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Feedback stability under simultaneous gap metric uncertainties in plant and controller *

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Abstract: The stability robustness of a feedback system is studied in this paper by assuming that the plant and the controller are subject to independent uncertainties and that the uncertainties are measured by the gap metric. A fairly complete solution is obtained by exploring the trigonometric structure of the graphs of the plant and the controller.

Keywords: Robust control; gap metric; simultaneous uncertainty in plant and controller.

1. Introduction

This paper studies the stability robustness of the feedback system shown in Figure 1, where the plant and the controller are assumed to be linear time-invariant finite dimensional systems. Describing the uncertainties in a linear system in terms of the gap metric [16,6,17,7], we give a complete characterization of the stability of the closed loop system with simultaneous uncertainties in both the plant and the controller. Such a stability robustness problem has, in fact, been studied since the gap metric was introduced to the control literature [16]. A recent thorough study can be found in [7], where a necessary and sufficient condition for the closed loop stability is obtained for the case when only the plant or the controller is subject to uncertainty; the simultaneous plant and controller uncertainty case is also considered in [7], where a necessary and sufficient condition for the closed loop stability robustness is obtained with respect to the sum of the plant uncertainty and the controller uncertainty. Our results assume that the plant space and controller space are independent; in this case, the necessary and sufficient condition obtained is in terms of the plant uncertainty and the controller uncertainty, looked upon as independent entities. A by-product of this paper is a necessary and sufficient condition for the closed loop stability robustness when the plant and the controller are subject to simultaneous normalized coprime factor uncertainties. This is made possible by the connection between gap metric uncertainties

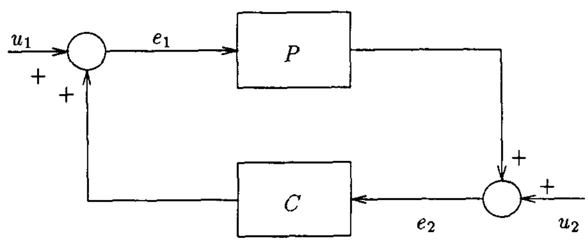

Fig. 1. The standard feedback system.

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and the normalized coprime factor uncertainties which has been recognized in [7]. For previous studies on coprime factor uncertainties, see [15,14,8].

We start with some standard definitions, Let $\mathscr{H}_2$ and $\mathscr{H}_\infty$ be the usual Hardy spaces with respect to $\mathbb{R}(s) > 0$. We assume that the functions in $\mathscr{H}_2$ and in $\mathscr{H}_\infty$ are respectively vector and matrix valued, but we suppress their dimensions from the notation with the assumption that all the operations are compatible. Let $\mathscr{RH}_\infty$ be the set of the real rational members of $\mathscr{H}_\infty$. We will not make formal distinctions between a system, its transfer matrix (a real rational matrix) and the (possibly unbounded) multiplication operator from $\mathscr{H}_2$ to $\mathscr{H}_2$ due to its transfer matrix. The feedback system shown in Figure 1 is said to be stable if the transfer matrix from $\left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right]$, which is given by

$$\mathbf{H}(P, C) = \begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-CP)^{-1} & C(I-PC)^{-1} \\ P(I-CP)^{-1} & (I-PC)^{-1} \end{bmatrix}, \tag{1}$$

is in $\mathscr{RH}_\infty$. For simplicity, we also say (P, C) is stable if $\mathbf{H}(P, C)$ is stable.

Let $\mathscr{X}$ and $\mathscr{Y}$ be two subspaces of a Hilbert space $\mathscr{H}$ and let $\Pi_{\mathscr{X}}$ and $\Pi_{\mathscr{Y}}$ be the orthogonal projections on $\mathscr{X}$ and $\mathscr{Y}$ respectively. The *gap* between $\mathscr{X}$ and $\mathscr{Y}$ is defined as

$$\gamma(\mathscr{X}, \mathscr{Y}) = \| \Pi_{\mathscr{X}} - \Pi_{\mathscr{Y}} \|.$$

This gap defines a metric on the set of all subspaces of $\mathscr{H}$.

Let NM^{-1} be any right coprime factorization of a real rational matrix P. By the *graph* of P, we mean

$$\mathscr{G}_P = \begin{bmatrix} M \\ N \end{bmatrix} \mathscr{H}_2.$$

The *gap* between two real rational matrices P_1 and P_2 is defined as the gap between their graphs, i.e.,

$$\delta(P_1, P_2) = \gamma(\mathscr{G}_{P_1}, \mathscr{G}_{P_2}).$$

The *T-gap* [7] between P_1 and P_2 is defined by

$$\delta_T(P_1, P_2) = \delta(P_1', P_2').$$

The gap and T-gap define two distinct metrics in the space of all real rational matrices of a fixed size. The gap metric ball and the T-gap metric ball centered at P_0 with radius r are then given by

$$\mathscr{B}_1(P, r) = \{P: \delta(P, P_0) < r\}, \qquad \mathscr{B}_2(P, r) = \{P: \delta_T(P, P_0) < r\}.$$

Our first main result shows that the stability robustness of a closed loop system with plant P and controller C can be measured by

$$\nu(P, C) = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \quad C] \right\|_\infty^{-1}. \tag{2}$$

Precisely, we show that for each predetermined $i, j \in \{1, 2\}$, all pairs in $\mathscr{B}_i(P_0, r_1) \times \mathscr{B}_j(C_0, r_2)$ are stable if and only if (r_1, r_2) is inside or on the boundary of the shaded area shown in Figure 2. The upper-right boundary of the shaded area is given by

$$\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0) = \tfrac{1}{2}\pi.$$

Our second main result shows that

$$\inf\{\nu(P, C): P \in \mathscr{B}_i(P_0, r_1), C \in \mathscr{B}_j(C_0, r_2)\} = \cos[\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0)] \tag{3}$$

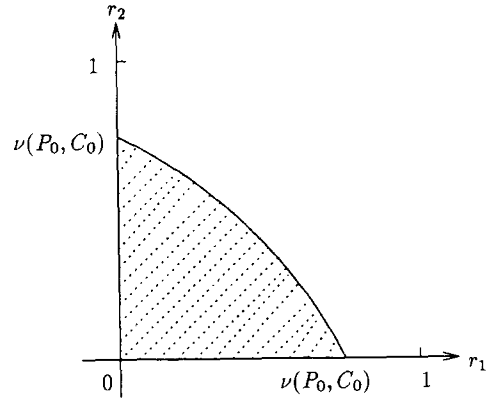

Fig. 2. The largest area containing only stable pairs.

for each (r_1, r_2) inside the shaded area of Figure 2. From our first main result, we see that $\nu(P, C)$ measures the stability robustness of the pair (P, C). Furthermore, each block of the the matrix

$$\begin{bmatrix} I \\ P \end{bmatrix}(I-CP)^{-1}[I \quad C] = \begin{bmatrix} (I-CP)^{-1} & (I-CP)^{-1}C \\ P(I-CP)^{-1} & P(I-CP)^{-1}C \end{bmatrix}$$

has its own physical significance: $(I-CP)^{-1}$ is the sensitivity function, $P(I-CP)^{-1}C$ is the complementary sensitivity function, $(I-CP)^{-1}C$ is the ratio of the control to the output noise, and $P(I-CP)^{-1}$ is the ratio of the output to the input noise. Therefore, $\nu(P, C)$ can be considered as a performance index of the feedback system formed by (P, C). (See [3] for more discussions.) The infimum (3) then corresponds to the worst case performance when P and C are in gap metric balls. An important feature of the right hand side of (3) is that it is monotonic in $\nu(P, C)$. Consequently, the worst case performance optimization problem is equivalent to the nominal performance optimization problem.

Finally in this section, we remark that both of these results have parallel versions if the uncertainties are measured by the pointwise gap metric [12], and they can be properly extended to more general situations which involve infinite dimensional systems or time-varying systems [4].

We will use, whenever possible, the conventional notation as used in the literature. For a subspace $\mathscr{X}$ of a Hilbert space $\mathscr{H}$, $\mathscr{X}^{\perp}$ is the orthogonal complement of $\mathscr{X}$ and $\Pi_{\mathscr{X}}$ is the orthogonal projection onto $\mathscr{X}$. If A is a bounded operator from $\mathscr{H}$ to another Hilbert space $\mathscr{K}$, then A^* is the adjoint of A, $\|A\|$ is the induced norm of A, and $\tau(A)$ is the so-called lower bound of A which is defined by $\tau(A) = \inf_{x \in \mathscr{H}, \|x\|=1} \|Ax\|$.

2. Preliminary results

In this section, we will first present some background knowledge on gaps between subspaces. We will see how gaps are connected with the angles (readers are referred to [2] for details). We will then generalize some of the elementary trigonometric relations. Finally we will show that the stability of (P, C) can be reflected by the relation between the graph of P and the graph of C.

Let $\mathscr{X}$ and $\mathscr{Y}$ be subspaces of a Hilbert space $\mathscr{H}$. From the definition of gap, we know that

$$\begin{aligned} \gamma(\mathscr{X}, \mathscr{Y}) &= \left\| \begin{bmatrix} \Pi_{\mathscr{X}} \\ \Pi_{\mathscr{X}^{\perp}} \end{bmatrix} (\Pi_{\mathscr{X}} - \Pi_{\mathscr{Y}}) [\Pi_{\mathscr{Y}} \Pi_{\mathscr{Y}^{\perp}}] \right\| = \left\| \begin{bmatrix} 0 & -\Pi_{\mathscr{X}} \Pi_{\mathscr{Y}^{\perp}} \\ \Pi_{\mathscr{X}^{\perp}} \Pi_{\mathscr{Y}} & 0 \end{bmatrix} \right\| \\ &= \max\{ \|\Pi_{\mathscr{X}} \Pi_{\mathscr{Y}^{\perp}}\|, \|\Pi_{\mathscr{X}^{\perp}} \Pi_{\mathscr{Y}}\| \} \\ &= \max\{ \|\Pi_{\mathscr{Y}^{\perp}} \Pi_{\mathscr{X}}\|, \|\Pi_{\mathscr{X}^{\perp}} \Pi_{\mathscr{Y}}\| \} \quad (\text{since } \Pi_{\mathscr{X}} \text{ and } \Pi_{\mathscr{Y}^{\perp}} \text{ are self-adjoint}). \end{aligned}$$

It will be convenient in the following development to have the concept of directed gaps. The *directed gap from $\mathscr{X}$ to $\mathscr{Y}$* is defined as

$$\vec{\gamma}(\mathscr{X}, \mathscr{Y}) = \| \Pi_{\mathscr{Y}^\perp} \Pi_{\mathscr{X}} \|.$$

Clearly,

$$\gamma(\mathscr{X}, \mathscr{Y}) = \max\{\vec{\gamma}(\mathscr{X}, \mathscr{Y}), \vec{\gamma}(\mathscr{Y}, \mathscr{X})\}.$$

Let X, $\tilde{X}$, Y and $\tilde{Y}$ be isometries with ranges $\mathscr{X}$, $\mathscr{X}^\perp$, $\mathscr{Y}$ and $\mathscr{Y}^\perp$ respectively. Then

$$\Pi_{\mathscr{X}} = XX^*, \quad \Pi_{\mathscr{X}^\perp} = \tilde{X}\tilde{X}^*, \quad \Pi_{\mathscr{Y}} = YY^*, \quad \Pi_{\mathscr{Y}^\perp} = \tilde{Y}\tilde{Y}^*.$$

It follows that

$$\vec{\gamma}(\mathscr{X}, \mathscr{Y}) = \| \tilde{Y}\tilde{Y}^*XX^* \| = \| \tilde{Y}^*X \| = \| X^*\tilde{Y} \|. \tag{4}$$

Note that the operator

$$\begin{bmatrix} X^* \\ \tilde{X}^* \end{bmatrix} \begin{bmatrix} Y & \tilde{Y} \end{bmatrix} = \begin{bmatrix} X^*Y & X^*\tilde{Y} \\ \tilde{X}^*Y & \tilde{X}^*\tilde{Y} \end{bmatrix}$$

is unitary. Therefore, we also have

$$\vec{\gamma}(\mathscr{X}, \mathscr{Y}) = \sqrt{1 - \tau^2(\tilde{X}^*\tilde{Y})} = \sqrt{1 - \tau^2(Y^*X)}. \tag{5}$$

Some of the elementary properties of the gap can be easily seen from formulae (4) and (5). For example, the gap is unitarily invariant, i.e., if A is a unitary operator on $\mathscr{H}$, then $\gamma(A\mathscr{X}, A\mathscr{Y}) = \gamma(\mathscr{X}, \mathscr{Y})$. If $\gamma(\mathscr{X}, \mathscr{Y}) < 1$, then $\tau(X^*Y) > 0$ and $\tau(Y^*X) > 0$. This implies that X^*Y is invertible [10, Problems 51 and 52]. It then follows that

$$\tau(X^*Y) = \|(X^*Y)^{-1}\|^{-1} = \|(Y^*X)^{-1}\|^{-1} = \tau(Y^*X),$$

and consequently $\vec{\gamma}(\mathscr{X}, \mathscr{Y}) = \vec{\gamma}(\mathscr{Y}, \mathscr{X})$.

It is well-known [1,2] that the concept of the gap between two subspaces has a close connection with the angles between vectors in these two subspaces. In fact, if we define $\theta(\mathscr{X}, \mathscr{Y}) = \arcsin \gamma(\mathscr{X}, \mathscr{Y}) \in [0, \frac{1}{2}\pi]$, then $\theta(\mathscr{X}, \mathscr{Y})$ can be considered as a generalization of the angle [1] between two one-dimensional subspaces (lines). Consider three one-dimensional subspaces, say in $\mathbb{R}^3$, with angle between them α, β, γ respectively. Clearly, we have $\gamma \leq \alpha + \beta$ and $\alpha \geq |\gamma - \beta|$. The former inequality is meaningful only when $\alpha + \beta \leq \frac{1}{2}\pi$ or, equivalently, $\sin^2\alpha + \sin^2\beta \leq 1$. If $\alpha + \beta \leq \frac{1}{2}\pi$, we obtain by applying the 'sin' addition formula

$$\sin \gamma \leq \sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha.$$

Similarly, we can obtain

$$\sin \alpha \geq \sin|\gamma - \beta| = |\sin \gamma \cos \beta - \sin \beta \cos \gamma|.$$

The generalizations of these relations appear very interesting; indeed, they are the foundation upon which our main results are obtained.

Proposition 1. *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be subspaces of a Hilbert space $\mathscr{H}$. Then*

$$\gamma(\mathscr{Y}, \mathscr{Z}) \geq |\gamma(\mathscr{X}, \mathscr{Z})\sqrt{1 - \gamma^2(\mathscr{X}, \mathscr{Y})} - \gamma(\mathscr{X}, \mathscr{Y})\sqrt{1 - \gamma^2(\mathscr{X}, \mathscr{Z})}|. \tag{6}$$

[1] There are two angles between two crossing lines; one is acute and the other is obtuse. Here we always refer to the acute one.

Proof. Let X, $\tilde{X}$, Y, $\tilde{Y}$, Z, $\tilde{Z}$ be isometries whose ranges are subspaces $\mathscr{X}$, $\mathscr{X}^\perp$, $\mathscr{Y}$, $\mathscr{Y}^\perp$, $\mathscr{Z}$, $\mathscr{Z}^\perp$ respectively. If we partition $\mathscr{H}$ into $\mathscr{X} \oplus \mathscr{X}^\perp$, then we can assume that

$$X = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \quad \tilde{Z} = \begin{bmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{bmatrix}.$$

If $\gamma(\mathscr{X}, \mathscr{Y}) = 1$, then either $\gamma(Y_1) = 0$ or $\tau(\tilde{Y}_1) = 0$. In the first case,

$$\tau(Z^*Y) = \tau(Z_1^*Y_1 + Z_2^*Y_2) \le \|Z_2^*Y_2\| \le \|Z_2\| = \vec{\gamma}(\mathscr{Z}, \mathscr{X}).$$

In the second case,

$$\tau(\tilde{Z}^*\tilde{Y}) = \tau\left(\tilde{Z}_1^*\tilde{Y}_1 + \tilde{Z}_2^*\tilde{Y}_2\right) \le \left\|\tilde{Z}_2^*\tilde{Y}_2\right\| \le \left\|\tilde{Z}_2\right\| = \vec{\gamma}(\mathscr{X}, \mathscr{Z}).$$

This shows that

$$\gamma(\mathscr{Y}, \mathscr{Z}) = \max\left\{\sqrt{1 - \tau^2(Z^*Y)}, \sqrt{1 - \tau^2(\tilde{Z}^*\tilde{Y})}\right\} \ge \sqrt{1 - \gamma^2(\mathscr{X}, \mathscr{Z})},$$

i.e., (6) holds if $\gamma(\mathscr{X}, \mathscr{Y}) = 1$. Similarly, we can show that (6) holds if $\gamma(\mathscr{X}, \mathscr{Z}) = 1$. So we have proved (6) for two special cases. Now we assume that $\gamma(\mathscr{X}, \mathscr{Y}) < 1$ and $\gamma(\mathscr{X}, \mathscr{Z}) < 1$. This implies that $\tau(Y_1)$, $\tau(Y_1^*)$, $\tau(Z_1)$ and $\tau(Z_1^*)$ are all greater than zero. Let Y_1 and Z_1 have polar decompositions $Y_1 = |Y_1|U$ and $Z_1 = |Z_1|V$ respectively. Since Y_1 and Z_1 are invertible, the partial isometries U and V are actually unitary [10, Problem 136]. Therefore, we can also assume, without loss of generality, that Y_1 and Z_1 are positive definite self-adjoint operators. Under this assumption, we have $Y_1 = (I - Y_2^*Y_2)^{1/2}$ and $Z_1 = (I - Z_2^*Z_2)^{1/2}$. A unitary dilation of Z is given by

$$\begin{bmatrix} (I - Z_2^*Z_2)^{1/2} & -Z_2^* \\ Z_2 & (I - Z_2Z_2^*)^{1/2} \end{bmatrix}.$$

Therefore its right column is an isometry with range $\mathscr{Z}^\perp$. By using formula (4),

$$\begin{aligned}
\gamma(\mathscr{Y}, \mathscr{Z}) &\ge \vec{\gamma}(\mathscr{Y}, \mathscr{Z}) \\
&= \left\|\begin{bmatrix} -Z_2 & (I - Z_2Z_2^*)^{1/2} \end{bmatrix}\begin{bmatrix} (I - Y_2^*Y_2)^{1/2} \\ Y_2 \end{bmatrix}\right\| \\
&= \left\|(I - Z_2Z_2^*)^{1/2}Y_2 - Z_2(I - Y_2^*Y_2)^{1/2}\right\| \\
&= \left\|(I - Z_2Z_2^*)^{1/2}\left[Y_2(I - Y_2^*Y_2)^{-1/2} - (I - Z_2Z_2^*)^{-1/2}Z_2\right](I - Y_2^*Y_2)^{1/2}\right\| \\
&\ge \tau\left[(I - Z_2Z_2^*)^{1/2}\right]\left|\left\|Y_2(I - Y_2^*Y_2)^{-1/2}\right\| - \left\|(I - Z_2Z_2^*)^{-1/2}Z_2\right\|\right|\tau\left[(I - Y_2^*Y_2)^{1/2}\right] \\
&= \tau\left[(I - Z_2^*Z_2)^{1/2}\right]\left|\left\|Y_2(I - Y_2^*Y_2)^{-1/2}\right\| - \left\|(I - Z_2Z_2^*)^{-1/2}Z_2\right\|\right|\tau\left[(I - Y_2^*Y_2)^{1/2}\right] \\
&= \tau(Z_1)\left|\frac{\|Y_2\|}{\sqrt{1 - \|Y_2\|^2}} - \frac{\|Z_2\|}{\sqrt{1 - \|Z_2\|^2}}\right|\tau(Y_1) \\
&= \tau(Z_1)\left|\frac{\sqrt{1 - \tau^2(Y_1)}}{\tau(Y_1)} - \frac{\sqrt{1 - \tau^2(Z_1)}}{\tau(Z_1)}\right|\tau(Y_1) \\
&= \left|\tau(Z_1)\sqrt{1 - \tau^2(Y_1)} - \tau(Y_1)\sqrt{1 - \tau^2(Z_1)}\right| \\
&= \left|\gamma(\mathscr{X}, \mathscr{Z})\sqrt{1 - \gamma^2(\mathscr{X}, \mathscr{Y})} - \gamma(\mathscr{X}, \mathscr{Y})\sqrt{1 - \gamma^2(\mathscr{X}, \mathscr{Z})}\right|.
\end{aligned}$$

This completes the proof. □

Corollary 1. *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be subspaces of a Hilbert space $\mathscr{H}$. Then*

$$\theta(\mathscr{Y}, \mathscr{Z}) \geq |\theta(\mathscr{X}, \mathscr{Z}) - \theta(\mathscr{X}, \mathscr{Y})|, \tag{7}$$

$$\theta(\mathscr{X}, \mathscr{Z}) \leq \theta(\mathscr{X}, \mathscr{Y}) + \theta(\mathscr{Y}, \mathscr{Z}). \tag{8}$$

Proof. (7) follows from (6) by applying the 'arcsin' function. (8) follows from (7). □

Corollary 2. *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be subspaces of a Hilbert space $\mathscr{H}$. If $\gamma^2(\mathscr{X}, \mathscr{Y}) + \gamma^2(\mathscr{Y}, \mathscr{Z}) \leq 1$, then*

$$\gamma(\mathscr{X}, \mathscr{Z}) \leq \gamma(\mathscr{X}, \mathscr{Y})\sqrt{1-\gamma^2(\mathscr{Y}, \mathscr{Z})} + \gamma(\mathscr{Y}, \mathscr{Z})\sqrt{1-\gamma^2(\mathscr{X}, \mathscr{Y})}. \tag{9}$$

Proof. (9) follows from (8) by applying the 'sin' function. The condition $\gamma^2(\mathscr{X}, \mathscr{Y}) + \gamma^2(\mathscr{Y}, \mathscr{Z}) \leq 1$, which is equivalent to $\theta(\mathscr{X}, \mathscr{Y}) + \theta(\mathscr{Y}, \mathscr{Z}) \leq \frac{1}{2}\pi$, is necessary since the 'sin' function is not monotonically increasing outside $[0, \frac{1}{2}\pi]$. □

The inequalities (6)–(9) are tight in the sense that if we fix $\gamma(\mathscr{X}, \mathscr{Y})$ and $\gamma(\mathscr{Y}, \mathscr{Z})$ with $\gamma^2(\mathscr{X}, \mathscr{Y}) + \gamma^2(\mathscr{Y}, \mathscr{Z}) \leq 1$ (or $\theta(\mathscr{X}, \mathscr{Y})$ and $\theta(\mathscr{Y}, \mathscr{Z})$ with $\theta(\mathscr{X}, \mathscr{Y}) + \theta(\mathscr{Y}, \mathscr{Z}) \leq \frac{1}{2}\pi$) and fix any one subspace among $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, then we can always choose the other two to achieve the equalities.

In the rest of this section, we relate the intended robustness measure $\nu(P, C)$ defined in (2) with the gap between two spaces. Let NM^{-1} and VU^{-1} be right coprime factorizations of P and C respectively. It is well known [5] that the pair (P, C) is stable if and only if the matrix

$$\begin{bmatrix} M & V \\ N & U \end{bmatrix}$$

is unimodular in $\mathscr{R}\mathscr{H}_\infty$, i.e., this matrix and its inverse are both in $\mathscr{R}\mathscr{H}_\infty$. Let us denote

$$\mathscr{G}'_C = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\mathscr{G}_C = \begin{bmatrix} V \\ U \end{bmatrix}\mathscr{H}_2$$

and call it the *inverse graph* of C. Then it follows from [10, Problem 52] that (P, C) is stable if and only if

$$\mathscr{G}_P \cap \mathscr{G}'_C = \{0\}, \qquad \mathscr{G}_P + \mathscr{G}'_C = \mathscr{H}_2 \oplus \mathscr{H}_2.$$

The following lemma is essentially Theorem 1 in [4], but we are going to give a proof based on formulae (4) and (5).

Lemma 1. *Let $\mathscr{X}$, $\mathscr{Y}$ be subspaces of a Hilbert space $\mathscr{H}$. Then $\mathscr{X} \cap \mathscr{Y} = \{0\}$ and $\mathscr{X} + \mathscr{Y} = \mathscr{H}$ if and only if $\gamma(\mathscr{X}, \mathscr{Y}^\perp) < 1$.*

Proof. Decompose $\mathscr{H}$ into $\mathscr{X} \oplus \mathscr{X}^\perp$. Then an isometry with range $\mathscr{X}$ is $[\begin{smallmatrix} I \\ 0 \end{smallmatrix}]$ and an isometry with range $\mathscr{Y}$ has the form $[\begin{smallmatrix} Y_1 \\ Y_2 \end{smallmatrix}]$. The condition that $\mathscr{X} \cap \mathscr{Y} = \{0\}$ and $\mathscr{X} + \mathscr{Y} = \mathscr{H}$ is equivalent to that $[\begin{smallmatrix} I & Y_1 \\ 0 & Y_2 \end{smallmatrix}]$ is invertible. This is true if and only if Y_2 is invertible. Since

$$\gamma(\mathscr{X}, \mathscr{Y}^\perp) = \max\left\{\sqrt{1-\tau^2(Y_2)}, \sqrt{1-\tau^2(Y_2^*)}\right\},$$

we can easily see that Y_2 is invertible if and only if $\gamma(\mathscr{X}, \mathscr{Y}^\perp) < 1$. □

Lemma 1 implies that (P, C) is stable if and only if

$$\gamma\left(\mathscr{G}_P, \mathscr{G}'^{\perp}_C\right) < 1.$$

Further, one might naturally conjecture that the distance of $\gamma(\mathscr{G}_P, \mathscr{G}'^{\perp}_C)$ from 1 has a connection with the stability robustness of (P, C). Indeed, the main results in the next section will establish such a connection. Here, we relate $\gamma(\mathscr{G}_P, \mathscr{G}_C'^{\perp})$ with the intended robustness measure $\nu(P, C)$ defined in the last section.

Proposition 2. *Let (P, C) be stable. Then*

$$\gamma(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) = \sqrt{1-\nu^2(P, C)}\,. \tag{10}$$

Proof. We have to introduce some local notation first. Let $\mathscr{L}_2$ and $\mathscr{L}_\infty$ be the standard Lebesgue spaces of vector and matrix valued functions with domain being the imaginary axis, and let $\mathscr{RL}_\infty$ be the set of real rational members of $\mathscr{L}_\infty$. For $F \in \mathscr{RL}_\infty$, we define $F^{\sim}$ by $F^{\sim}(s) = F'(-s)$. The orthogonal projection from $\mathscr{L}_2$ to $\mathscr{H}_2$ is denoted by Π_+.

Since (P, C) is stable, it follows from Lemma 1 that $\gamma(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) < 1$, which implies that $\gamma(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) = \vec{\gamma}(\mathscr{G}_P, \mathscr{G}_C'^{\perp})$. Let NM^{-1} and VU^{-1} be normalized right coprime factorizations of P and C respectively. Since $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ is an isometry from $\mathscr{H}_2$ to $\mathscr{G}_P$ and $\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]$ is an isometry from $\mathscr{H}_2$ to $\mathscr{G}_C'$, formula (4) leads to

$$\gamma(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) = \left\| \begin{bmatrix} V \\ U \end{bmatrix}^* \begin{bmatrix} M \\ N \end{bmatrix} \right\| = \left\| \Pi_+ [V^{\sim} \quad U^{\sim}] \begin{bmatrix} M \\ N \end{bmatrix} \right\|.$$

The above norm is the induced $\mathscr{H}_2$ to $\mathscr{H}_2$ operator norm of a Topolitz operator with an $\mathscr{RL}_\infty$ matrix symbol. It is known [10, Problem 245] that this norm is equal to the $\mathscr{L}_\infty$ norm of the symbol. Hence,

$$\gamma(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) = \left\| [V^{\sim} \quad U^{\sim}] \begin{bmatrix} M \\ N \end{bmatrix} \right\|_\infty.$$

Let $\tilde{U}^{-1}\tilde{V}$ be a left coprime factorization of C. Since the matrix $\left[\begin{smallmatrix} V^{\sim} & U^{\sim} \\ \tilde{U} & -\tilde{V} \end{smallmatrix}\right]$ defines a unitary operator on $\mathscr{L}_2$, we then have

$$\sqrt{1-\gamma^2(\mathscr{G}_P, \mathscr{G}_C'^{\perp})} = \tau\left\{ [\tilde{U} \quad -\tilde{V}] \begin{bmatrix} M \\ N \end{bmatrix} \right\} = \|(\tilde{U}M - \tilde{V}N)^{-1}\|_\infty^{-1}$$
$$= \left\| \begin{bmatrix} M \\ N \end{bmatrix} (\tilde{U}M - \tilde{V}N)^{-1} [\tilde{U} \quad \tilde{V}] \right\|_\infty^{-1} = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \quad C] \right\|_\infty^{-1} = \nu(P, C).$$

□

An alternative expression for (10) is

$$\theta(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) = \arccos \nu(P, C).$$

3. The main results

In the statements of the following theorems, we always assume that $i, j \in \{1, 2\}$ are arbitrary but fixed and that r_1 and r_2 are positive real numbers.

Theorem 1. *Let (P_0, C_0) be stable. Then (P, C) is stable for all $P \in \mathscr{B}_i(P_0, r_1)$ and $C \in \mathscr{B}_j(C_0, r_2)$ if and only if*

$$\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0) \le \tfrac{1}{2}\pi. \tag{11}$$

Inequality (11) gives rise to the shaded area in Figure 2. Some of the useful features of this area can be exposed more clearly by recognizing that inequality (11) is equivalent to

$$r_1^2 + r_2^2 + 2r_1r_2\sqrt{1-\nu^2(P_0, C_0)} \le \nu^2(P_0, C_0), \tag{12}$$

which is the condition derived in [12] for the pointwise gap metric case. It can be easily seem from (12) that the up-right boundary is part of an ellipse. This boundary is always inside the arc centered at the origin with radius $\nu(P_0, C_0)$ and outside the segment described by $r_1 + r_2 = \nu(P_0, C_0)$.

We emphasize that the balls used in above theorems are open balls. However, Theorem 1 is true even if one of the balls is replaced by a closed ball [2]. We can verify this by simple modification of the 'if' part of the proof; the 'only if' part remains true automatically. If we replace one of the balls in Theorem 1 by a closed ball and let the closed ball have radius zero, then we obtain Theorem 5, Theorem 3′ and Theorem 6 of [7], which are the conditions for the stability robustness of the closed loop system when only the plant or the controller is subject to gap metric or T-gap metric uncertainty. When we refer to Theorem 1 in the following development, we sometimes mean the version with one of the open balls being replaced by a closed ball.

Theorem 2. *Let* (P_0, C_0) *be stable and let* (r_1, r_2) *satisfy* (11). *Then*

$$\inf\{\nu(P, C)\colon P \in \mathscr{B}_i(P_0, r_1) \text{ and } C \in \mathscr{B}_j(C_0, r_2)\} = \cos[\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0)]. \tag{13}$$

In a similar fashion, one of the balls in (13) can be replaced by a closed ball, and the closed ball is allowed to have radius zero. This version of Theorem 2 will be useful in the following development and will be meant sometimes when we refer to Theorem 2.

4. Proof of the main results

We need two lemmas before proceeding to the proofs.

Given a real rational matrix P, let NM^{-1} and $\tilde{M}^{-1}\tilde{N}$ be its normalized right and left coprime factorizations. The right and left normalized factor balls centered at P with radius r are the following sets of real rational matrices:

$$\mathscr{C}_1(P, r) = \left\{(N + \Delta_N)(M + \Delta_M)^{-1}\colon \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < r,\ M + \Delta_M \text{ is invertible}\right\},$$

$$\mathscr{C}_2(P, r) = \left\{\left(\tilde{M} + \Delta_{\tilde{M}}\right)^{-1}\left(\tilde{N} + \Delta_{\tilde{N}}\right)\colon \|[\Delta_{\tilde{M}} \quad \Delta_{\tilde{N}}]\|_\infty < r,\ \tilde{M} + \Delta_{\tilde{M}} \text{ is invertible}\right\},$$

respectively. The following lemma is a combination of the Lemma 2 and Lemma 3 in [7], and their dual versions.

Lemma 2. $\mathscr{B}_1(P, r) = \mathscr{C}_1(P, r)$ *for each* $r \le \inf_{\mathbb{R}(s)>0}\left[\begin{smallmatrix} M(s) \\ N(s) \end{smallmatrix}\right]$, *and* $\mathscr{B}_2(P, r) = \mathscr{C}_2(P, r)$ *for each* $r \le \inf_{\mathbb{R}(s)>0}[\tilde{M}(s) \quad \tilde{N}(s)]$.

The following lemma follows from Theorem 3′ in [7].

Lemma 3. (P, C) *is stable for all* $P \in \mathscr{B}_i(P_0, r_1)$; $C \in \mathscr{B}_j(C_0, r_2)$ *and for all* $i, j \in \{1, 2\}$ *if and only if the statement is true for any one particular* $i, j \in \{1, 2\}$.

[2] A closed ball centered at P_0 with radius r is defined to be, e.g., $\{P\colon \delta(P, P_0) \le r\}$. One should note here however that a closed ball is not necessarily the closure of the open ball with the same center and radius. For example, the closure of $\mathscr{B}_1(0, 1)$ is not equal to $\{P\colon \delta(P, 0) \le 1\}$.

Proof of Theorem 1. Lemma 3 implies that in the following proof, we can choose the actual value of i and j as we feel convenient.

(I) 'If' part. We prove this part for $i=j=1$. Let

$$\theta_0 = \arccos \nu(P_0, C_0), \qquad \theta_1 = \arcsin r_1, \qquad \theta_2 = \arcsin r_2.$$

It follows from Proposition 2 that $\theta_0 = \arcsin \gamma(\mathscr{G}_{P_0}, \mathscr{G}_{C_0}'^{\perp})$. For each (P, C) with $P \in \mathscr{B}_1(P_0, r_1)$ and $C \in \mathscr{B}_1(C_0, r_2)$,

$$\theta(\mathscr{G}_P, \mathscr{G}_{P_0}) < \theta_1, \qquad \theta\left(\mathscr{G}_{C_0}'^{\perp}, \mathscr{G}_C'^{\perp}\right) = \theta(\mathscr{G}_{C_0}', \mathscr{G}_C') = \theta(\mathscr{G}_{C_0}, \mathscr{G}_C) < \theta_2.$$

The first equality above follows from the definition easily; the second follows from the unitary invariance of the gap. By using Corollary 1,

$$\theta(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) \le \theta(\mathscr{G}_P, \mathscr{G}_{P_0}) + \theta\left(\mathscr{G}_{P_0}, \mathscr{G}_{C_0}'^{\perp}\right) + \theta\left(\mathscr{G}_{C_0}'^{\perp}, \mathscr{G}_C'^{\perp}\right) < \theta_1 + \theta_2 + \theta_0 \le \tfrac{1}{2}\pi,$$

which means $\gamma(\mathscr{G}_P, \mathscr{G}_C'^{\perp}) < 1$, i.e., (P, C) is stable.

(II) 'Only if' part. We prove this part for $i=1$ and $j=2$. Assume that a pair (r_1, r_2) is given which satisfies

$$\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0) > \tfrac{1}{2}\pi.$$

We have to show that there exist $P \in \mathscr{B}_1(P_0, r_1)$ and $C \in \mathscr{B}_2(C_0, r_2)$ with $\delta_1(P, P_0) < r_1$ and $\delta_2(C, C_0) < r_2$ such that (P, C) is unstable. The assumption made implies that there exist t_1 and t_2 with $0 < t_1 < r_1$ and $0 < t_2 < r_2$ such that

$$\arcsin t_1 + \arcsin t_2 + \arccos \nu(P_0, C_0) = \tfrac{1}{2}\pi.$$

Let

$$\theta_0 = \arccos \nu(P_0, C_0), \qquad \theta_1 = \arcsin t_1, \qquad \theta_2 = \arcsin t_2.$$

Then we have $\theta_0 + \theta_1 + \theta_2 = \frac{1}{2}\pi$. Let $N_0 M_0^{-1}$ be a normalized right coprime factorization of P_0 and let $\tilde{U}_0^{-1}\tilde{V}_0$ be a normalized left coprime factorization of C_0. Since

$$\nu(P_0, C_0) = \left\| \left(\tilde{U}_0 M_0 - \tilde{V}_0 N_0\right)^{-1} \right\|_\infty^{-1} = \inf_{\omega \in \mathbb{R}} \underline{\sigma}\left[\tilde{U}_0(j\omega) M_0(j\omega) - \tilde{V}_0(j\omega) N_0(j\omega)\right],$$

there must exist $\bar{\omega} \in [0, \infty]$ such that $\underline{\sigma}[\tilde{U}_0(j\bar{\omega})M_0(j\bar{\omega}) - \tilde{V}_0(j\bar{\omega})N_0(j\bar{\omega})] = \nu(P_0, C_0)$. Assume the following singular value decomposition:

$$\begin{bmatrix} M_0(j\bar{\omega}) \\ N_0(j\bar{\omega}) \end{bmatrix} = X \begin{bmatrix} I \\ 0 \end{bmatrix} Y^*,$$

where X, Y are unitary matrices. In this case, we can always choose X, Y and an additional unitary matrix Z such that

$$Z^*\left[\tilde{U}_0(j\bar{\omega}) \quad -\tilde{V}_0(j\bar{\omega})\right] X = [A \quad B]$$

where

$$A = \operatorname{diag}(a_1, a_2, \ldots, a_m) \in \mathbb{R}^{m \times m}, \qquad B = \operatorname{diag}\left(\sqrt{1-a_1^2}, \sqrt{1-a_2^2}, \ldots, \sqrt{1-a_{\min\{p,m\}}^2}\right) \in \mathbb{R}^{m \times p}$$

and $a_1 = \underline{\sigma}(A)$. (Here we assume that P_0 is $p \times m$ and C_0 is $m \times p$.) Then

$$a_1 = \underline{\sigma}(A) = \underline{\sigma}(ZAY^*) = \underline{\sigma}\left[\tilde{U}_0(j\bar{\omega}) M_0(j\bar{\omega}) - \tilde{V}_0(j\bar{\omega}) N_0(j\bar{\omega})\right] = \nu(P_0, C_0).$$

Let

$$W=\left[\begin{array}{cccc|cccc} \sin\theta_1 & 0 & \cdots & 0 & \cos\theta_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \hline \cos\theta_1 & 0 & \cdots & 0 & -\sin\theta_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array}\right],$$

which is clearly a real orthogonal matrix. Then

$$W\begin{bmatrix} I \\ 0 \end{bmatrix}=\left[\begin{array}{cccc} \sin\theta_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline \cos\theta_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right]$$

and

$$[A \quad B]W'=\left[\begin{array}{cccc|cccc} \sqrt{1-x^2} & 0 & \cdots & 0 & x & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 & 0 & \sqrt{1-a_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_m & 0 & 0 & \cdots & . \end{array}\right]$$

where $x=\cos\theta_0\cos\theta_1-\sin\theta_0\sin\theta_1$. Hence $x=\cos(\theta_0+\theta_1)=\sin\theta_2$.

For a fixed number $\bar{\omega}\in(0,\infty)$, define a map $\phi_{\bar{\omega}}$ from $\mathbb{C}$ to real rational functions by

$$\phi_{\bar{\omega}}(\lambda)=\begin{cases} \alpha\dfrac{s-\beta}{s+\beta} & \text{if } \lambda \text{ is not real,} \\ \lambda & \text{if } \lambda \text{ is real,} \end{cases}$$

where $\alpha\in\mathbb{R}$ and $\beta\in(0,\infty)$ are determined from

$$\lambda=\alpha\frac{\mathrm{j}\bar{\omega}-\beta}{\mathrm{j}\bar{\omega}+\beta}.$$

This map maps a complex number λ to a stable all-pass real rational function whose value at $\bar{\omega}$ is λ.

Let u_1 be the first column of XW', u_2 the first row of Y^*, u_3 the first column of Z and u_4 the $(m+1)$-th row of WX^*. If $\bar{\omega}$ is 0 or ∞, they can be made all real; otherwise, we replace their elements by the images of themselves under map $\phi_{\bar{\omega}}$.

Let

$$\Delta = \begin{cases} u_1(s) \sin \theta_1 \left(\dfrac{s-1}{s+1} \right)^{2n} u_2(s) & \text{if } \bar{\omega} = 0 \text{ or } \infty, \\ u_1(s) \sin \theta_1 \left(\dfrac{s-\bar{\omega}}{s+\bar{\omega}} \right)^{2n} u_2(s) & \text{if } \bar{\omega} \in (0, \infty), \end{cases}$$

$$\tilde{\Delta} = \begin{cases} u_3(s) \sin \theta_2 \left(\dfrac{s-1}{s+1} \right)^{2n} u_4(s) & \text{if } \bar{\omega} = 0 \text{ or } \infty, \\ u_3(s) \sin \theta_2 \left(\dfrac{s-\bar{\omega}}{s+\bar{\omega}} \right)^{2n} u_4(s) & \text{if } \bar{\omega} \in (0, \infty), \end{cases}$$

and let

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix} - \Delta, \qquad \begin{bmatrix} \tilde{U}(s) & -\tilde{V}(s) \end{bmatrix} = \begin{bmatrix} \tilde{U}_0(s) & -\tilde{V}_0(s) \end{bmatrix} - \tilde{\Delta}.$$

If n is large enough, both M and $\tilde{U}$ will be invertible. For each $\mathbb{R}(s) \geq 0$,

$$\| \Delta \| \leq \sin \theta_1 = t_1 < \nu(P_0, C_0) \quad \text{and} \quad \| \tilde{\Delta} \| \leq \sin \theta_2 = t_2 < \nu(P_0, C_0).$$

By [7, Lemma 1] and its dual version,

$$\nu(P_0, C_0) \leq \inf_{s \in \mathbb{C}^+} \underline{\sigma}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix} \right) \quad \text{and} \quad \nu(P_0, C_0) \leq \inf_{s \in \mathbb{C}^+} \underline{\sigma}\left(\begin{bmatrix} \tilde{U}_0(s) & -\tilde{V}_0(s) \end{bmatrix} \right).$$

This implies that $[\begin{smallmatrix} M(s) \\ N(s) \end{smallmatrix}]$ has full column rank and $[\tilde{U}(s) \ -\tilde{V}(s)]$ has full now rank. Hence M, N are right-coprime and $\tilde{U}$, $\tilde{V}$ are left-coprime.

Let $P = NM^{-1}$ and $C = \tilde{U}^{-1}\tilde{V}$. Then $P \in \mathscr{C}_1(P_0, r_1)$ and $C \in \mathscr{C}_2(C_0, r_2)$. By Lemma 2, we obtain $P \in \mathscr{B}_1(P_0, r_1)$ and $c \in \mathscr{B}_2(C_0, r_2)$. Since

$$\begin{bmatrix} \tilde{U}(j\bar{\omega}) & -\tilde{V}(j\bar{\omega}) \end{bmatrix} \begin{bmatrix} M(j\bar{\omega}) \\ N(j\bar{\omega}) \end{bmatrix}$$

is singular, it follows that (P, C) is unstable. □

Proof of Theorem 2. Let us first prove

$$\inf\{\nu(P, C_0) \colon P \in \mathscr{B}_i(P_0, r_1)\} = \cos[\arcsin r_1 + \arccos \nu(P_0, C_0)]$$

for $i = 1, 2$. Let $\theta_0 = \arccos \nu(P_0, C_0)$, $\theta_1 = \arcsin r_1$. It follows from Corollary 1 that for each $P \in \mathscr{B}_1(P_0, r_1)$,

$$\theta\left(\mathscr{G}_P, \mathscr{G}_{C_0}^{\prime \perp}\right) \leq \theta\left(\mathscr{G}_P, \mathscr{G}_{P_0}\right) + \theta\left(\mathscr{G}_{P_0}, \mathscr{G}_{C_0}^{\prime \perp}\right) < \theta_1 + \theta_0.$$

Therefore, for each $P \in \mathscr{B}_1(P_0, r_1)$,

$$\nu(P, C_0) = \cos \theta\left(\mathscr{G}_P, \mathscr{G}_{C_0}^{\prime \perp}\right) > \cos(\theta_1 + \theta_0)$$

which implies that

$$\inf\{\nu(P, C_0) \colon P \in \mathscr{B}_1(P_0, r_1)\} \geq \cos(\theta_1 + \theta_0).$$

Now suppose we have '>' in the above inequality. Then there exists $\epsilon > 0$ such that for each $P \in \mathscr{B}_1(P_0, r_1)$,

$$\nu(P, C_0) > \cos(\theta_1 + \theta_0 - \epsilon).$$

It then follows from Theorem 1 that (P, C) is stable for all $P \in \mathscr{B}_1(P_0, r_1)$ and $C \in \mathscr{B}_1[C_0, \cos(\theta_1 + \theta_0 - \epsilon)]$. However,

$$\arcsin r_1 + \arcsin \cos(\theta_1 + \theta_0 - \epsilon) + \arccos \nu(P_0, C_0) = \theta_1 + \tfrac{1}{2}\pi - (\theta_1 + \theta_0 - \epsilon) + \theta_0 = \tfrac{1}{2}\pi + \epsilon > \tfrac{1}{2}\pi.$$

This contradicts Theorem 1. Therefore, we must have

$$\inf\{\nu(P, C_0): P \in \mathscr{B}_1(P_0, r_1)\} = \cos(\theta_1 + \theta_0) = \cos[\arcsin r_1 + \arccos \nu(P_0, C_0)].$$

From the identity $\nu(P, C) = \nu(P', C')$ [7, Corollary 1], we obtain

$$\inf\{\nu(P, C_0): P \in \mathscr{B}_2(P_0, r_1)\} = \inf\{\nu(P', C_0'): P' \in \mathscr{B}_1(P_0', r_2)\} = \cos[\arcsin r_1 + \arccos \nu(P_0, C_0)].$$

Similalry, we can show that

$$\inf\{\nu(P_0, C): C \in \mathscr{B}_j(P_0, r_2)\} = \cos[\arcsin r_2 + \arccos \nu(P_0, C_0)]$$

for $j = 1, 2$.

Now we proceed to the proof of Theorem 2:

$$\inf\{\nu(P, C): P \in \mathscr{B}_i(P_0, r_1) \text{ and } C \in \mathscr{B}_j(C_0, r_2)\}$$
$$= \inf_{C \in \mathscr{B}_j(C_0, r_2)} \left\{ \inf_{P \in \mathscr{B}_i(P_0, r_1)} \nu(P, C) \right\} = \inf_{C \in \mathscr{B}_j(C_0, r_2)} \cos[\arcsin r_1 + \arccos \nu(P_0, C)]$$
$$= \cos\left[\arcsin r_1 + \arccos \inf_{C \in \mathscr{B}_j(C_0, r_2)} \nu(P_0, C)\right] = \cos[\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0)].$$

This completes the proof. □

5. Implications of the main results

One possible way to study closed loop stability robustness under simultaneous plant and controller uncertainties is to metrize the space of all plant-controller pairs in terms of the gap metrics in the individual plant space and controller space. A class of metrics in the plant-controller pair space of all (P,C) can be chosen as

$$d_p[(P_1, C_1), (P_2, C_2)] = [\delta^p(P_1, P_2) + \delta^p(C_1, C_2)]^{1/p}$$

for $p \in [1, \infty]$.

Corollary 3. *Let (P_0, C_0) be stable. Then all (P, C) satisfying*

$$d_1[(P, C), (P_0, C_0)] < r$$

are stable if and only if $r \leq \nu(P, C)$.

This is exactly Theorem 7 in [7].

Corollary 4. *Let (P_0, C_0) be stable and $p \in [2, \infty]$. Then all (P, C) satisfying*

$$d_p[(P, C), (P_0, C_0)] < r$$

are stable if and only if $r \leq 2^{1/p}(\frac{1}{2}[1 - \sqrt{1 - \nu^2(P, C)}])^{1/2}$.

We can also use a combination of gap metrics and T-gap metrics in Corollaries 3 and 4. The proof of these two corollaries follows on computing the smallest Hölder p-norm among the points in the boundary of the shaded area in Figure 2; the smallest 1-norm occurs at either end, while the smallest p-norm for $p \in [2, \infty]$ occurs in the middle.

It is also of interest to consider the stability robustness of an open loop system, which can be regarded as a special case of the closed loop system when the controller is set to be zero.

Corollary 5. *Let P_0 be stable. Then all $P \in \mathscr{B}_i(P_0, r)$ are stable if and only if* $r \leq 1/\sqrt{1 + \| P_0 \|_\infty^2}$. *Furthermore, if the inequality is satisfied, then*

$$\sup\{\| P \|_\infty : P \in \mathscr{B}_i(P_0, r)\} = \frac{r + \sqrt{1 - r^2}\, \| P_0 \|_\infty}{\sqrt{1 - r^2} - r \| P_0 \|_\infty}.$$

Proof. If P is stable, then

$$\nu(P, 0) = \left\| \begin{bmatrix} I \\ P \end{bmatrix} \right\|_\infty^{-1} = \frac{1}{\sqrt{1 + \| P \|_\infty^2}}.$$

The first part of the corollary then follows from Theorem 1. The above equality can also be expressed as

$$\arccos \nu(P, 0) = \arctan \| P \|_\infty.$$

Since $\inf\{\nu(P, 0): P \in \mathscr{B}_j(P_0, r)\} = \cos[\arcsin r + \arccos \nu(P_0, 0)]$, it follows that

$$\begin{aligned} \sup\{\| P \|_\infty : P \in \mathscr{B}_i(P_0, r)\} &= \tan(\arcsin r + \arctan \| P_0 \|_\infty) \\ &= \left(\frac{r}{\sqrt{1 - r^2}} + \| P_0 \|_\infty \right) \Big/ \left(1 - \frac{r}{\sqrt{1 - r^2}} \| P_0 \|_\infty \right) \\ &= \left(r + \sqrt{1 - r^2}\, \| P_0 \|_\infty \right) / \left(\sqrt{1 - r^2} - r \| P_0 \|_\infty \right). \qquad \square \end{aligned}$$

Finally, it should be noted that an immediate consequence of Lemma 2 is that Theorem 1 and Theorem 2 remain valid if we replace the gap metric balls $\mathscr{B}_i$ by normalized coprime factor balls $\mathscr{C}_i$. Therefore, as a by-product, we obtain a complete characterization of the stability robustness of the closed loop system, when both the plant and the controller are subject to normalized coprime factor uncertainties.

6. Conclusions

The major purpose of this paper is to use the gap metric to analyze the stability robustness of a feedback system which has both an uncertain plant and an uncertain controller. However, one of the important contributions of this paper is the proof of fact that the function θ satisfies the triangular inequality. One of the implications of this fact is that the function θ also defines a metric on the set of subspaces of a Hilbert space; this metric, which might be called the angular gap metric, has the advantage that the triangular inequality is tight. In fact, a number of metrics other than the gap metric can be derived along this line. For example, 2 sin $\frac{1}{2}\theta$, which is called spherical gap metric in [9], has a nice geometric meaning which can be extended to Banach spaces [2,11]. Since all of these metrics define the same balls (with different radius), they are basically the same. It appears that the gap metric, although popular, is a somewhat *ad hoc* choice.

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