Pergamon

PII: S0005–1098(98)00080–6

Automatica, Vol. 34, No. 11, pp. 1431–1437, 1998

Printed in Great Britain
0005-1098/98 $—see front matter

Brief Paper

Direct State Space Solution of Multirate Sampled-Data $\mathscr{H}_2$ Optimal Control*

LI QIU† and KAN TAN†

Key Words—Multirate; sampled-data systems; lifting technique; $\mathscr{H}_2$ optimization; causality constraint; nest algebra.

Abstract—In solving the multirate sampled-data $\mathscr{H}_2$ control problem using the lifting approach, one needs to solve a constraint discrete-time $\mathscr{H}_2$ optimal control problem for a generalized plant with infinite dimensional input/output spaces. To solve this problem, the existing sampled-data $\mathscr{H}_2$ design technique computes an equivalent finite dimensional discrete-time system and then designs the optimal $\mathscr{H}_2$ controller for the equivalent system. In this paper, we will show that this problem can be solved using state space formulas by dealing with operators directly. The operator compositions are computed explicitly using discrete multirate lifting and matrix exponentials. The advantages of the direct method are: it is straightforward, it has clear physical meanings, and it is more efficient computationally. A sufficient condition for the existence and uniqueness of multirate sampled-data $\mathscr{H}_2$ optimal controller is given in terms of the continuous-time plant. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

For a sampled-data control system, the plant is in general a continuous-time LTI system, the controller is composed of A/D converters (samplers), a digital computer, and D/A converters (holds). Hence, a sampled-data control system is a hybrid system involving both continuous-time and discrete-time signals. In many applications, the samplers and the holds do not necessarily operate in the same rate. In such cases, the system is called a multirate sampled-data control system. Since the plant evolves in the continuous-time, performance criteria are most readily formulated in the continuous-time domain. The studies of the single-rate sampled-data $\mathscr{H}_2$ design include (Bamieh and Pearson, 1992; Chen and Francis, 1991, 1995; Khargonekar and Sivashankar, 1991; Trentelman and Stoorvogel, 1995), where the sampled-data $\mathscr{H}_2$ design problem is converted to a pure discrete-time $\mathscr{H}_2$ design for an equivalent discrete-time system. For the multirate case, after lifting, causality constraint arises (Colaneri and Nicolao, 1995; Chen and Qiu, 1994; Qiu and Chen, 1994; Al-Rahmani and Franklin, 1992; Shu and Chen, 1996; Voulgaris and Bamieh, 1993; Voulgaris *et al*., 1994). Addressing $\mathscr{H}_2$ optimal control in particular, references (Colaneri and Nicolao, 1995; Voulgaris and Bamieh, 1993) find an equivalent pure discrete-time system and then design a controller with the causality constraint for the equivalent discrete-time system, references (Chen and Qiu, 1994; Qiu and Chen, 1994) give a direct method based on the frequency-domain technique and the nest algebra. Reference (Shu and Chen, 1996) treats a multirate pure discrete-time system by a state-space method. This paper gives a direct multirate sampled-data $\mathscr{H}_2$ design: when a system arises with infinite-dimensional input/output spaces by continuous-time lifting, we treat it directly instead of converting it to another equivalent pure discrete-time system with finite dimensional input/output spaces. It is shown that two Riccati equations are to be solved which contain matrix-valued operator compositions, and these compositions can be computed explicitly in state space formulas. We will be interested in the conditions guaranteeing the existence and uniqueness of the optimal sampled-data controller. Results in the single-rate case were given by (Khargonekar and Sivashankar, 1991; Trentelman and Stoorvogel, 1995). We generalize them to get a sufficient condition for the multirate case. The advantages of the direct state-space solution are: physical meanings are preserved so it is conceptionally more clear, and less computation is needed because the conversion to the pure discrete-time system is no longer necessary.

The results in this paper have been implemented in a MATLAB toolbox for multirate systems and control currently under development (Qiu *et al*., 1996).

The organization of this paper is as follows: Section 2 presents the multirate sampled-data configuration and the continuous-time lifting. Section 3 derives the direct state space solution of multirate sampled-data $\mathscr{H}_2$ optimal control. Section 4 addresses the computational issues of the operator compositions involved in Section 3. Section 5 gives a sufficient condition in terms of the continuous-time plant for the existence and uniqueness of the $\mathscr{H}_2$ optimal sampled-data control.

The notation used in this paper will be standard. For operator

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

from space $\mathscr{W} \oplus \mathscr{U}$ to space $\mathscr{Z} \oplus \mathscr{Y}$ and operator Q from $\mathscr{Y}$ to $\mathscr{U}$, the linear fractional transformation $P_{11} + P_{12}Q(I - P_{22}Q)^{-1}P_{21}$ is denoted by $\mathscr{F}(P, Q)$. For discrete-time signals and systems, the λ-transform, which is obtained from the z-transform by replacing z with $1/\lambda$, is used.

2. *Setup*

The setup of a multirate sampled-data control system is shown in Fig. 1. Here G_a is an analog generalized plant with two (vector) inputs, the exogenous input w and the control input u, and two (vector) outputs, the signal z to be regulated and the measured signal y. We assume that G_a is LTI with a state-space model

$$\hat{G}_a(s) = \left[\begin{array}{c|cc} A_a & B_{a1} & B_{a2} \\ \hline C_{a1} & 0 & D_{a12} \\ C_{a2} & 0 & 0 \end{array}\right]. \tag{1}$$

Three blocks in the direct feedthrough matrix of G_a are assumed to be zero with $D_{a11} = 0$ for the finiteness of the $\mathscr{H}_2$ norm which

* Received 6 May 1997; revised 16 December 1997; received in final form 28 April 1998. A preliminary version of this paper was presented at the 2nd Asian Control Conference, which was held in Seoul, Korea, during July, 1997. This paper was recommended for publication in revised form by Associate Editor André L. Tits under the direction of Editor Tamer Başar. *Corresponding author* Professor Li Qiu. Tel. + 852 2358 7067; Fax + 852 2358 1485; E-mail eeqiu@ee.ust.hk.

†Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

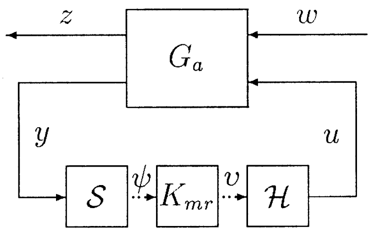

Fig. 1. The general multirate sampled-data setup.

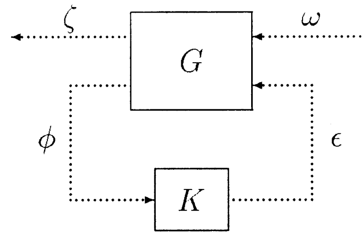

Fig. 2. The lifted system.

will be introduced later, $D_{a21} = 0$ for the proper functioning of the samplers when the exogenous input is an impulsive function, and $D_{a22} = 0$ for simplicity. We furthermore assume that all matrices in the state space model (1) are real. Symbols $\mathcal{S}$ and $\mathcal{H}$ represent multirate sampling (A/D) and hold (D/A) operations and are defined as follows:

$$\mathcal{S} = \begin{bmatrix} S_{m_1 h} & & \\ & \ddots & \\ & & S_{m_p h} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H_{n_1 h} & & \\ & \ddots & \\ & & H_{n_q h} \end{bmatrix}.$$

These correspond to performing the A/D conversions for the p channels of y periodically with periods $m_i h$, respectively, and the D/A conversions for the q channels of υ with periods $n_j h$, respectively. Here m_i and n_j are integers and h is a real number referred to as the *base period*. The linear multirate controller K_{mr} is assumed to satisfy three properties: periodicity, causality, and finite dimensionality; then they can be implemented in the form of some difference equations (Chen and Qiu, 1994).

The closed-loop system in Fig. 1 can be converted to an LTI discrete-time system with infinite dimensional input/output spaces by the lifting technique. Let $\sigma = lh$ with l the least common multiple of all m_i and n_j. Let L_σ be the continuous lifting operator mapping a continuous signal to a discrete sequence taking values in $\mathcal{K} := \mathcal{L}_2[0, \sigma)$ (Bamieh and Pearson, 1992), and L_m the m-fold discrete lifting operator (Khargonekar *et al.*, 1985). Define $\bar{m}_i = l/m_i$, $\bar{n}_j = l/n_j$, and

$$\mathcal{L}_M = \begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix}, \quad \mathcal{L}_N = \begin{bmatrix} L_{\bar{n}_1} & & \\ & \ddots & \\ & & L_{\bar{n}_q} \end{bmatrix}.$$

Then the multirate system of Fig. 1 can be converted into a single-rate discrete system in Fig. 2, where

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} L_\sigma & \\ & \mathcal{L}_M \mathcal{S} \end{bmatrix} G_a \begin{bmatrix} L_\sigma^{-1} & \\ & \mathcal{H} \mathcal{L}_N^{-1} \end{bmatrix}, \tag{2}$$

$$K = \mathcal{L}_N K_{mr} \mathcal{L}_M^{-1}. \tag{3}$$

When K is LTI, which is true if and only if $\mathcal{H} K_{mr} \mathcal{S}$ is σ-periodic in continuous-time (Chen and Qiu, 1994), the closed-loop system of G and K is an LTI discrete-time system. Note that ω and ζ are $\mathcal{K}$-valued sequences by lifting w and z.

We adopt the generalized $\mathcal{H}_2$ measure proposed for the periodic systems in (Bamieh and Pearson, 1992; Khargonekar and Sivashankar, 1991). Let F_a be a strictly causal σ-periodic system. Then the lifted system $F = L_\sigma F_a L_\sigma^{-1}$ is an LTI discrete-time system. It can be shown that F has a Hilbert–Schmidt operator-valued transfer function $\hat{F}$. Then F_a is said to be in $\mathcal{H}_2$ if $\hat{F}$ is in $\mathcal{H}_2$ in the sense of (Sz.-Nagy and Foias, 1970), and the $\mathcal{H}_2$ norm of F_a is defined to be the $\mathcal{H}_2$ norm of $\hat{F}$.

The multirate sampled-data $\mathcal{H}_2$ optimal control problem can then be stated as follows. Given a finite-dimensional analog plant G_a and sampling and hold schemes, design a multirate controller K_{mr}, which is causal, finite-dimensional, and σ-periodic in real time, such that the system shown in Fig. 1 is internally stabilized and the $\mathcal{H}_2$ norm of the closed loop map $\mathcal{F}(G_a, \mathcal{H} K_{mr} \mathcal{S})$ is minimized.

This $\mathcal{H}_2$ optimal control problem can be translated to the lifted domain. Due to causality of G_a and K_{mr}, the lifted systems G and K must be causal and satisfy some causality constraint characterized by nest operators (Chen and Qiu, 1994). Let $\mathcal{U}$ and $\mathcal{Y}$ be the spaces spanned by the parts of υ and ψ, respectively, occurring in the time interval $[k\sigma, (k+1)\sigma)$. For $0 \le r \le l$, let the subspace $\mathcal{U}_r$ be spanned by the part of υ occurring during the interval $[k\sigma + (l - r)h, (k+1)\sigma)$; similarly for $\mathcal{Y}_r$ and ψ. Then the causality of G_a implies that

$$\hat{G}_{22}(0)\mathcal{U}_r \subseteq \mathcal{Y}_{r+1}, \quad r = 0, 1, \ldots, l-1,$$

and the causality of K_{mr} requires that

$$\hat{K}(0)\mathcal{Y}_r \subseteq \mathcal{U}_r, \quad r = 0, 1, \ldots, l,$$

where $\hat{K}(0)$ is the D-matrix in the lifted controller. Using the nest operator terminology (Chen and Qiu, 1994), these can be rewritten as $\hat{G}_{22}(0) \in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\})$, the set of strict nest operators from $\{\mathcal{U}_r\}$ to $\{\mathcal{Y}_r\}$, and $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$, the set of nest operators from $\{\mathcal{Y}_r\}$ to $\{\mathcal{U}_r\}$.

The equivalent multirate sampled-data $\mathcal{H}_2$ optimal control problem in the lifted domain is then as follows: Given the lifted generalized plant G, design an internally stabilizing K satisfying $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$ to minimize the $\mathcal{H}_2$ norm of the closed loop transfer function $\mathcal{F}(\hat{G}, \hat{K})$ of the system in Fig. 2.

3. *$\mathcal{H}_2$ design for multirate sampled-data system*

Let

$$\hat{G}(\lambda) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_a & D_{21} & D_{22} \end{array}\right] \tag{4}$$

be the state-space model of the lifted plant as in equation (2). Here B_1, C_1, D_{11}, D_{12}, and D_{21} are (Hilbert–Schmidt) operators between appropriate spaces. The causality of G_a implies that $D_{22} \in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\})$.

The following assumptions are made.

(A1) (A, B_2, C_2) is stabilizable and detectable.

(A2)

$$\ker \begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix} = \{0\} \quad \text{for all } |\lambda| = 1.$$

(A3)

$$\left(\text{range} \begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right)^{\perp} = \{0\} \quad \text{for all } |\lambda| = 1.$$

The connection of these assumptions with the original analog plant data will be addressed in Section 5.

The solution of $\mathcal{H}_2$ optimal control depends, as usual, on the following Riccati equations:

$$X = A^*XA + C_1^*C_1 - (A^*XB_2 + C_1^*D_{12}) \times (B_2^*XB_2 + D_{12}^*D_{12})^{-1}(B_2^*XA + D_{12}^*C_1), \tag{5}$$

$$Y = AYA^* + B_1B_1^* - (AYC_2^* + B_1D_{21}^*) \times (C_2YC_2^* + D_{21}D_{21}^*)^{-1}(C_2YA^* + D_{21}B_1^*). \tag{6}$$

The solutions of X and Y of the two equations (5) and (6) are said to be stabilizing if

$$A - B_2(B_2^* X B_2 + D_{12}^* D_{12})^{-1} (B_2^* X A + D_{12}^* C_1)$$

and

$$A - (AYC_2^* + B_1 D_{21}^*)(C_2 Y C_2^* + D_{21} D_{21}^*)^{-1} C_2$$

are stable, respectively.

Proposition 1. Suppose the lifted plant G satisfies the assumptions (A1)–(A3). Then there exist unique stabilizing solutions to Riccati equations (5) and (6).

The proof of this proposition is the same as that for the case when B_1, C_1, D_{12}, and D_{21} are matrices, with the simple modification of replacing matrix transposes by operator adjoints wherever appropriate.

Let X and Y be the stabilizing solutions of Riccati equations (5) and (6). Then as usual it can be shown that $B_2^* X B_2 + D_{12}^* D_{12} > 0$ and $C_2 Y C_2^* + D_{21} D_{21}^* > 0$. We define

$$F = -(B_2^* X B_2 + D_{12}^* D_{12})^{-1}(B_2^* X A + D_{12}^* C_1),$$

$$L = -(AYC_2^* + B_1 D_{21}^*)(C_2 Y C_2^* + D_{21} D_{21}^*)^{-1},$$

$$H = -R^{-1}\Pi_{\mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})}[R^{*-1}(B_2^* XAYC_2^* + D_{12}^* C_1 Y C_2^* + B_2^* X B_1 D_{21}^* + D_{12}^* D_{11} D_{21}^*)\, S^{*-1}]S^{-1},$$

where $\Pi_{\mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})}$ is the orthogonal projection from $\mathcal{L}(\mathcal{Y},\mathcal{U})$ onto $\mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$ and $R \in \mathcal{N}(\{\mathcal{U}_r\})$ and $S \in \mathcal{N}(\{\mathcal{Y}_r\})$ satisfies

$$R^* R = B_2^* X B_2 + D_{12}^* D_{12}. \tag{7}$$

$$SS^* = C_2 Y C_2^* + D_{21} D_{21}^*. \tag{8}$$

The factorizations in equations (7) and (8) are always possible (Chen and Qiu, 1994). A simple choice might be the Cholesky factorization.

Theorem 1. Assume the plant G in the form of equation (4) satisfies assumptions (A1)–(A3). Then the lifted multirate sampled-data $\mathcal{H}_2$ optimal controller is given by

$$\hat{K}_{\text{opt}}(\lambda) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right],$$

where

$$A_K = A + B_2 F + LC_2 - B_2 H C_2 - (-L + B_2 H) D_{22}(I + HD_{22})^{-1}(F - HC_2),$$

$$B_K = (-L + B_2 H)(I + D_{22} H)^{-1},$$

$$C_K = (I + HD_{22})^{-1}(F - HC_2),$$

$$D_K = (I + HD_{22})^{-1} H.$$

The optimal $\mathcal{H}_2$ norm is

$$\|\mathcal{F}(\hat{G}, \hat{K}_{\text{opt}})\|_2^2 = \text{tr}(A^* XAY + XB_1 B_1^* + C_1^* C_1 Y - XY) + \|D_{11}\|_{\text{HS}}^2 - \|RHS\|_2^2.$$

Proof. The proof will be sketchy since it involves many standard materials as given in Chen and Francis (1995). The emphasis will be on the handling of the causality constraint. Let X and Y be the stabilizing solutions of equations (5) and (6). Then all stabilizing controllers without causality constraint are characterized by a linear fractional transformation

$$\hat{K} = \mathcal{F}(\hat{J}, \hat{Q}), \quad \hat{Q} \in \mathcal{RH}_\infty, \tag{9}$$

where

$$\hat{J}(\lambda) = \left[\begin{array}{c|cc} A + B_2 F + LC_2 - LD_{22} F & -L & B_2 + LD_2 \\ \hline F & 0 & I \\ -C_2 - D_{22} F & I & -D_{22} \end{array}\right].$$

Note that

$$\hat{K}(0) = \mathcal{F}(\hat{J}(0), \hat{Q}(0)) = \hat{Q}(0)(I + D_{22}\hat{Q}(0))^{-1}.$$

Since $D_{22} \in \mathcal{N}_s(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$, it is easy to show (Chen and Qiu, 1994) that $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$ if and only if $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$. Now it follows from the standard analysis, see e.g. Chen and Francis (1995), that under the controllers characterized in equation (9), the closed-loop transfer function is

$$\mathcal{F}(\hat{G}, \hat{K}) = \mathcal{F}[\hat{G}, \mathcal{F}(\hat{J}, \hat{Q})] = \hat{T}_{11} + \hat{T}_{12}\hat{Q}\hat{T}_{21},$$

where

$$\hat{T}_{11}(\lambda) = \left[\begin{array}{cc|c} A + B_2 F & B_2 F & B_1 \\ 0 & A + LC_2 & -B_1 - LD_{21} \\ \hline C_1 + D_{12} F & D_{12} F & D_{11} \end{array}\right],$$

$$\hat{T}_{12}(\lambda) = \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array}\right],$$

$$\hat{T}_{21}(\lambda) = \left[\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right].$$

Let $\hat{T}^{\sim}(\lambda) = \hat{T}(1/\lambda)'$ be the adjoint of $\hat{T}(\lambda)$. Then for $\hat{T}_{11}$, $\hat{T}_{12}$, $\hat{T}_{21}$ defined by above equations, we have (Chen and Francis, 1995)

$$\hat{T}_{12}^{\sim}\hat{T}_{12} = B_2^* X B_2 + D_{12}^* D_{12}, \tag{10}$$

$$\hat{T}_{21}\hat{T}_{21}^{\sim} = C_2 Y C_2^* + D_{21} D_{21}^*, \tag{11}$$

and $(\hat{T}_{12}^{\sim}\hat{T}_{11}\hat{T}_{21}^{\sim})^{\sim} \in \mathcal{H}_2$ with

$$\Pi_{\mathcal{H}_2}(\hat{T}_{12}^{\sim}\hat{T}_{11}\hat{T}_{21}^{\sim}) = B_2^* XAYC_2^* + D_{12}^* C_1 Y C_2^* + B_2^* X B_1 D_{21}^* + D_{12}^* D_{11} D_{21}^*. \tag{12}$$

Now carry out matrix factorizations in equations (7) and (8). Define

$$\hat{U} = \begin{bmatrix} R^{*-1}\hat{T}_{12}^{\sim} \\ I - \hat{T}_{12} R^{-1} R^{*-1} \hat{T}_{12}^{\sim} \end{bmatrix},$$

$$\hat{V} = [\hat{T}_{21}^{\sim} S^{*-1} \quad I - \hat{T}_{21}^{\sim} S^{*-1} S^{-1} \hat{T}_{21}].$$

Then equations (10) and (11) imply $\hat{U}^{\sim}\hat{U} = I$ and $\hat{V}\hat{V}^{\sim} = I$. Hence

$$\begin{aligned} \|\mathcal{F}(\hat{G}, \hat{K})\|_2^2 &= \|\hat{T}_{11} + \hat{T}_{12}\hat{Q}\hat{T}_{21}\|_2^2 \\ &= \|U(\hat{T}_{11} + \hat{T}_{12}\hat{Q}\hat{T}_{21})V\|_2^2 \\ &= \|R^{*-1}\hat{T}_{12}^{\sim}\hat{T}_{11}\hat{T}_{21}^{\sim}S^{*-1} + R\hat{Q}S\|_2^2 \\ &\quad + \|\hat{W}_{12}\|_2^2 + \|\hat{W}_{21}\|_2^2 + \|\hat{W}_{22}\|_2^2, \end{aligned}$$

where

$$\begin{bmatrix} R^{*-1}\hat{T}_{12}^{\sim}\hat{T}_{11}\hat{T}_{21}^{\sim}S^{*-1} & \hat{W}_{12} \\ \hat{W}_{21} & \hat{W}_{22} \end{bmatrix} = \hat{U}\hat{T}_{11}\hat{V}.$$

Note that $\hat{W}_{12}$, $\hat{W}_{21}$, $\hat{W}_{22}$ are independent of $\hat{Q}$. So minimizing $\|\mathcal{F}(\hat{G}, \hat{K})\|_2^2$ is equivalent to minimizing $\|R^{*-1}\hat{T}_{12}^{\sim}\hat{T}_{11}\hat{T}_{21}^{\sim}S^{*-1} + R\hat{Q}S\|_2^2$ for $\hat{Q} \in \mathcal{H}_\infty$ satisfying $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$. From equation (12), we know that

$$R^{*-1}\hat{T}_{12}^{\sim}\hat{T}_{11}\hat{T}_{21}^{\sim}S^{*-1} \in R^{*-1}(B_2^* XAYC_2^* + D_{12}^* C_1 Y C_2^* + B_2^* X B_1 D_{21}^* + D_{12}^* D_{11} D_{21}^*) S^{*-1} + \mathcal{H}_2^{\perp}.$$

Since $R\hat{Q}S \in \mathcal{H}_\infty$, it follows that $R\hat{Q}S$ can only be used to cancel part of the constant term of $R^{*-1}\hat{T}_{12}^{\sim}\hat{T}_{11}\hat{T}_{21}^{\sim}S^{*-1}$. Consider the causality constraint $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$ and that $R \in \mathcal{N}(\{\mathcal{U}_r\})$ and $S \in \mathcal{N}(\{\mathcal{Y}_r\})$ are invertible. The optimal $\hat{Q}$ is given by

$$R\hat{Q}_{\text{opt}}S = -\Pi_{\mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r}[R^{*-1}(B_2^* XAYC_2^* + D_{12}^* C_1 Y C_2^* + B_2^* X B_1 D_{21}^* + D_{12}^* D_{11} D_{21}^*)\, S^{*-1}].$$

Hence $\hat{Q}_{\rm opt} = H$. The optimal $\mathscr{H}_2$ norm is given by

$$\|\mathscr{F}(\hat{G}, \hat{K})\|_2^2 = \|\hat{T}_{11}\|_2^2 - \|R\hat{Q}_{\rm opt}S\|_2^2 = \|\hat{T}_{11}\|_2^2 - \|RHS\|_2^2.$$

It remains to show that

$$\|\hat{T}_{11}\|_2^2 = \mathrm{tr}(A^*XAY + XB_1B_1^* + C_1^*C_1Y - XY) + \|D_{11}\|_{HS}^2.$$

It is easy to verify that

$$\hat{T}_{11}(\lambda) = \hat{T}_{\rm F}(\lambda) + \hat{T}_{12}(\lambda)\hat{T}_{\rm L}(\lambda),$$

where

$$\hat{T}_F(\lambda) = \left[\begin{array}{c|c} A + B_2F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array}\right],$$

$$\hat{T}_L(\lambda) = \left[\begin{array}{c|c} A + LC_2 & -B_1 - LD_{21} \\ \hline F & 0 \end{array}\right].$$

Straightforward computation shows that $\hat{T}_F$ and $\hat{T}_{12}\hat{T}_L$ are orthogonal to each other and equation (10) implies $\|\hat{T}_{12}\hat{T}_L\|_2 = \|R\hat{T}_L\|_2$. Hence,

$$\begin{aligned}\|\hat{T}_{11}\|_2^2 &= \|\hat{T}_F\|_2^2 + \|R\hat{T}_L\|_2^2 \\ &= \|D_{11}\|_{\rm HS}^2 + \mathrm{tr}B_1^*XB_1 + \mathrm{tr}\, RFYF^*R^* \\ &= \|D_{11}\|_{\rm HS}^2 + \mathrm{tr}\, XB_1B_1^* \\ &\quad + \mathrm{tr}(A^*XB_2 + C_1^*D_{12})(B_2^*XB_2 + D_{12}^*D_{12})^{-1} \\ &\quad \times (B_2^*XA + D_{12}^*C_1)\; Y \\ &= \|D_{11}\|_{\rm HS}^2 + \mathrm{tr}\, XB_1B_1^* + \mathrm{tr}\,(A^*XA + C_1^*C_1 - X)Y. \quad \square\end{aligned}$$

The optimal control formula in Theorem 1 first appeared in Qiu *et al.* (1996). A (slightly less general) complete version of this theorem is also independently obtained in Mirkin and Palmor (1997). The proof here is different from that in Mirkin and Palmor (1997).

4. *Computation of the operator compositions*

From the development in the last section, it is seen that to compute the multirate sampled-data $\mathscr{H}_2$ optimal controller and the optimal $\mathscr{H}_2$ norm using the direct state space solution we need from the lifted system G matrices A, B_2, C_2, D_{22}, operator compositions:

$$\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} [B_1^*\; D_{21}^*], \qquad \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} [C_1\; D_{12}],$$

$D_{12}^*D_{11}D_{21}^*$, and norm $\|D_{11}\|_{\rm HS}$ as the input data. The matrices A, B_2, C_2, D_{22} are easy to obtain. A way to compute $\|D_{11}\|_{\rm HS}$ using matrix exponentials is given in Bamieh and Pearson (1992). The computation of the required operator compositions, however, is rather nontrivial. For a special case when all m_i, $i = 1, \ldots, p$, are the same and all $n_j, j = 1, \ldots, q$, are the same (the dual rate case), integral formulas for these operator compositions are obtained in Qiu and Chen (1994). There are characteristic functions involved in integral formulas, which make the computation quite complicated. The characteristic functions arise due to the multirate nature of the controller. To avoid this complication, we will show that these operator compositions can be obtained through a two-step lifting: first lift the plant $G_{\rm a}$ in the base period h and then lift l-fold in discrete-time. The resulted system can be related to G easily and because of this the data on the lifted system G can be obtained from some data associated with the intermediate system obtained after lifting $G_{\rm a}$ in the base period. In the end it is shown that the required data of G can also be obtained using matrix exponentials.

Recall equation (2) in Section 2:

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} L_\sigma & \\ & \mathscr{L}_{\rm M}\mathscr{S} \end{bmatrix} \begin{bmatrix} G_{a11} & G_{a12} \\ G_{a21} & G_{a22} \end{bmatrix} \begin{bmatrix} L_\sigma^{-1} & \\ & \mathscr{H}\mathscr{L}_N^{-1} \end{bmatrix}.$$

It is possible to find matrices L_H and L_S such that

$$\mathscr{L}_M\mathscr{S} = L_SL_lS_h \text{ and } \mathscr{H}\mathscr{L}_N^{-1} = H_hL_l^{-1}L_H.$$

Actually, in the first system period $[0, \sigma]$, $L_{\rm H}$ and $L_{\rm S}$ are required to satisfy

$$L_S \begin{bmatrix} y_1(0) \\ \vdots \\ y_p(0) \\ y_1(h) \\ \vdots \\ y_p(h) \\ \vdots \\ y_1[(l-1)h] \\ \vdots \\ y_p[(l-1)h] \end{bmatrix} = \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_1(\bar{m}-1) \\ \vdots \\ \psi_p(0) \\ \vdots \\ \psi_p(\bar{m}_p - 1) \end{bmatrix},$$

$$L_H \begin{bmatrix} v_1(0) \\ \vdots \\ v_1(\bar{n}-1) \\ \vdots \\ v_q(0) \\ \vdots \\ v_q(\bar{n}_q - 1) \end{bmatrix} = \begin{bmatrix} u_1(0) \\ \vdots \\ u_q(0) \\ u_1(h) \\ \vdots \\ u_q(h) \\ \vdots \\ u_1[(l-1)h] \\ \vdots \\ u_q[(l-1)h] \end{bmatrix}.$$

In the subsequent system periods, things are the same except possible time shifts. Therefore, $L_{\rm S}$ is a $\sum_{i=1}^p \bar{m}_i \times lp$ block matrix with all blocks equal to zero matrices except the $(\sum_{k=1}^{i-1}\bar{m}_k + k_i + 1, k_im_ip + i)$th blocks which are equal to identity matrices and $L_{\rm H}$ is an $lq \times \sum_{j=1}^q \bar{n}_j$ matrix with all blocks equal to zero matrices except the $(rq + j, \sum_{k=1}^{j-1}\bar{n}_k + \lfloor r/n_j \rfloor)$th blocks which are equal to identity matrices. Here $k_i = 0, \ldots, m_i - 1$, $r = 0, \ldots, (l-1)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$ and $\lfloor \cdot \rfloor$ means the integer part.

Now it is clear that

$$G = \begin{bmatrix} L_\sigma L_h^{-1} L_l^{-1} L_l L_h & \\ & L_{\rm S}L_lS_h \end{bmatrix} \begin{bmatrix} G_{a11} & G_{a12} \\ G_{a21} & G_{a22} \end{bmatrix} \times \begin{bmatrix} L_h^{-1} L_l^{-1} L_l L_h L_\sigma & \\ & H_h L_l^{-1} L_{\rm H} \end{bmatrix}.$$

Let

$$G_h = \begin{bmatrix} G_{h11} & G_{h12} \\ G_{h21} & G_{h22} \end{bmatrix} = \begin{bmatrix} L_h & \\ & S_h \end{bmatrix} \begin{bmatrix} G_{a11} & G_{a12} \\ G_{a21} & G_{a22} \end{bmatrix} \begin{bmatrix} L_h^{-1} & \\ & H_h \end{bmatrix}.$$

Then G_h is the equivalent discrete time system of G_a lifted in base period h and it is well known (Chen and Francis, 1995) that G_h has state-space model

$$\hat{G}_h(\lambda) = \left[\begin{array}{c|cc} A_h & B_{h1} & B_{h2} \\ \hline C_{h1} & D_{h11} & D_{h12} \\ C_{h2} & 0 & 0 \end{array}\right]$$

which is formed by matrices

$$A_h = \exp(A_{\rm a}h), \quad B_{h2} = \int_0^h \exp(A_at)\,\mathrm{d}t\, B_{a2}, \quad C_{h2} = C_{a2},$$

and operators

$$B_{h1} : B_{h1}\omega = \int_0^h \exp(A_a(h-t))B_{a1}\omega(t)\,\mathrm{d}t,$$

$$C_{h1} : (C_{h1}\xi)(t) = C_{a1}\exp(A_at)\xi, \quad t \in [0, h),$$

$$D_{h11} : (D_{h11}\omega)(t) = C_{a1}\int_0^t \exp(A_a(t-\tau))B_{a1}\omega(\tau)\,\mathrm{d}\tau, \quad t \in [0, h)$$

$$D_{h12} : (D_{h12}\upsilon)(t) = D_{a12}\upsilon + C_{a1}\int_0^t \exp(A_a\tau)\,\mathrm{d}\tau B_{a2}\upsilon, \quad t \in [0, h).$$

Therefore,

$$\tilde{G} = \begin{bmatrix} L_l & \\ & L_l \end{bmatrix} \begin{bmatrix} G_{h11} & G_{h12} \\ G_{h21} & G_{h22} \end{bmatrix} \begin{bmatrix} L_l^{-1} & \\ & L_l^{-1} \end{bmatrix}$$

has state-space model

$$\hat{\tilde{G}}(\lambda) = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right],$$

where

$$\tilde{A} = A_h^l,$$

$$\tilde{B}_i = [A_h^{l-1} B_{hi}\ A_h^{l-2} B_{hi} \cdots B_{hi}],$$

$$\tilde{C}_i = \begin{bmatrix} C_{hi} \\ C_{hi}A_h \\ \vdots \\ C_{hi}A_h^{l-1} \end{bmatrix},$$

$$\tilde{D}_{ij} = \begin{bmatrix} D_{hij} & 0 & \cdots & 0 \\ C_{hi}B_{hj} & D_{hij} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ C_{hi}A_h^{l-2}B_{hj} & C_{hi}A_h^{l-3}B_{hj} & \cdots & D_{hij} \end{bmatrix}, \quad i, j = 1, 2.$$

Since

$$G = \begin{bmatrix} L_\sigma L_h^{-1} L_l^{-1} & \\ & L_S \end{bmatrix} \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} \begin{bmatrix} L_l L_h L_\sigma & \\ & L_H \end{bmatrix}$$

and L_σ, L_h, L_l are unitary operators, its state-space model

$$\hat{G}(\lambda) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

satisfies

$$\begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix} = \begin{bmatrix} I & \\ & L_S \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B}_2 \\ \tilde{C}_2 & \tilde{D}_{22} \end{bmatrix} \begin{bmatrix} I & \\ & L_H \end{bmatrix},$$

$$\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} [B_1^*\ D_{21}^*] = \begin{bmatrix} \tilde{B}_1\tilde{B}_1^* & \tilde{B}_1\tilde{D}_{21}^* L_S^* \\ L_S\tilde{D}_{21}\tilde{B}_1^* & L_S\tilde{D}_{21}\tilde{D}_{21}^* L_S^* \end{bmatrix},$$

$$\begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} [C_1\ D_{12}] = \begin{bmatrix} \tilde{C}_1^*\tilde{C}_1 & \tilde{C}_1^*\tilde{D}_{12}L_H \\ L_H^*\tilde{D}_{12}^*\tilde{C}_1 & L_H^*\tilde{D}_{12}^*\tilde{D}_{12}L_H \end{bmatrix},$$

$$D_{12}^* D_{11} D_{21}^* = L_H^* \tilde{D}_{12}^* \tilde{D}_{11} \tilde{D}_{21}^* L_S^*.$$

The detailed structures of $\tilde{B}_1$, $\tilde{C}_1$, $\tilde{D}_{11}$ $\tilde{D}_{12}$, and $\tilde{D}_{21}$ reveal that the required operator compositions can be computed if operator compositions $B_{h1}B_{h1}^*$,

$$\begin{bmatrix} C_{h1}^* \\ D_{h12}^* \end{bmatrix} [C_{h1}\ D_{h12}], \quad \begin{bmatrix} C_{h1}^* \\ D_{h12}^* \end{bmatrix} D_{h11}\ B_{h1}^*$$

can be computed.

Proposition 2. Let

$$P = \exp\left(\begin{bmatrix} -A_a^* & 0 & C_{a1}^*C_{a1} & 0 \\ -B_{a2}^* & 0 & D_{a12}^*C_{a1} & 0 \\ 0 & 0 & A_a & B_{a1}B_{a1}^* \\ 0 & 0 & 0 & -A_a^* \end{bmatrix} h\right), \tag{13}$$

$$P = \exp\left(\begin{bmatrix} -A_a^* & 0 & C_{a1}^*C_{a1} & C_{a1}^*D_{a12} \\ -B_{a2}^* & 0 & D_{a12}^*C_{a1} & D_{a12}^*D_{a12} \\ 0 & 0 & A_a & B_{a2} \\ 0 & 0 & 0 & 0 \end{bmatrix} h\right), \tag{14}$$

and let P and Q be partitioned into 4×4 block matrices compatibly with the right-hand side matrices in equations (13) and (14), respectively. Then

$$A_h = P_{33},$$

$$B_{h2} = Q_{34},$$

$$C_{h2} = C_{a2},$$

$$B_{h1}B_{h1}^* = P_{34}P_{33}^*,$$

$$\begin{bmatrix} C_{h1}^* \\ D_{h12}^* \end{bmatrix} [C_{h1}\ D_{h12}] = \begin{bmatrix} Q_{33} & Q_{34} \\ Q_{43} & Q_{44} \end{bmatrix}^* \begin{bmatrix} Q_{13} & Q_{14} \\ Q_{23} & Q_{24} \end{bmatrix},$$

$$\begin{bmatrix} C_{h1}^* \\ D_{h12}^* \end{bmatrix} D_{h11}B_{h1}^* = \begin{bmatrix} Q_{33} & Q_{34} \\ Q_{43} & Q_{44} \end{bmatrix}^* \begin{bmatrix} P_{14} \\ P_{24} \end{bmatrix} P_{33}^*.$$

The following lemma is needed in proving Proposition 2.

Lemma 1 (Van Loan, 1978). Let A_{11} and A_{22} both be square and define

$$\begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix} := \exp\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} t\right), \quad t > 0.$$

Then $F_{11}(t) = \exp(A_{11}t)$, $F_{22}(t) = \exp(A_{22}t)$, and

$$F_{12}(t) = \int_0^t \exp(A_{11}(t-\tau)) A_{12} \exp(A_{22}\tau)\,d\tau$$

$$= \int_0^t \exp(A_{11}\tau) A_{12} \exp(A_{22}(t-\tau))\,d\tau.$$

Proof (Proposition 2). The first five equalities are actually proved in Chen and Francis (1995). We only need to prove the last equality. It can be shown that

$$\begin{bmatrix} C_{h1}^* \\ D_{h12}^* \end{bmatrix} D_{h11}\ B_{h1}^* = \int_0^h \begin{bmatrix} (C_{a1}\exp(A_a t))^* \\ (D_{a12} + \int_0^t C_{a1}\exp(A_a(t-\tau))B_{a2}d\tau)^* \end{bmatrix}$$

$$\times C_{a1} \int_0^t \exp(A_a(t-\tau)) B_{a1}B_{a1}^* \exp(A_a^*(h-\tau))\,d\tau\,dt.$$

From Lemma 1 and equation (14), we have

$$\exp\left(\begin{bmatrix} A_a^* & 0 \\ B_{a2}^* & 0 \end{bmatrix} h\right) \begin{bmatrix} P_{13} & P_{14} \\ P_{23} & P_{24} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$= \int_0^h \exp\left(\begin{bmatrix} A_a^* & 0 \\ B_{a2}^* & 0 \end{bmatrix} t\right) \begin{bmatrix} C_{a1}^*C_{a1} & 0 \\ D_{a12}^*C_{a1} & 0 \end{bmatrix}$$

$$\times \exp\left(\begin{bmatrix} A_a & B_{a1}B_{a1}^* \\ 0 & -A_a^* \end{bmatrix} t\right) dt \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$= \int_0^h \begin{bmatrix} \exp(A_a^* t) & 0 \\ B_{a2}^* \int_0^t \exp(A_a^*(t-\tau))\,d\tau & I \end{bmatrix} \begin{bmatrix} C_{a1}^*C_{a1} & 0 \\ D_{a12}^*C_{a1} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \int_0^t \exp(A_a(t-\tau))B_{a1}B_{a1}^* \exp(-A_a^*\tau)\,d\tau \\ \exp(-A_a^* t) \end{bmatrix} dt$$

$$= \int_0^h \begin{bmatrix} \exp(A_a^* t) C_{a1}^* C_{a1} & 0 \\ D_{a12}^*C_{a1} + B_{a2}^* \int_0^t \exp(A_a^*(t-\tau))\,d\tau C_{a1}^*C_{a1} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \int_0^t \exp(A_a(t-\tau))B_{a1}B_{a1}^* \exp(-A_a^*\tau)\,d\tau \\ \exp(-A_a^* t) \end{bmatrix} dt$$

$$= \int_0^h \begin{bmatrix} \exp(A_a^* t) C_{a1}^* C_{a1} \\ D_{a12}^*C_{a1} + B_{a2}^* \int_0^t \exp(A_a^*(t-\tau))\,d\tau C_{a1}^*C_{a1} \end{bmatrix}$$

$$\int_0^t \exp(A_a^*(t-\tau)) B_{a1}B_{a1}^* \exp(-A_a^*\tau)\,d\tau dt.$$

Therefore

$$\begin{bmatrix} C_{h1}^* \\ D_{h12}^* \end{bmatrix} D_{h11} B_{h1}^* = \exp\left(\begin{bmatrix} A_a & B_{a2} \\ 0 & 0 \end{bmatrix}^* h \right) \begin{bmatrix} P_{13} & P_{14} \\ P_{23} & P_{24} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \exp(A_a^* h)$$

$$= \begin{bmatrix} Q_{33} & Q_{34} \\ Q_{43} & Q_{44} \end{bmatrix}^* \begin{bmatrix} P_{14} \\ P_{24} \end{bmatrix} P_{33}^*. \qquad \square$$

5. *On the existence and uniqueness of the $\mathscr{H}_2$ optimal controller*

In this section, we address the condition in terms of the continuous-time plant G_a which ensures assumptions (A1)–(A3) in terms of lifted system G. Since the existence and uniqueness of multirate sampled-data $\mathscr{H}_2$ optimal controller are guaranteed by assumptions (A1)–(A3), we wish to have a sufficient condition for assumptions (A1)–(A3) to hold. Our results generalize those in Trentelman and Stoorvogel (1995) (with some errors fixed), where single-rate sampled-data $\mathscr{H}_2$ optimal control is investigated.

Proposition 3. Assumptions (A1)–(A3) hold if the plant G_a in equation (1) and σ satisfy the following conditions:

(C1) (A_a, B_{a2}, C_{a2}) is stabilizable and detectable and σ is non-pathological with respect to A_a.
(C2) (C_{a1}, A_a) has no unobservable modes on the imaginary axis,
$(A_a, B_{a2}, C_{a1}, D_{a12})$ is right-invertible and has no zero at 0.
(C3) (A_a, B_{a1}) has no uncontrollable modes on the imaginary axis and $(A_a, B_{a1}, C_{a2}, 0)$ is left-invertible.

Proof. (C1) implies that

$$(C_{a2}, \tilde{A}, \textstyle\int_0^\sigma \exp(A_a t)\,\mathrm{d}t\, B_{a2})$$
$$= (C_{h2}, \tilde{A}, (A_h^{l-1} + A_h^{l-2} + \cdots + I) B_{h2})$$

is stabilizable and detectable.

Define the function

$$g(s) = \exp(s(l-1)h) + \exp(s(l-2)h) + \cdots + 1$$
$$= \frac{\exp(slh) - 1}{\exp(sh) - 1}.$$

It is analytic everywhere (the "poles" are all canceled by "zeros" there) and

$$\{\text{zeros of } g\} = \{s\colon \exp(shl) = 1, \exp(sh) \neq 1\}$$
$$= \{\mathrm{j}k2\pi/\sigma,\ k \neq 0,\ \pm l,\ \pm 2l, \dots\}$$

The spectral mapping theorem says that the eigenvalues of the matrix $g(A_a)$ are precisely the values of g at eigenvalues of A_a. Hence, $g(A_a)$ is singular if and only if A_a has an eigenvalue at $\mathrm{j}k2\pi/\sigma$ for some $k \neq 0, \pm l, \pm 2l, \dots$. This is impossible since σ is non-pathological and A_a is real. This shows that $g(A_a) = A_h^{l-1} + A_h^{l-2} + \cdots + I$ is nonsingular. Noting the fact that $\tilde{A}$ commutes with $A_h^{l-1} + A_h^{l-2} + \cdots + I$, we conclude that (C1) implies that $(C_{h2}, \tilde{A}, B_{h2})$ is stabilizable and detectable. Since $(C_{h2}, \tilde{A}, B_{h2})$ is obtained from (C, A, B) by deleting some inputs and outputs, the stabilizability and detectability of $(C_{h2}, \tilde{A}, B_{h2})$ implies those of (C, A, B) .

Next, we show that (C1) and (C2) imply (A2). Actually, we will show a stronger statement: (C1) and (C2) imply

$$\ker \begin{bmatrix} \tilde{A} - \lambda I & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix} = \{0\}$$

for all $|\lambda| = 1$. Assume that (C1) and (C2) are true but

$$\ker \begin{bmatrix} \tilde{A} - \lambda & I\tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix} \neq \{0\}$$

for some $|\lambda| = 1$. Then at this λ, there exists

$$[x^* \ \ u_1^* \ \ u_2^* \ \ \cdots \ \ u_l^*]^* \neq 0$$

such that

$$\begin{bmatrix} A_h^l - \lambda I & A_h^{l-1} B_{h2} & A_h^{l-2} B_{h2} & \cdots & B_{h2} \\ C_{h1} & D_{h12} & 0 & \cdots & 0 \\ C_{h1} A_h & C_{h1} B_{h2} & D_{h12} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C_{h1} A_h^{l-1} & C_{h1} A_h^{l-2} B_{h2} & C_{h1} A_h^{l-3} B_{h2} & \cdots & D_{h12} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \\ \vdots \\ u_l \end{bmatrix} = 0. \tag{15}$$

Consider the second row of equation (15):

$$C_{a1} \exp(A_a t) x + D_{a12} u_1 + C_{a1} \int_0^t \exp(A_a \tau)\,\mathrm{d}\tau B_{a2} u_1 = 0. \tag{16}$$

Evaluating equation (16) at $t = 0$, we obtain $C_{a1}x + D_{a12}u_1 = 0$. Differentiating equation (16) and then evaluating at $t = 0$, we obtain $A_a x + B_{a2} u_1 \in \langle \ker C_{a1} | A_a \rangle$. where $\langle \ker C_{a1} | A_a \rangle$ is the unobservable subspace of (C_{a1}, A_a) which is A_a-invariant. Now suppose

$$C_{a1}x + D_{a12}u_1 = 0, \dots, C_{a1}x + D_{a12}u_{r-1} = 0$$

and $A_a x + B_{a2} u_1 \in \langle \ker C_{a1} | A_a \rangle, \dots, A_a x + B_{a2} u_{r-1} \in \langle \ker C_{a1} | A_a \rangle$. Consider the $(r+1)$th row of equation (15):

$$C_{a1} \exp(A_a t)[A_h^{r-1} x + A_h^{r-2} B_{h2} u_1 + \cdots + B_{h2} u_{r-1}]$$
$$+ D_{a12} u_r + C_{a1} \int_0^t \exp(A_a \tau)\,\mathrm{d}\tau\, B_{a2} u_r = 0. \tag{17}$$

Evaluating at $t = 0$, we get

$$C_{a1}[A_h^{r-1} x + A_h^{r-2} B_{h2} u_1 + \cdots + B_{h2} u_{r-1}] + D_{a12} u_r = 0. \tag{18}$$

Note that

$$A_h^{r-1} = (A_h^{r-1} - A_h^{r-2}) + (A_h^{r-2} - A_h^{r-3}) + \cdots + (A_h - I) + I, \tag{19}$$

and

$$(A_h^{r-k} - A_h^{r-k-1}) x + A_h^{r-k-1} B_{h2} u_k$$
$$= A_h^{r-k-1} \int_0^h \exp(A_a t)\,\mathrm{d}t (A_a x + B_{a2} u_k), \tag{20}$$

for $k = 1, 2, \dots, r-1$. Hence equation (18) leads to $C_{a1}x + D_{a12}u_r = 0$. Differentiating equation (17) and evaluating at $t = 0$, we obtain $A_a[A_h^{r-1}x + A_h^{r-2}B_{h2}u_1 + \cdots + B_{h2}u_{r-1}] + B_{a2}u_r \in \langle \ker C_{a1} | A_a \rangle$. Noting equations (19) and (20), we have $A_a x + B_{a2} u_r \in \langle \ker C_{a1} | A_a \rangle$. By deduction, we have shown that for $r = 1, 2, \dots, l$,

$$A_a x + B_{a2} u_r \in \langle \ker C_{a1} | A_a \rangle, \quad C_{a1} x + D_{a12} u_r = 0. \tag{21}$$

If there are $r_1 \neq r_2$ such that $u_{r_1} \neq u_{r_2}$, then

$$B_{a2}(u_{r_1} - u_{r_2}) \in \langle \ker C_{a1} | A_a \rangle, \quad D_{a12}(u_{r_1} - u_{r_2}) = 0.$$

Let $x_0 = (\mathrm{j}\omega I - A_a)^{-1} B_{a2}(u_{r_1} - u_{r_2})$ for any $\mathrm{j}\omega$ not being an eigenvalue of A_a. Then $x_0 \in \langle \ker C_{a1} | A_a \rangle$. Hence

$$\begin{bmatrix} A_a - \mathrm{j}\omega I & B_{a2} \\ C_{a1} & D_{a12} \end{bmatrix} \begin{bmatrix} x_0 \\ u_{r_1} - u_{r_2} \end{bmatrix} = 0,$$

which contradicts (C2). This shows that $u_1 = u_2 = \cdots = u_l =: u$.

Now the first row of equation (15) becomes $(A_h^l - \lambda I)x + (A_h^{l-1} + A_h^{l-2} + \cdots + I) B_{h2} u = 0$ which can be rewritten as

$$(A_h^{l-1} + A_h^{l-2} + \cdots + I)[(A_h - I)x + B_{h2}u] = (\lambda - 1)x. \tag{22}$$

Since $(A_h - I)x + B_{h2}u = \int_0^h \exp(A_a t)\,\mathrm{d}t (A_a x + B_{a2} u)$ and $A_a x + B_{a2} u \in \langle \ker C_{a1} | A_a \rangle$, we get $(\lambda - 1)\ x \in \langle \ker C_{a1} | A_a \rangle$. There are possibly two cases: $\lambda \neq 1$ and $\lambda = 1$.

For the case when $\lambda \neq 1$, we have $x \in \langle \ker C_{a1} | A_a \rangle$. If $u \neq 0$, equation (21) implies $D_{a12} u = 0$ and $B_{a2} u \in \langle \ker C_{a1} | A_a \rangle$. Let

$x_0 = (\mathrm{j}\omega I - A_a)^{-1} B_{a2} u \in \langle \ker C_{a1} | A_a \rangle$ for any $\mathrm{j}\omega$ not being an eigenvalue of A_a. Then

$$\begin{bmatrix} A_a - \mathrm{j}\omega I & B_{a2} \\ C_{a1} & D_{a12} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} = 0,$$

which contradicts (C2). On the other hand if $u = 0$, then $x \neq 0$ and

$$\begin{bmatrix} A_h^l - \lambda I \\ C_{h1} \end{bmatrix} x = 0.$$

The second row immediately gives $C_{a1}x = 0$. The first row implies that λ and x form an eigenvalue and eigenvector pair of $\exp(A_a hl) = \exp(A_a \sigma)$. Since σ is non-pathological, there is a unique $w \in \mathrm{Im}$ such that $\mathrm{j}\omega h$ is an eigenvalue of A_a and $\exp(A_a \sigma)$ has the same Jordan chains as A_a. Hence, we have $(A_a - \mathrm{j}\omega I)\, x = 0$. Therefore,

$$\begin{bmatrix} A_a - \mathrm{j}\omega I \\ C_{a1} \end{bmatrix} x = 0,$$

which also contradicts (C2).

For the case when $\lambda = 1$, equation (22) becomes

$$(A_h^{l-1} + A_h^{l-2} + \cdots + I)[(A_h - I)x + B_{h2}u] = 0. \quad (23)$$

We have shown that $A_h^{l-1} + A_h^{l-2} + \cdots + I$ is nonsingular. Hence equation (23) implies $(A_h - I)\, x + B_{h2}u = 0$, which is $\int_0^h \exp(A_a t)\,\mathrm{d}t (A_a x + B_{a2}u) = 0$. Since σ is non-pathological, so is h. Hence $\mathrm{j}(2k\pi/h)$, $k = \pm 1, \pm 2, \ldots$, are not eigenvalues of A_a. This implies that $\int_0^h \exp(A_a t)$ is nonsingular. Hence $A_a x + B_{a2}u = 0$. Therefore,

$$\begin{bmatrix} A_a & B_{a2} \\ C_{a1} & D_{a12} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0,$$

which contradicts (C2).

It remains to show that (A3) is implied by (C1) and (C3). Again, we are going to show a stronger statement: (C1) and (C3) imply

$$\left(\mathrm{range} \begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right)^{\perp} = \{0\} \quad \text{for all } |\lambda| = 1.$$

Assume now that (C1) and (C3) hold but

$$\left(\mathrm{range} \begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix} \right)^{\perp} \neq \{0\}$$

for some $|\lambda| = 1$. Then at this λ, there exists $[x^* \; u_1^* \; u_2^* \; \cdots \; u_l^*] \neq 0$ such that

$$[x^* \; u_1^* \; u_2^* \; \cdots \; u_l^*] \begin{bmatrix} A_h^l - \lambda I & A_h^{l-1}B_{h1} & A_h^{l-2}B_{h1} & \cdots & B_{h1} \\ C_{h2} & 0 & 0 & \cdots & 0 \\ C_{h2}A_h & C_{h2}B_{h1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ C_{h2}A_h^{l-1} & C_{h2}A_h^{l-2}B_{h1} & C_{h2}A_h^{l-3}B_{h1} & \cdots & 0 \end{bmatrix} = 0. \quad (24)$$

The last column gives $x^* \int_0^h \exp(A_a(h-t)) B_{a1}\omega(t)\,\mathrm{d}t = 0$ for all $\omega \in \mathscr{K}$, which means that $x \in \langle A_a | \mathrm{range} B_{a1} \rangle^{\perp}$, where $\langle A_a | \mathrm{range} B_{a1} \rangle$ is the reachable subspace of (A_a, B_{a1}), which is A_a-invariant. The lth column of equation (24) gives $x^* A_h B_{h1} + u_l^* C_{h2} B_{h1} = 0$. Since $\mathrm{range} A_h B_{h1} \subset \langle A_a | \mathrm{range} B_{a1} \rangle$, it follows $u_l^* C_{h2} B_{h1} = 0$, so $C_{h2}^* u_l \in \langle A_a | \mathrm{range} B_{a1} \rangle^{\perp}$. By induction, we can show that $C_{h2}^* u_k \in \langle A_a | \mathrm{range} B_{a1} \rangle^{\perp}$ for $k = 1, 2, \ldots, l$. If u_k is nonzero for one of $k = 1, 2, \ldots, l$, let $x^* = u_k^* C_{h2}(\mathrm{j}\omega I - A_a)^{-1}$ for any $\mathrm{j}\omega$ not being an eigenvalue of A_a. Since $\langle A_a | \mathrm{range} B_{a1} \rangle$ is A_a-invariant, it follows that $x \in \langle A_a | \mathrm{range} B_{a1} \rangle^{\perp}$ and

$$[x^* \; u_k^*] \begin{bmatrix} A_a - \mathrm{j}\omega I & B_{a1} \\ C_{a2} & 0 \end{bmatrix} = 0,$$

which contradicts (C3). If $u_k = 0$ for all $k = 1, 2, \ldots, l$, then $x \neq 0$ and $x^* A_h^l - \lambda I B_{h1} = 0$. The first column of the above equation implies that λ and x form an eigenvalue and left eigenvector pair of $\exp(A_a \sigma)$. Since σ is non-pathological, there is a unique $\omega \in R$ such that $\mathrm{j}\omega\sigma$ is an eigenvalue of A_a and $\exp(A_a \sigma)$ has the same left Jordan chains as A_a. Hence $x^*(A_a - \mathrm{j}\omega I) = 0$. Therefore $x^*[A_a - \mathrm{j}\omega I \;\; B_{a1}] = 0$, which also contradicts (C3). □

6. *Conclusions*

The main contribution of this paper is the direct state-space solution of the multirate sampled-data $\mathscr{H}_2$ optimal control. This new method avoids converting the sampled-data problem to an equivalent discrete-time problem and it also reduces the effort in dealing with the causality issue due to the multirate sampling. It enjoys more theoretic elegance and at the same time leads to less computational effort. The same idea can be applied to multirate sampled-data $\mathscr{H}_\infty$ control, which is currently under study by the authors.

Acknowledgements—This work is supported by Hong Kong Research Grant Council and Sino Software Research Center.

References

Al-Rahmani, H. M. and G. F. Franklin (1992). Multirate control: A new approach. *Automatica*, **28,** 35–44.

Bamieh, B., J. B. Pearson, B. A. Francis and A. Tannenbaum (1991). A lifting technique for linear periodic systems with applications to sampled-data control. *Systems Control Lett.*, **17,** 79–88.

Bamieh, B. and J. B. Pearson (1992). The $\mathscr{H}_2$ problem for sampled-data systems. *Systems Control Lett.*, **19,** 1–12.

Chen, T. and B. A. Francis (1991). Linear time-varying $\mathscr{H}_2$ optimal control of sampled-data systems. *Automatica*, **27,** 963–74.

Chen, T. and L. Qiu (1994). $\mathscr{H}_\infty$ design of general multirate sampled-data control systems. *Automatica*, **30,** 1139–1152.

Chen, T. and B. A. Francis (1995). *Optimal Sampled-Data Control Systems.* Springer, London.

Colaneri, P. and G. De Nicolao (1995). Multirate LQG control of continuous-time stochastic systems. *Automatica*, **31,** 591–596.

Khargonekar, P. P., K. Poolla and A. Tannenbaum (1985). Robust control of linear time-invariant plants using periodic compensation. *IEEE Trans. Automat. Control*, **AC-30,** 1088–1096.

Khargonekar, P. P. and N. Sivashankar (1991). $\mathscr{H}_2$ optimal control of sampled-data systems. *Systems Control Lett.*, **17,** 425–436.

Meyer, D. G. (1992). A parameterization of stabilizing controllers for multirate sampled-data systems. *IEEE Trans. Automat. Control*, **AC-37,** 1411–1415.

Mirkin, L. and Z. J. Palmor (1997). Mixed discrete/continuous specifications in sampled-data $\mathscr{H}_2$-optimal control. *Automatica*, **33,** 1997–2014.

Qiu, L. and T. Chen (1994). $\mathscr{H}_2$-optimal design of multirate sampled-data systems. *IEEE Trans. Automat. Control*, **AC-39,** 2506–2511.

Qiu, L., T. Chen, A. Potvin, J. Gu, K. Tan, Y. Sun and H. Shu (1996). Development of MATLAB tools and case study for multirate control design. *Proc. 35th IEEE Conf. on Decision and Control*, pp. 1278–1283.

Shu, H. and T. Chen (1996). State-space approach to discrete-time $\mathscr{H}_2$ optimal control with a causality constraint. *Systems Control Lett.*, **26,** 69–77.

Stoorvogel, A. A. (1992). *The H_∞ Control Problem.* Printice-Hall, New York.

Sz.-Nagy, B. and C. Foias (1970), *Harmonic Analysis of Operators on Hilbert Space.* North-Holland, Amsterdam.

Trentelman, H. L. and A. A. Stoorvogel (1995). Sampled-data and discrete-time $\mathscr{H}_2$ optimal control. *SIAM J. Control Optim.*, **33,** 834–862.

Van Loan, C. F. (1978). Computing integrals involving the matrix exponential. *IEEE Trans. Automat. Control*, **AC-23,** 395–404.

Voulgaris, P. G. and B. Bamieh (1993). Optimal $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control of hybrid multirate systems. *Systems Control Lett.*, **20,** 249–261.

Voulgaris, P. G., M. A. Dahleh and L. S. Valavani (1994). $\mathscr{H}_\infty$ and $\mathscr{H}_2$ optimal controllers for periodic and multirate systems. *Automatica*, **30,** 251–263.