Proceedings of the 40th IEEE
Conference on Decision and Control
Orlando, Florida USA, December 2001

Extended Argument Principle and Integral Design Constraints Part II: New Integral Relations

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Abstract

This paper studies performance limitation and design tradeoff issues in the analysis and design of linear time-invariant, single-input single-output feedback control systems. We develop a number of integral constraints, which extend the classical Bode/Poisson sensitivity and complementary sensitivity integrals. The new integral relations lead to new insights into the study of fundamental limitation and design tradeoff issues, and together with the classical results, enable a more refined and more informative performance analysis.

1 Introduction

Bode/Poisson integrals of the sensitivity and complementary sensitivity functions are critical in the study of fundamental limitation and tradeoff issues in feedback control design. Substantial progress has been made on this subject after the pioneering work of Bode [1] and Horowitz [11], especially in 1980s and 1990s. A thorough review of the recent key developments can be found in [13]. For single-input single-output (SISO) systems, Freudenberg and Looze [8, 9] extended the classical Bode integral relations to open-loop unstable continuous-time systems. Extensions to discrete-time systems were made by Sung and Hara [14], and by Middleton [12]. More recently, growing attention has been devoted to multi-input multi-output (MIMO) systems, which has led to various generalizations of Bode/Poisson integral relations, obtained by Hara and Sung [10], Chen [2, 5, 6], and Chen and Nett [3]. Notably, these integral results all share the common feature that they characterize how the performance of feedback control systems may be constrained by undesirable system properties, such as, nonminimum phase zeros, unstable poles, and time delays in the plant, and how such constraints necessitate tradeoffs of feedback properties at different frequencies.

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In the earlier companion paper [7], the authors established a link between Bode/Poisson integrals and the well-known argument principle and its extended versions, which consequently unify the classical Bode/Poisson integrals under a single category. It was suggested in [7] that the extended argument principle, for its generality, may aid in searching for new integral relations of significance to the control context.

In this paper we continue the investigation in [7]. Using the mathematical tools developed in [7], we derive several additional sensitivity and complementary sensitivity integral relations of Bode type. Our motive herein lies in the consideration that when certain specific properties of a given plant are known, more specific integral constraints may be available to capture the properties. Such constraints will then allow more refined analysis of and shed new lights into performance tradeoff and limitation issues. In turn, they complement the classical results and add to the repertoire of the tools available in control performance analysis.

The rest of this paper is organized as follows. In Section 2, we provide a brief review of several extended versions of the argument principle developed in [7]. Section 3 presents our main results, where a number of new integral relations are derived for the sensitivity and complementary sensitivity functions. Section 4 concludes our discussion.

2 Preliminaries

Consider the single-input single-output linear time-invariant feedback system depicted in Figure 1. Let

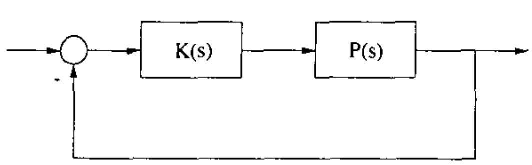

Figure 1: The feedback system

the plant and compensator transfer functions be de-

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noted by $P(s)$ and $K(s)$, respectively. We shall assume that $P(s)$ and $K(s)$ are both proper rational functions. Define the open-loop transfer function by $L(s) := P(s)K(s)$, and the sensitivity and complementary sensitivity function by

$$S(s) := \frac{1}{1+L(s)}, \quad T(s) := \frac{L(s)}{1+L(s)}.$$

With no loss of generality, we assume that there exists no unstable pole-zero cancellation in $L(s)$. Whenever this is the case, the closed-loop stability of the system implies that both $S(s)$ and $T(s)$ are analytic in the closed right half plane. We assume throughout that the system is stable. Furthermore, we assume that $L(s)$ satisfies the conjugate symmetry property (cf. Assumption A 2.2).

Denote the open right half plane by $\mathbb{C}_+$ and its closure by $\overline{\mathbb{C}}_+$. Suppose that the open-loop transfer function $L(s)$ has right half plane poles $p_i \in \mathbb{C}_+$, $i = 1, \cdots, N_p$, counting the multiplicities. Suppose also that it has right half plane zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_z$, counting the multiplicities. Then $L(s)$ can be factored as

$$L(s) = L_m(s)B_p^{-1}(s)B_z(s), \tag{1}$$

where $B_p(s)$ and $B_z(s)$ are the Blaschke products associated with the zeros and poles of $L(s)$, respectively, defined by

$$B_p(s) = \prod_{i=1}^{N_p} \frac{p_i - s}{\overline{p}_i + s}, \quad B_z(s) = \prod_{i=1}^{N_z} \frac{z_i - s}{\overline{z}_i + s}.$$

Here for a complex number s, we denote its conjugate by $\overline{s}$. Hence, the sensitivity and complementary sensitivity functions admit the factorizations

$$S(s) = S_m(s)B_p(s), \qquad T(s) = T_m(s)B_z(s). \tag{2}$$

We shall assume throughout this paper that $L(s)$ has neither zero nor pole on the imaginary axis. Under this assumption, $L_m(s)$ is stable and minimum phase, so are $S_m(s)$ and $T_m(s)$. It is worth noting that imaginary zeros or poles of $L(s)$ have no effect on integral relations of all known kinds, whenever the integrals in question are appropriately defined, specifically in terms of the so-called Cauchy principal values [2, 8].

Next, we list several extended forms of the argument principle developed in Part I [7] of this series. These lemmas will be used in the sequel for developing new integral formulae for the sensitivity and complementary sensitivity functions. First, for $f(s)$ and $g(s)$, which will be used explicitly in Lemma 2.1-2, we make the following assumptions.

A 2.1 *$f(s)$ is meromorphic in $\mathbb{C}_+$, which does not have zero or pole on the imaginary axis.*

A 2.2 *$f(s)$ satisfies the conjugate symmetry property, $\overline{f(s)} = f(\overline{s})$.*

A 2.3 *$g(s)$ is odd on the imaginary axis, $g(j\omega) = -g(-j\omega)$.*

A 2.4 *For $s \in \mathbb{C}_+$, $\lim_{s\to\infty} sg'(s)\log f(s)$ exists.*

A 2.5 *At any singularity $j\omega_0$ of $g(s)$ on the imaginary axis, $\lim_{s\to j\omega_0} (s - j\omega_0)\frac{f'(s)}{f(s)}g(s)$ exists.*

A 2.6 *At any singularity $j\omega_0$ of $g(s)$ on the imaginary axis, $\lim_{\omega\to\omega_0} g(j\omega)\log|f(j\omega)| = 0$.*

Here in making the assumptions A 2.1 and A 2.2, we intend to take $f(s)$ to be a certain system transfer function. The assumptions can then be imposed with no loss of generality. The other assumptions, A 2.3-2.6, are also rather general and are widely applicable. In particular, the assumption A 2.3 is oriented specifically for deriving Bode type integral relations, and it may be relaxed when $g(s)$ is certain logarithm function (cf. Lemma 2.2). The assumption A 2.4 is standard in the control performance studies, which is needed to insure that the functions $f(s)$ and $g(s)$ behave appropriately at infinity. The assumptions A 2.5-2.6 are necessary for relevant integrals to be well-defined when singularities of $g(s)$ do occur on the imaginary axis.

The following preliminary lemmas are adopted from [7].

Lemma 2.1 *Suppose that $f(s)$ has N_z zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_z$, and N_p poles $p_i \in \mathbb{C}_+$, $i = 1, \cdots, N_p$, all counting the multiplicities. Suppose also that $g(s)$ has N_s singularities at $j\omega_i$, $i = 1, \cdots, N_s$, but is analytic in $\mathbb{C}_+$. Then under the assumptions A 2.1-2.6,*

$$\int_{-\infty}^{\infty} g'(j\omega)\log f(j\omega)d\omega$$

$$= 2\pi\left(\sum_{i=1}^{N_z} g(z_i) - \sum_{i=1}^{N_p} g(p_i)\right) + \pi\sum_{i=1}^{N_s}\gamma_i + \pi\beta, \tag{3}$$

where

$$\gamma_i := \lim_{s\to j\omega_i} (s - j\omega_i)\frac{f'(s)}{f(s)}g(s),$$
$$\beta := \lim_{s\to\infty} sg'(s)\log f(s).$$

A mathematical result of interest in its own right, Lemma 2.1 can be further simplified for applications to a control setting. One special case arises when $g(s)$ is a logarithmic function with singularities on the imaginary axis. In particular, it will be of interest to examine

$$g(s) = \log\left(\frac{s + j\omega_c}{s - j\omega_c}\right), \tag{4}$$

for some $\omega_c > 0$, which extends trivially to the function

$$g(s) = \log\left(\prod_{i=1}^{N_s} \frac{s + j\omega_i}{s - j\omega_i}\right). \tag{5}$$

Note further that in performance analysis of control systems, our main interest is on the magnitude frequency response of the system. Thus, when $f(s)$ is chosen to be a system's transfer function, which satisfies the conjugate symmetry property, $g(s)$ is to be selected such that $g'(j\omega)$ is even, so that $g'(j\omega)\log|f(j\omega)|$ is even. It is easy to verify that when $g(s)$ is the logarithm of a rational function with singularities on the imaginary axis, $g'(j\omega)$ is even if and only if $g(s)$ possesses the form of (5). As a result, we may address this case by considering $g(s)$ given by (4), without loss of generality.

Lemma 2.2 *Let $f(s)$ be analytic in $\mathbb{C}_+$. Suppose that $f(s)$ has N_z zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_z$, all counting the multiplicities. Then under the assumptions A 2.1-2.2,*

$$\int_{-\infty}^{\infty} \frac{\omega_c}{\omega^2 - \omega_c^2} \log f(j\omega) d\omega$$

$$= \pi \arg f(j\omega_c) - \pi j \sum_{i=1}^{N_z} \log \frac{z_i + j\omega_c}{z_i - j\omega_c}. \tag{6}$$

This lemma can be seen as an easy consequence of Lemma 2.1, with $g(s)$ given by (4). The following lemma is also quoted from [7] under a related, albeit different, assumption.

A 2.7 $\forall\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\lim_{R\to\infty} R\left|\frac{f'(Re^{j\theta})}{f(Re^{j\theta})} g(Re^{j\theta})\right| = 0.$

Lemma 2.3 *Let $f(s)$ be a meromorphic function in $\mathbb{C}_+$. Suppose that $f(s)$ has N_z zeros $z_i \in \overline{\mathbb{C}}_+$, $i = 1, \cdots, N_{I_z}, \cdots N_z$, and N_p poles $p_i \in \overline{\mathbb{C}}_+$, $i = 1, \cdots, N_{I_p}, \cdots, N_p$, all counting the multiplicities, in which z_i, $i = 1, \cdots, N_{I_p}$, and p_i, $i = 1, \cdots, N_{I_z}$, are on the imaginary axis. Then under the assumption A 2.7, and whenever $g(s)$ is analytic in $\overline{\mathbb{C}}_+$,*

$$\int_{-\infty}^{\infty} \frac{f'(j\omega)}{f(j\omega)} g(j\omega) d\omega = -\pi\left(\sum_{i=1}^{N_{I_z}} g(z_i) - \sum_{i=1}^{N_{I_p}} g(p_i)\right)$$

$$-2\pi\left(\sum_{i=N_{I_z}+1}^{N_z} g(z_i) - \sum_{i=N_{I_p}+1}^{N_p} g(p_i)\right). \tag{7}$$

3 New Integral Constraints

Based on the preliminary results in Section 2, we now derive a number of new integral relations for the sensitivity and complementary sensitivity functions. The general spirit in our development is that with extra information available on the system, additional integral relations may be obtained.

Theorem 3.1 *Suppose that $L(0) \neq \infty$. Then*

$$\int_0^{\infty} \log\left|\frac{S(j\omega)}{S(0)}\right| \frac{d\omega}{\omega^2} = \frac{\pi}{2}\frac{S'(0)}{S(0)} + \pi \sum_{i=1}^{N_p} \frac{1}{p_i}. \tag{8}$$

Note that $L(0) \neq \infty$ whenever the system has no integrator. Furthermore, if $L(s)$ contain a double (or more) zero at $s = 0$, then $S'(0)/S(0) = 0$.

Proof. Since $L(0) \neq \infty$, $S(0) \neq 0$. Let

$$f(s) = \frac{S(s)}{S(0)}, \qquad g(s) = \frac{1}{s}.$$

We have

$$\beta = -\lim_{s\to\infty} \frac{1}{s} \log \frac{S(s)}{S(0)} = 0,$$

and thus the assumption A 2.4 is satisfied. Note that $g(s)$ has a singularity at $s = 0$, where

$$\lim_{s\to 0} s \frac{f'(s)}{f(s)} g(s) = \frac{S'(0)}{S(0)}.$$

Hence the assumption A 2.5 is satisfied. It is easy to verify that $\log|f(j\omega)|$ converges to 0 at a rate of (or higher than) ω^2 when $\omega \to 0$. Therefore the assumption A 2.6 is satisfied. Invoking Lemma 2.1 yields the desired result (8). ∎

Theorem 3.2 *Suppose that $L(\infty) \neq 0$. Then*

$$\int_0^{\infty} \log\left|\frac{T(j\omega)}{T(\infty)}\right| d\omega$$

$$= \frac{\pi}{2} \lim_{s\to\infty} s \frac{T(s) - T(\infty)}{T(\infty)} + \pi \sum_{i=1}^{N_z} z_i. \tag{9}$$

In this case, $L(s)$ cannot be strictly proper.

Proof. Under the condition $L(\infty) \neq 0$, $\lim_{s\to\infty} T(s) \neq 0$, and we may construct,

$$f(s) = \frac{T(s)}{T(\infty)}, \qquad g(s) = s.$$

It is straightforward to check that the assumptions A 2.1-3 are satisfied. Since,

$$\begin{aligned}\beta &= \lim_{s\to\infty} s\log\frac{T(s)}{T(\infty)} \\ &= \lim_{s\to\infty} s\frac{T(s)-T(\infty)}{T(\infty)},\end{aligned}$$

the result follows by directly applying Lemma 2.1. ■

Theorem 3.3 *Assume that*

$$\lim_{R\to\infty} R^{4n+1} \sup_{\theta\in[-\pi/2,\ \pi/2]} |L(Re^{j\theta})| = 0, \qquad (10)$$

for some integer $n \geq 0$. *Then,*

$$\int_0^\infty \omega^{4n}\log|S(j\omega)|d\omega = \frac{\pi}{4n+1}\sum_{i=1}^{N_p} p_i^{4n+1}. \qquad (11)$$

Assumption (10) imposes a constraint on the relative degree of $L(s)$.

Proof. Define

$$f(s) = S(s), \qquad g(s) = s^{4n+1}.$$

Under (10), $\log S(s)$ can be expanded at ∞ as

$$\begin{aligned}\log S(s) &= -\log(1+L(s)) \\ &= -L(s) + \frac{1}{2}L^2(s) - \cdots.\end{aligned}$$

As a result,

$$\beta = \lim_{s\to\infty}(4n+1)s^{4n+1}\log S(s) = 0.$$

The proof is then completed by using Lemma 2.1. ■

Theorem 3.4 *Assume that for* $i = 1,\ \cdots, 4n+1$,

$$\begin{aligned}T(0) &\neq 0, & (12)\\ \frac{d^i}{ds^i}\log\frac{T(s)}{T(0)}\bigg|_{s=0} &= 0, & (13)\end{aligned}$$

for some integral $n > 0$. *Then,*

$$\int_0^\infty \frac{1}{\omega^{4n+2}}\log\left|\frac{T(j\omega)}{T(0)}\right|d\omega = \frac{\pi}{4n+1}\sum_{i=1}^{N_z}\frac{1}{z_i^{4n+1}}. \quad (14)$$

The conditions (12) and (13) together imply that $L(s)$ has an integrator up to the order of $4n+1$.

Proof. Let

$$f(s) = \frac{T(s)}{T(0)}, \qquad g(s) = \frac{1}{s^{4n+1}}.$$

We have

$$\beta = \lim_{s\to\infty} -\frac{4n+1}{s^{4n+1}}\log\frac{T(s)}{T(0)} = 0.$$

Thus the assumption A 2.4 is satisfied. Since under the conditions (12) and (13),

$$\begin{aligned}\lim_{s\to 0} s\frac{f'(s)}{f(s)}g(s) &= \lim_{s\to 0}\frac{1}{s^{4n}}\frac{T'(s)}{T(s)} \\ &= \frac{1}{(4n)!}\frac{d^{4n+1}}{ds^{4n+1}}\log\frac{T(s)}{T(0)}\bigg|_{s=0} \\ &= 0,\end{aligned}$$

the assumption A 2.5 is satisfied. Similarly as in the proof of Theorem 3.1, the assumption A 2.6 is also satisfied. The result is then immediate by invoking Lemma 2.1. ■

Remarks 3.1 Theorem 3.3-4 constitute some generalized forms of the Bode sensitivity and complementary sensitivity integrals. For $n = 0$, both (11) and (14) reduce to the classical Bode sensitivity and complementary sensitivity integrals, respectively. It is clear that, for $n \neq 0$, the imaginary parts of the nonminimum phase zeros and unstable poles will affect the corresponding integrals, unlike in the classical results. ■

Remarks 3.2 In [4], it was shown that $\log S_m(s)$ may be represented as the one-sided Laplace transform of some function $f(t)$, and the initial value, $f(0^+)$, is equal to the Bode sensitivity integral. It is straightforward to show that the integral in (11) is equal to the $4nth$-order derivative of $f(t)$ at $t = 0^+$. Thus, Theorem 3.3 provides a more detailed description of the initial behavior of $f(t)$, in a way resembling to that provided by moments. The same interpretation can be made for Theorem 3.4. ■

Remarks 3.3 As in [8, 13], assume that the open-loop gain satisfies the bandwidth restriction

$$|L(j\omega)| \leq \delta\left(\frac{\omega_c}{\omega}\right)^{k+1}, \quad \omega \geq \omega_c,$$

where $\delta < 1/2$. Note that k, ω_c and δ can be adjusted to vary magnitude roll-off behavior. Select $k = 4n+1$. Then, a straightforward calculation gives rise to

$$\left|\int_{\omega_c}^\infty \omega^{4n}\log|S(j\omega)|d\omega\right| \leq \frac{3\delta\omega_c^{4n+1}}{2}. \qquad (15)$$

Suppose now that there is a disturbance attenuation requirement such that

$$|S(j\omega)| \leq \alpha < 1, \qquad \omega \leq \omega_1 < \omega_c. \qquad (16)$$

In light of Theorem 3.3 and the inequalities (15) and (16), we immediately obtain the lower bound

$$\sup_{\omega\in[\omega_1,\omega_c]} \log|S(j\omega)| \geq \frac{1}{\omega_c^{4n+1}-\omega_1^{4n+1}}$$

$$\left(\pi\sum_{i=1}^{N_p}\mathrm{Re}(p_i^{4n+1})+\omega_1^{4n+1}\log\frac{1}{\alpha}-\frac{3(4n+1)}{2}\delta\omega_c^{4n+1}\right). \tag{17}$$

The interpretation of (17) is similar to those given in [8, 13]. ■

Theorem 3.5 *Suppose that $S(j\omega_c)\neq 0$ for any $\omega_c>0$. Then,*

$$\int_0^\infty \frac{1}{\omega^2-\omega_c^2}\log\left|\frac{S(j\omega)}{S(j\omega_c)}\right|d\omega$$

$$=\frac{\pi}{2}\frac{\arg S(j\omega_c)}{\omega_c}-\frac{\pi}{2\omega_c}\arg B_p(j\omega_c). \tag{18}$$

Similarly, suppose that $T(j\omega_c)\neq 0$ for any $\omega_c>0$. Then,

$$\int_0^\infty \frac{1}{\omega^2-\omega_c^2}\log\left|\frac{T(j\omega)}{T(j\omega_c)}\right|d\omega$$

$$=\frac{\pi}{2}\frac{\arg T(j\omega_c)}{\omega_c}-\frac{\pi}{2\omega_c}\arg B_z(j\omega_c). \tag{19}$$

Proof. We shall only prove (18), and the proof for (19) is similar. Toward this end, it suffices to choose

$$f(s)=\frac{S(s)}{|S(j\omega_c)|}.$$

The proof is then completed by applying Lemma 2.2.■

Remarks 3.4 An alternative proof for Theorem 3.5 can be pursued by invoking Lemma 2.3. For (18), we may select $f(s)$ and $g(s)$ as

$$f(s)=\frac{s+j\omega_c}{s-j\omega_c}, \qquad g(s)=\log\frac{S_m(s)}{|S(j\omega_c)|}.$$

As such, $f(s)$ is analytic in $\mathbb{C}_+$ and has imaginary zero $-j\omega_c$ and imaginary pole $j\omega_c$. An application of Lemma 2.3 gives rise to

$$\int_{-\infty}^\infty \frac{2j\omega_c}{\omega^2-\omega_c^2}\log\frac{S_m(j\omega)}{|S(j\omega_c)|}d\omega$$

$$=-\pi\left(\log\frac{S_m(-j\omega_c)}{|S(j\omega_c)|}-\log\frac{S_m(j\omega_c)}{|S(j\omega_c)|}\right).$$

The desired result then follows by a straightforward calculation, which results in (18). ■

Remarks 3.5 Here a close observation reveals that the Bode integrals for $S(s)$ and $T(s)$ can be considered as limiting cases of (18) and (19), when in the latter results ω_c goes to ∞ and 0, respectively. For example, when $\omega_c\to 0$, $\frac{1}{\omega^2-\omega_c^2}$ converges uniformly to $1/\omega^2$. Thus, (19) holds in the limit as

$$\int_0^\infty \log\left|\frac{T(j\omega)}{T(0)}\right|\frac{d\omega}{\omega^2}=\frac{\pi}{2}\frac{d\arg T(j\omega)}{d\omega}\bigg|_{\omega=0}+\pi\sum_{i=1}^{N_z}\frac{1}{p_i}.$$

It is then trivial to verify that

$$\frac{d\arg T(j\omega)}{d\omega}\bigg|_{\omega=0}=\frac{T'(0)}{T(0)}.$$

Note that similar derivation also applies to the limiting case of (18). ■

4 Conclusion

In this paper we have derived several new integral relations using the extended argument principles developed in the first part [7] of this two-part series. While these integral relations share much in common with the classical results by nature, the novelty lies both in the technical derivation and in the fact that they help exhibit new features of fundamental design limitations, which complement those already known. The fact that these results are obtained in a simple manner and under strong conditions points to the possibility that more alternative meaningful integral results may still be available. Indeed, as our development hinges entirely on the selection of two functions, which can be rather general and may be selected from a wide variety, there seems to be many possibilities that remain to be explored, toward which a deeper investigation appears to be warranted. Table 1 summarizes the new integrals and indicate how they may be derived, based on the extended argument principle developed in [7].

Acknowledgement

This research was supported in part by the NSF and in part by the Hong Kong Research Grants Council.

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Table 1:

$f(s)$	$g(s)$	Integral Relation
$\frac{S(s)}{S(0)}$	$\frac{1}{s}$	$\int_0^\infty \log\left\lvert\frac{S(j\omega)}{S(0)}\right\rvert \frac{d\omega}{\omega^2} = \frac{\pi}{2}\frac{S'(0)}{S(0)} + \pi\sum_{i=1}^{N_p}\frac{1}{p_i}$
$\frac{T(s)}{T(\infty)}$	s	$\int_0^\infty \log\left\lvert\frac{T(j\omega)}{T(\infty)}\right\rvert d\omega = \frac{\pi}{2}\lim_{s\to\infty} s\frac{T(s)-T(\infty)}{T(\infty)} + \pi\sum_{i=1}^{N_z} z_i$
$S(s)$	s^{4n+1}	$\int_0^\infty \omega^{4n}\log\lvert S(j\omega)\rvert d\omega = \frac{\pi}{4n+1}\sum_{i=1}^{N_p} p_i^{4n+1}$
$\frac{T(s)}{T(0)}$	$\frac{1}{s^{4n+1}}$	$\int_0^\infty \frac{1}{\omega^{4n+2}}\log\left\lvert\frac{T(j\omega)}{T(0)}\right\rvert d\omega = \frac{\pi}{4n+1}\sum_{i=1}^{N_z}\frac{1}{z_i^{4n+1}}$
$\frac{S(s)}{\lvert S(j\omega_c)\rvert}$	$\log\frac{s+j\omega_c}{s-j\omega_c}$	$\int_0^\infty \log\left\lvert\frac{S(j\omega)}{S(j\omega_c)}\right\rvert \frac{d\omega}{\omega^2-\omega_c^2} = \frac{\pi}{2}\frac{\arg S(j\omega_c)}{\omega_c} - \frac{\pi}{2\omega_c}\arg B_p(j\omega_c)$
$\frac{T(s)}{\lvert T(j\omega_c)\rvert}$	$\log\frac{s+j\omega_c}{s-j\omega_c}$	$\int_0^\infty \log\left\lvert\frac{T(j\omega)}{T(j\omega_c)}\right\rvert \frac{d\omega}{\omega^2-\omega_c^2} = \frac{\pi}{2}\frac{\arg T(j\omega_c)}{\omega_c} - \frac{\pi}{2\omega_c}\arg B_z(j\omega_c)$

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