrix}
    \psi_1(0) \\
    \psi_2(0) \\
    \vdots \\
    \psi_1(h - 1) \\
    \vdots \\
    \psi_p(h - 1)
\end{bmatrix}.
\]

In the subsequent system periods, things are the same except possible time shifts. Therefore, $L_{1}$ is a $[n_{i,j} \times p] \times p$ block matrix with all blocks equal to zero matrices except the $(\sum_{i=1}^{n} \sum_{j=1}^{m} \delta_{i,j} + 1, k, m, p + i)$ blocks which are equal to identity matrices and $L_{N_{\mathcal{F}}}$ is an $l_{i} \times \sum_{i=1}^{n} \sum_{j=1}^{m} \delta_{i,j}$ matrix with all blocks equal to zero matrices except the $(\sum_{i=1}^{n} \sum_{j=1}^{m} \delta_{i,j} + 1, k, m, p + i)$ blocks which are equal to identity matrices. Here $k = 0, \ldots, m_i - 1, r = 0, \ldots, l - 1, t = 1, \ldots, p, j = 1, \ldots, q$, and $\delta_{i,j}$ means the integer part.

Now it is clear that
\[
G = \begin{bmatrix}
    L_{0} & L_{1}^{-1}L_{1}L_{N_{\mathcal{F}}} & \cdots \\
    \vdots & \ddots & \vdots \\
    L_{p} & L_{1}^{-1}L_{1}L_{N_{\mathcal{F}}} & \cdots \\
    H_{1}L_{1}^{-1}L_{N_{\mathcal{F}}}
\end{bmatrix}.
\]

Let
\[
G_{S} = \begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix} = \begin{bmatrix}
    L_{0} & \cdots \\
    \vdots & \ddots \\
    \cdots & \cdots
\end{bmatrix} = \begin{bmatrix}
    G_{11} & G_{12} & \cdots \\
    G_{21} & G_{22} & \cdots
\end{bmatrix} = \begin{bmatrix}
    L_{0}^{-1} & \cdots \\
    \vdots & \ddots \\
    \cdots & \cdots
\end{bmatrix} = \begin{bmatrix}
    L_{1}^{-1} & \cdots \\
    \vdots & \ddots \\
    \cdots & \cdots
\end{bmatrix} = \begin{bmatrix}
    H_{S}
\end{bmatrix}
\]

Then $G_{S}$ is the equivalent discrete time system of $G_0$ lifted in base period $h$ and it is well known (Chen and Francis, 1995) that $G_{S}$ has state-space model
\[
\hat{G}_{S}(t) = \begin{bmatrix}
    A_{1} & B_{11} & B_{12} \\
    C_{11} & D_{11} & D_{12} \\
    C_{21} & 0 & 0
\end{bmatrix}
\]

which is formed by matrices
\[
A_{1} = \exp(A_{1}h), \quad B_{12} = \int_{0}^{h} \exp(A_{1}t) dt B_{12}, \quad C_{21} = C_{21}h,
\]

and operators
\[
\begin{align*}
B_{11} : B_{11}(t) &= \int_{0}^{t} \exp(A_{1}(h - t))B_{11}(t) dt, \\
C_{21} : (C_{21}h)(t) &= C_{21}\exp(A_{1}t)C_{21}, \quad t \in [0, h), \\
D_{11} : (D_{11}h)(t) &= \int_{0}^{t} \exp(A_{1}(h - t))B_{11}(t) dt, \quad t \in [0, h), \\
D_{12} : (D_{12}h)(t) &= D_{12}h + C_{21}h \int_{0}^{t} \exp(A_{1}t) dt B_{12}, \quad t \in [0, h).
\end{align*}
\]
Therefore,

\[
\delta = \begin{bmatrix}
L_t & G_{h1} & G_{h2} & -L_t^{-1}
\end{bmatrix}
\]

has state-space model

\[
\delta(\lambda) = \begin{bmatrix}
\bar{A} & \bar{B}_1 & \bar{B}_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix},
\]

where

\[
\bar{A} = A_c, \\
\bar{B}_i = [A^{-1}_h B_i, A^{-1}_h B_{a_1} \cdots B_{a_i}], \\
C_i = \begin{bmatrix}
C_{a_1} \\
C_{a_2} \\
\vdots \\
C_{a_n} A^{-1}_h
\end{bmatrix}, \\
\bar{D}_{ij} = \begin{bmatrix}
D_{a_1} & 0 & \cdots & 0 \\
C_{a_1} B_{a_1} & D_{a_1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{a_n} A^{-1}_h B_{a_1} & C_{a_n} A^{-1}_h B_{a_2} & \cdots & D_{a_n}
\end{bmatrix}, \quad i, j = 1, 2.
\]

Since

\[
G = \begin{bmatrix}
L_t L_{a_1} L_{a_2} & \bar{G}_1 & \bar{G}_2 & \bar{G}_3 & \bar{G}_4 & \bar{G}_5 & \bar{G}_6 & \bar{G}_7 & \bar{G}_8 & \bar{G}_9
\end{bmatrix}
\]

and \(L_{a_1}, L_{a_2}, L_t\) are unitary operators, its state-space model

\[
\tilde{\delta}(\lambda) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\]

satisfies

\[
\begin{bmatrix}
A & B_1 & B_2 \\
C_3 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} = \begin{bmatrix}
I & \frac{1}{2} \bar{A} & \frac{1}{2} \bar{B}_2 \\
\frac{1}{2} \bar{C}_2 & D_{12} & D_{11} \\
\frac{1}{2} \bar{C}_1 & D_{22} & D_{21}
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_1 \\
D_{11} \\
C_1 & D_{11}
\end{bmatrix} \bar{B}_1^T \bar{D}_{11} = \begin{bmatrix}
\bar{B}_1 \bar{B}_1^T & \bar{B}_1 \bar{B}_1^T L_2 & \bar{B}_1 \bar{D}_{11} L_2 \\
L_2 \bar{D}_{11} & L_2 \bar{D}_{11}^T L_2 & L_2 \bar{D}_{11}^T L_2 \\
C_1 & D_{11} & D_{11}
\end{bmatrix}
\]

\[
C^T \bar{D}_{12} = \begin{bmatrix}
C_1 & D_{12} & D_{11} & D_{11}
\end{bmatrix}
\]

\[
D_{12} D_{12}^T = L_2^T D_{12} \bar{B}_1 \bar{D}_{11} \bar{D}_{11}^T L_2.
\]

The detailed structures of \(\bar{B}_1, \bar{C}_1, \bar{C}_2, \bar{D}_{11}, 12, \bar{D}_{21}, \bar{D}_{22}\) reveal that the required operator compositions can be computed if operator compositions \(B_{a_1} B_{a_2}\),

\[
\begin{bmatrix}
C_{a_1} \\
D_{a_1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_{a_2} \\
D_{a_2}
\end{bmatrix}
\]

can be computed.

**Proposition 2.** Let

\[
\begin{bmatrix}
A \ 0 \\
0 \ A
\end{bmatrix}
\]

and let \(P\) and \(Q\) be partitioned into \(4 \times 4\) block matrices compatibly with the right-hand side matrices in equations (13) and (14), respectively. Then

\[
A_h = P_{33}, \\
B_{a_2} = Q_{34}, \\
C_{a_2} = C_{a_2}, \\
B_{a_1}^* B_{a_1} = P_{34} Q_{34}.
\]

\[
\begin{bmatrix}
C_{a_1} D_{a_1} \\
D_{a_1}
\end{bmatrix}
\begin{bmatrix}
Q_{13} & Q_{14}^T \\
Q_{13} & Q_{14}^T
\end{bmatrix}
= \begin{bmatrix}
Q_{13} & Q_{14}^T \\
Q_{13} & Q_{14}^T
\end{bmatrix}
\begin{bmatrix}
C_{a_1} D_{a_1} \\
D_{a_1}
\end{bmatrix}
\]

The following lemma is needed in proving Proposition 2.

**Lemma 1** (Van Loan, 1978). Let \(A_{11}\) and \(A_{22}\) both be square and define

\[
F_{11}(t) = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}, \quad t > 0.
\]

Then \(F_{11}(t) = \exp(A_{11} t) F_{12}(t) = \exp(A_{12} t)\), and

\[
F_{22}(t) = \int_0^t \exp(A_{11} (t - \tau)) A_{12} \exp(A_{22} t) \, d\tau.
\]

**Proof (Proposition 2).** The first five equalities are actually proved in Chen and Francis (1995). We only need to prove the last equality. It can be shown that

\[
\begin{bmatrix}
C_{a_1} \\
D_{a_1}
\end{bmatrix}
\begin{bmatrix}
Q_{13} & Q_{14}^T \\
Q_{13} & Q_{14}^T
\end{bmatrix}
= \int_0^t \left( A_{11} - A_{12} \right) \exp(A_{11} t - \tau) A_{12} \exp(A_{22} t) \, d\tau.
\]

From Lemma 1 and equation (14), we have

\[
\exp \left( \begin{bmatrix}
A_{22}^T & 0 \\
B_{22}^T & 0
\end{bmatrix} t \right) P_{22} = \begin{bmatrix}
P_{22} & 0 \\
P_{23} & P_{24}
\end{bmatrix} I
\]

\[
= \int_0^t \exp \left( \begin{bmatrix}
A_{22}^T & 0 \\
B_{22}^T & 0
\end{bmatrix} t \right) \left( \begin{bmatrix}
C_{a_1} \\
D_{a_1}
\end{bmatrix}
\begin{bmatrix}
Q_{13} & Q_{14}^T \\
Q_{13} & Q_{14}^T
\end{bmatrix}
\begin{bmatrix}
C_{a_1} \\
D_{a_1}
\end{bmatrix}
\begin{bmatrix}
Q_{13} & Q_{14}^T \\
Q_{13} & Q_{14}^T
\end{bmatrix}
\begin{bmatrix}
C_{a_1} \\
D_{a_1}
\end{bmatrix} \right) \left( \begin{bmatrix}
C_{a_1} \\
D_{a_1}
\end{bmatrix} \right) \left( \begin{bmatrix}
Q_{13} & Q_{14}^T \\
Q_{13} & Q_{14}^T
\end{bmatrix} \right) \left( \begin{bmatrix}
C_{a_1} \\
D_{a_1}
\end{bmatrix} \right) \left( \begin{bmatrix}
Q_{13} & Q_{14}^T \\
Q_{13} & Q_{14}^T
\end{bmatrix} \right)
\]

\[
\int_0^t \exp(A_{22} t - \tau) B_{a_2} B_{a_2}^T \exp(-A_{22}^T \tau) \, d\tau = \int_0^t \exp(-A_{22}^T \tau) \, d\tau
\]

\[
\int_0^t \exp(A_{22} t - \tau) B_{a_2} B_{a_2}^T \exp(-A_{22}^T \tau) \, d\tau = \int_0^t \exp(-A_{22}^T \tau) \, d\tau
\]

\[
\int_0^t \exp(A_{22} t - \tau) B_{a_2} B_{a_2}^T \exp(-A_{22}^T \tau) \, d\tau = \int_0^t \exp(-A_{22}^T \tau) \, d\tau
\]

\[
\int_0^t \exp(A_{22} t - \tau) B_{a_2} B_{a_2}^T \exp(-A_{22}^T \tau) \, d\tau = \int_0^t \exp(-A_{22}^T \tau) \, d\tau
\]

\[
\int_0^t \exp(A_{22} t - \tau) B_{a_2} B_{a_2}^T \exp(-A_{22}^T \tau) \, d\tau = \int_0^t \exp(-A_{22}^T \tau) \, d\tau
\]
5. On the existence and uniqueness of the $\mathcal{H}_2$ optimal controller

In this section, we address the condition in terms of the continuous-time plant $G_0$, which ensures assumptions (A1)–(A3) in terms of lifted system $G$. Since the existence and uniqueness of multirate sampled-data $\mathcal{H}_2$ optimal controller are guaranteed by assumptions (A1)–(A3), we wish to have a sufficient condition for assumptions (A1)–(A3) to hold. Our results generalize those in Trentelman and Stoorvogel (1995) (with some errors fixed), where single-rate sampled-data $\mathcal{H}_2$ optimal control is investigated.

**Proposition 3.** Assumptions (A1)--(A3) hold if the plant $G_0$ in equation (1) and $\sigma$ satisfy the following conditions:

(C1) $(A_0, B_0, C_0)$ is stabilizable and detectable and $\sigma$ is non-pathological with respect to $A_0$.

(C2) $(C_0, A_0)$ has no uncontrollable modes on the imaginary axis.

(C3) $(A_0, B_0)$ has no uncontrollable modes on the imaginary axis and $(A_0, B_0, C_0, D_{12})$ is left-invertible.

**Proof.** (C1) implies that

$$
(C_{20}, \tilde{A}, [\exp(A_0) t] B_{20})
$$

is stabilizable and detectable. Define the function

$$
g(s) = \exp(s(-1)h) + \exp(s(1)h) + \cdots + 1
\frac{\exp(s(h) - 1)}{\exp(s(h) - 1)}.
$$

It is analytic everywhere (the “poles” are all canceled by “zeros” there) and

$$
\{\text{zeros of } g(s)\} = \{s: \exp(s(h)) = 1, \exp(s(h)) \neq 1\}
$$

$$
= \{0, 2\pi/\alpha, \pm 2\pi, \pm 4\pi, \ldots\}
$$

The spectral mapping theorem says that the eigenvalues of the matrix $g(A_0)$ are precisely the values of $g$ at eigenvalues of $A_0$. Hence, $g(A_0)$ is singular if and only if $A_0$ has an eigenvalue at $jk\pi/\alpha$ for some $k \neq 0, \pm 1, \pm 2, \ldots$. This is impossible since $\sigma$ is non-pathological and $A_0$ is real. This shows that $g(A_0) = A_0^{k+1} + A_0^{k+2} + \cdots + I$ is nonsingular. Noting the fact that $A_0$ commutes with $A_0^{k+1} + A_0^{k+2} + \cdots + I$, we conclude that (C1) implies that $(C_{20}, \tilde{A}, B_{20})$ is stabilizable and detectable. Since $(C_{20}, \tilde{A}, B_{20})$ is obtained from $(C, \tilde{A}, B_{20})$ by deleting some inputs and outputs, the stabilizability and detectability of $(C_{20}, \tilde{A}, B_{20})$ implies that of $(C, \tilde{A}, B_{20})$.

Next, we show that (C1) and (C2) imply (A2). Actually, we will show a stronger statement: (C1) and (C2) imply

$$
\ker \begin{bmatrix}
\tilde{A} - I
C_1
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
B_2
D_{12}
\end{bmatrix}
\end{bmatrix}
= \{0\}
$$

for all $|\lambda| = 1$. Assume that (C1) and (C2) are true but

$$
\ker \begin{bmatrix}
\tilde{A} - I
C_1
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
B_2
D_{12}
\end{bmatrix}
\end{bmatrix}
\neq \{0\}
$$

for some $|\lambda| = 1$. Then at this $\lambda$, there exists

$$
[x^* u^* u^2 \cdots u^p]^* \neq 0
$$

such that

$$
\begin{bmatrix}
A_0 - \lambda I
A_0^{-1} B_{30}
A_0^{-2} B_{30}
\vdots
\end{bmatrix}
\begin{bmatrix}
C_{41}
G_{41}
A_0 B_{30}
D_{31}
\vdots
\end{bmatrix}
\begin{bmatrix}
x
C_{40} A_0
D_{31}
\vdots
\end{bmatrix}
= 0.
$$

(15)

Consider the second row of equation (15):

$$
C_{40} \exp(A_0 t)x + D_{31} u_1 + C_{40} \int_0^t \exp(A_0 \tau) d\tau B_{20} u_1 = 0.
$$

(16)

Evaluating equation (16) at $t = 0$, we obtain $C_{40} x + D_{31} u_1 = 0$. Differentiating equation (16) and then evaluating at $t = 0$, we obtain $A_0 x + B_{30} u_1 \in \ker(C_{40} A_0)$, which is the observable subspace of $(C_{40}, A_0)$ which is $A_0$-invariant. Now suppose

$$
C_{40} x + D_{31} u_1 = 0, \ldots, C_{40} x + D_{31} u_{r-1} = 0
$$

and $A_0 x + B_{30} u_1 \in \ker(C_{40} A_0)$, and $A_0 x + B_{30} u_{r-1} \in \ker(C_{40} A_0)$.

Consider the $(r + 1)$th row of equation (15):

$$
C_{40} \exp(A_0 t)\begin{bmatrix}
A_0^{-1} x + A_0^{-2} B_{30} u_1 + \cdots + B_{30} u_{r-1}
\end{bmatrix}
+ D_{31} u_r + C_{40} \int_0^t \exp(A_0 \tau) d\tau B_{20} u_r = 0.
$$

(17)

Evaluating at $t = 0$, we get

$$
C_{40} \begin{bmatrix}
A_0^{-1} x + A_0^{-2} B_{30} u_1 + \cdots + B_{30} u_{r-1}
\end{bmatrix} + D_{31} u_r = 0.
$$

(18)

Note that

$$
A_0^{-1} x = (A_0^{-1} - A_0^{-2}) + (A_0^{-2} - A_0^{-3}) + \cdots + (A_0 - I) + I,
$$

and

$$
(A_0^{-k} - A_0^{-k-1}) x + A_0^{-k-1} B_{30} u_k
= A_0^{-k-1} \int_0^t \exp(A_0 \tau) d\tau B_{20} u_k.
$$

(20)

for $k = 1, 2, \ldots, r - 1$. Hence equation (18) leads to $C_{40} x + D_{31} u_r = 0$. Differentiating equation (17) and evaluating at $t = 0$, we obtain $A_0 x + A_0^{-1} B_{30} u_1 + \cdots + B_{30} u_{r-1} + B_{30} u_r \in \ker(C_{40} A_0)$. Noting equations (19) and (20), we have $A_0 x + B_{30} u_r \in \ker(C_{40} A_0)$. By deduction, we have shown that for $r = 1, 2, \ldots, l$,

$$
A_0 x + B_{30} u_r \in \ker(C_{40} A_0), \quad C_{40} x + D_{31} u_r = 0.
$$

(21)

If there are $r_1 \neq r_2$ such that $u_{r_1} = u_{r_2}$, then

$$
B_{30} u_{r_1} - B_{30} u_{r_2} \in \ker(C_{40} A_0), \quad D_{31} u_{r_1} - u_{r_2} = 0.
$$

Let $x_0 = (j \omega t - A_0^{-1}) B_{30} u_r - u_r$ for any $j \omega$ not being an eigenvalue of $A_0$. Then $x_0 \in \ker(C_{40} A_0)$. Hence

$$
\begin{bmatrix}
A_0 - j \omega I
C_{40}
\end{bmatrix}
\begin{bmatrix}
x_0
u_{r-1} - u_r
\end{bmatrix}
= 0,
$$

which contradicts (C2). This shows that $u_1 = u_2 = \cdots = u_l = u_0$.

Now the first row of equation (15) becomes

$$
(A_0^{-1} - A_0^{-2}) x + (A_0^{-2} - A_0^{-3}) + \cdots + (I + A_0 - I) x = 0.
$$

(22)

Since $(A_0 - I) x + B_{30} u_r = \int_0^t \exp(A_0 t) d\tau B_{20} u_r = 0$, we have $x \in \ker(C_{40} A_0)$. Let

$$
C_{40} = 0
$$

and

$$
B_{30} u_r \in \ker(C_{40} A_0).
$$

(19)

For the case when $j \neq 1$, we have $x \in \ker(C_{40} A_0)$. If $u_0 \neq 0$, equation (21) implies $D_{31} u_r = 0$ and $B_{30} u_r \in \ker(C_{40} A_0)$. Let
\[ x_0 = (\omega I - A_x)B_{st} \in \ker \{ A_0 | A_x \} \text{ for any } \omega \text{ not being an eigenvalue of } A_x. \]

Then
\[ \begin{bmatrix} A_x - j \omega I & B_{st} \\ C_0 & D_{st} \end{bmatrix} x_0 = 0, \]

which contradicts (C2). On the other hand if \( u = 0 \), then \( x \neq 0 \) and
\[ \begin{bmatrix} A_x - j \omega I \\ C_0 \end{bmatrix} x = 0. \]

The second row immediately gives \( C_0 x = 0 \). The first row implies that \( \omega \) and \( x \) form an eigenvalue and eigenvector pair of \( \exp(0) = \exp(A_x \omega) \). Since \( \omega \) is non-pathological, there is a unique \( w \in \text{im} \{ \omega \} \) such that \( \omega \) is an eigenvalue of \( A_x \) and \( \exp(A_x \omega) \) has the same left Jordan chains as \( A_x \). Hence \( \exp(x^* \omega) \) has the same left Jordan chains as \( A_x \). Hence \( x^* \omega = 0 \). Therefore \( x^* \omega = 0 \), which also contradicts (C3).

6. Conclusions
The main contribution of this paper is the direct state-space solution of the multirate sampled-data \( x \) optimal control. This new method avoids converting the sampled-data problem to an equivalent discrete-time problem and it also reduces the effort in dealing with the causality issue due to the multirate sampling. It enjoys more theoretic elegance and at the same time leads to less computational effort. The same idea can be applied to multirate sampled-data \( x \) control, which is currently under study by the authors.

Acknowledgements—This work is supported by Hong Kong Research Grant Council and Sino Software Research Center.

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