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Fundamental limit of discrete-time systems in tracking multi-tone sinusoidal signals☆

Weizhou Su[a,*], Li Qiu[b], Jie Chen[c]

[a]College of Automation Science and Engineering, South China University of Technology, Guangzhou, China
[b]Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Hong Kong, China
[c]Department of Electrical Engineering, University of California, Riverside, CA 92521-0425, USA

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Abstract

This paper studies the tracking performance of linear time-invariant multi-variable discrete-time systems. The specific problem under consideration is to track a multi-tone sinusoidal reference signal consisting of linear combinations of a step and several sinusoidal signals, whereas the tracking performance is measured by the energy of the error response between the output of the plant and the reference signal. Our purpose is to find the fundamental limit for the best attainable performance, under any control structures and parameters, and we seek to determine this limit analytically in terms of the given plant and reference characteristics. Both the full-information and partial-information tracking schemes are formulated and investigated to address these goals, which are concerned with whether or not the reference information is fully available for tracking. Analytical expressions are developed in full generality under full-information tracking, and for a more specialized case under partial-information scheme. In addition, an optimal cheap control design is constructed to show that the performance limit can be attained asymptotically in the full-information case. The results show that in general plant nonminimum phase zeros and reference modes can interact to fundamentally constrain a system's tracking ability. They also show that absence of full reference information can degrade the tracking performance, thus demonstrating an intrinsic trade-off between the tracking objective and the availability of the reference information.

Keywords: Performance limitation; Optimal control; Tracking; Nonminimum phase zeros

1. Introduction

Solutions to optimal control problems are often in the form of numerical algorithms that do not readily exhibit the relationship between the optimal performance and the properties of the plant to be controlled, whereas knowing such relationship is useful for a number of purposes, for example, in assessing the limitation of the plant, in understanding the trade-offs in the design task, and in knowing the fundamental design limits. Recently, there have been significant research activities devoted to such control limitation and trade-off issues, focusing on the study of the optimal performance achievable by feedback control, and especially on how the performance may be intrinsically constrained by the properties of the plant. A strong indication of the ongoing research in this area is seen by the publication of the recent Special Issue of the IEEE Transactions on Automatic Control, on New Developments and Applications in Performance Limitation of Feedback Control (cf. IEEE Transactions on Automatic Control, vol. AC-48, no. 8, August 2003).

Performance limitation studies in the aforementioned spirit were pursued in the context of optimal control by Kwakernaak and Sivan (1972). They showed that for a right-invertible linear time-invariant (LTI) system, the best achievable performance of the cheap optimal LQR control is zero if and only if the plant is minimum phase; in other words, perfect regulation can be attained whenever the plant is minimum

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* Corresponding author.
E-mail address: wzhsu@scut.edu.cn (W. Su).

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phase. This result was subsequently extended by Francis (1979) to general multi-input multi-output (MIMO) LTI systems. In a similar manner, optimal tracking problems have also been under heavy scrutiny. Davison and Scherzinger (1987) showed that with a specified class of reference and disturbance signals, it is possible to achieve perfect asymptotic tracking and disturbance rejection for minimum phase square systems. Furthermore, Morari and Zafiriou (1989), Qiu and Davison (1993), and Chen, Qiu, and Toker (2000) obtained analytical expressions of the optimal tracking performance, for a variety of reference inputs including step and sinusoidal signals. It became clear that the best tracking performance depends on the nonminimum phase zeros of the plant and the interaction between such zeros and the frequency of the reference signal, and that for a MIMO system, it also depends on how the zero and input directions may be aligned.

This paper studies optimal tracking problems concerning MIMO discrete-time systems. Earlier work on the tracking performance of discrete-time systems includes Qiu and Chen (1999), Toker, Chen, and Qiu (2002), and Jemaa and Davison (2003), for such reference inputs as step and sinusoidal signals. The present paper continues the development of Su, Qiu, and Chen (2003), investigating, in a MIMO discrete-time setting, the limitation in tracking linear combinations of a step and several sinusoidal signals of different frequencies. The study has in it the goal to understand how the reference input information may affect tracking ability. For this purpose, two cases will be considered. In the first case, we assume that the tracking controller has full access of the reference information; more specifically, with the reference signal posed as the output of an exosystem, we assume that the state of the exosystem is fully accessible for control. Alternatively, in the second case, we assume that the controller is only driven by the reference signal itself and no state information is available. We refer to these two cases as *full-information* and *partial-information* tracking, respectively.

The case of full-information tracking was fully explored by Su et al. (2003) for continuous-time systems. This paper develops parallel results for MIMO discrete-time systems. Analytical expressions are obtained for the best tracking performance, whether it is measured with respect to a fixed reference signal or averaged over all references. The results reinforce the findings of Su et al. (2003), demonstrating that a fundamental limit exists for tracking performance, which is determined by the interaction between the plant nonminimum phase zeros and the harmonic frequencies of the multi-tone reference signal. Additionally, the performance is also dictated by the mutual orientations between certain directions of the nonminimum phase zeros and the directions of the harmonic components of the reference signal.

The partial-information case, on the other hand, is significantly different and adds a distinctly new ingredient in our study of the tracking performance. While in general it appears highly nontrivial to obtain an analytical result of desired insight and simplicity, we derive a simple expression of the best tracking performance in a special, yet still rather meaningful situation. Specifically, we consider a single-tone sinusoidal signal. We show that in this case the tracking performance will be additionally impeded, in addition to the limit imposed in the full-information case. This additional limitation is attributed to the lack of full reference information available for tracking, and consequently indicates a possible trade-off between the availability of the information and the quality in tracking. The result thus leads to a more in-depth understanding of the tracking performance limitations. It shows that for reference signals of sufficient harmonic contents, not only the characteristics of the reference signal and the tracking system, but also how the signal information may be accessed can play an important role.

In an additional result of this paper, we show that the performance limit of the full-information tracking can be achieved in the limit by optimal cheap LQR control design. We construct explicitly a control law to demonstrate this effect, which consists of a standard observer-based state feedback and a feedforward of the full information of the reference states. While this optimal controller is LTI, the result indicates that it is in fact the optimal among all possible stabilizing nonlinear, time-varying controllers for the tracking problem under consideration; in other words, more general nonlinear and time-varying controllers have no advantage for the given problem, as far as tracking performance is concerned.

The remainder of this paper is organized follows. In Section 2, we formulate the tracking performance problem. Both the directional and average tracking performance are introduced. In Section 3, we present a preliminary result on the inner–outer factorization of right-invertible MIMO systems. The factorization is developed specifically to facilitate the subsequent study of the optimal tracking performance, and it consists of a cascade connection of first-order inner factors and a minimum phase factor. In Section 4, we solve analytically the general full-information tracking problem and a partial-information tracking problem for MIMO systems. In Section 5, we show that the performance limit obtained can be approached in the limit by an observer-based linear controller designed via the cheap linear quadratic control method. An illustrative example is given in Section 6. Section 7 concludes the paper. To streamline the presentation, all the proofs are relegated to the appendices at the end of the paper.

The notation used throughout this paper is fairly standard. For any complex number, vector and matrix, denote their conjugate, transpose, conjugate transpose, real and imaginary part by $\bar{(\cdot)}$, $(\cdot)^T$, $(\cdot)^*$, $Re(\cdot)$ and $Im(\cdot)$, respectively. Denote the pseudo-inverse of a matrix by $(\cdot)^\dagger$. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open unit disk and the unit circle be denoted by $\mathbb{D}$ and $\mathbb{T}$, respectively. The usual Lebesgue space of vector-valued square integrable functions on $\mathbb{T}$ is denoted by $\mathscr{L}_2$. The set of those functions in $\mathscr{L}_2$ which are analytic in $\mathbb{D}$ is denoted by $\mathscr{H}_2$ and the set of those in $\mathscr{L}_2$ that are analytic on the complement of $\mathbb{D} \cup \mathbb{T}$ and vanish at the origin is denoted by $\mathscr{H}_2^\perp$. It is well-known that $\mathscr{H}_2$ and $\mathscr{H}_2^\perp$ form an orthogonal complement as subspaces of $\mathscr{L}_2$. The Euclidean vector norm and the norm in the space $\mathscr{L}_2$ are both denoted by $\|\cdot\|_2$. The symbol $\mathscr{RH}_\infty$ denotes the set of all stable, rational transfer matrices. Finally, the inner product of two complex vectors u, v is defined as $\langle u, v\rangle := u^*v$.

2. Problem statement

The tracking system under consideration is depicted in Fig. 1. In this configuration, λ is the unit delay operator, $P(\lambda)$ the given plant transfer matrix (the λ-transform of the plant impulse response sequence), $K(\lambda)$ the controller transfer matrix, and $S(\lambda)$ the exosystem which is excited by the unit impulse $\delta(k)$ and generates the reference signal $r(k)$. We assume that certain information $w(k)$ of the signal generator is available to the controller. The full-information tracking scheme corresponds to $w(k) = v(k)$, where $v(k)$ represents the state of the exosystem (not shown in the figure), while for partial-information tracking, $w(k) = r(k)$. The measurement $y(k)$ of the plant output is used for feedback. Whether or not $y(k)$ contains the full information of the plant, i.e., the state of the plant, is unimportant.

The reference signal $r(k)$ is given in the form

$$r(k) = \sum_{l=-n}^{n} v_l e^{j\omega_l k}, \tag{1}$$

where $\omega_l \in [0, \pi]$, $l = 0, \pm 1, \ldots, \pm n$, are distinct real frequencies satisfying $\omega_{-l} = -\omega_l$, $\omega_l \in [0, \pi]$, $i = 0, 1, \ldots, n$; and v_l, $l = 0, \pm 1, \ldots, \pm n$, are complex vectors satisfying the relations $v_{-l} = \bar{v}_l$. Implicitly, we take $\omega_0 = 0$ and v_0 to be a real vector. Clearly, the reference defined in this way is a real-valued signal, a multi-tone combination of sinusoids. We use the vector

$$v = [v_{-n}^* \ \cdots \ v_{-1}^* \ v_0^* \ v_1^* \ \cdots \ v_n^*]^*$$

to capture the magnitude and phase information of all frequency components of the reference signal.

The tracking performance is measured by the energy of the transient tracking error response

$$J(v) = \sum_{k=0}^{\infty} \|r(k) - z(k)\|^2 = \sum_{k=0}^{\infty} \|e(k)\|^2.$$

Denote the set of all internally stabilizing controllers of the system by $\mathscr{K}$. Our problem is to find an explicit expression

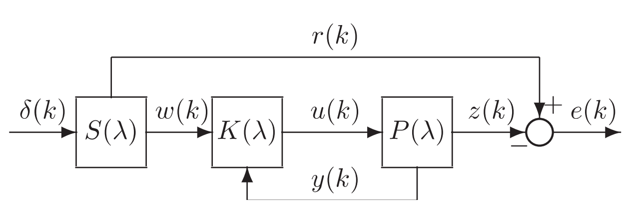

Fig. 1. A general two-parameter control structure.

of the smallest tracking error, i.e., the limit of performance achievable, defined as

$$J_{\text{opt}}(v) = \inf_{K \in \mathscr{K}} J(v).$$

The explicit dependence of $J(v)$ and $J_{\text{opt}}(v)$ on the vector v emphasizes the fact that in general the performance varies with v. Alternatively, it is also of interest to investigate the average measure of performance given below, which is independent of v. Assume that v is a random vector with entries as zero-mean and mutually uncorrelated random variables of unit variance. The average tracking performance is then defined by

$$E = \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\}.$$

Accordingly, the optimal average tracking performance is defined as

$$E_{\text{opt}} = \inf_{K \in \mathscr{K}} E.$$

These indices, averaged over all input vectors v of given first and second order statistics, serve as certain uniform measures and benchmarks, which can be especially useful in circumstances where v is unknown a priori.

The authors studied in Su et al. (2003) similar tracking performance problems for MIMO continuous-time systems, wherein explicit expressions were obtained for the best achievable performance with full-information tracking of multi-tone continuous-time sinusoidal signals. Our purpose in the present paper is twofold. We shall first attempt to derive explicit expressions for $J_{\text{opt}}(v)$ and E_{opt} with full-information tracking, that is, under the assumption that the controller input $w(k)$ is the state of the reference generator. This corresponds to the case that the controller has the maximal access of the reference information. Secondly, we consider the partial-information tracking problem, i.e., when $w(k) = r(k)$. In this case, the controller has the minimal access of the reference information.

3. A preliminary result

Let $G(\lambda)$ be a right-invertible real rational matrix representing a discrete-time LTI system. We define the poles and zeros (including multiplicity) of $G(\lambda)$ according to its Smith–McMillan form. A zero q of $G(\lambda)$ is said to be a nonminimum phase zero if $q \in \mathbb{D}$, and $G(\lambda)$ is said to be nonminimum phase if it has at least one nonminimum phase zero, and minimum phase otherwise. With this convention, zeros at the origin are considered nonminimum phase and are taken into account in our analysis.

Conduct the right coprime factorization $G(\lambda) = N(\lambda)M^{-1}(\lambda)$, where $M(\lambda), N(\lambda) \in \mathscr{RH}_\infty$. It is known that q is a nonminimum phase zero of $G(\lambda)$ if and only if it is also a nonminimum phase zero of $N(\lambda)$. For any nonminimum phase zero q of $N(\lambda)$, there exists a unit vector η such that

$$\eta^* N(q) = 0.$$

We call the vector η the (left or output) zero vector of $G(\lambda)$ associated with the nonminimum phase zero q.

Suppose that $G(\lambda)$ has nonminimum phase zeros $q_1, q_2, \ldots, q_m$. Order the zeros so that each pair of complex conjugate zeros are in adjacent order. For the given frequency $\omega_l \in \mathbb{R}$, assume that $q_i \neq e^{j\omega_l}$, $i = 1, 2, \ldots, m$. We may find a unitary vector $\eta_{\omega_l 1}$ and construct

$$G_{\omega_l 1}(\lambda) = I - \eta_{\omega_l 1} \frac{1-|q_1|^2}{e^{j\omega_l} - q_1} \frac{\lambda - e^{j\omega_l}}{\lambda q_1^* - 1} \eta_{\omega_l 1}^*$$
$$= U_{\omega_l 1} \begin{bmatrix} \frac{e^{j\omega_l} q_1^* - 1}{e^{j\omega_l} - q_1} \frac{\lambda - q_1}{\lambda q_1^* - 1} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l 1}^*, \quad (2)$$

where $\eta_{\omega_l 1}$ is selected as the zero vector of $G(\lambda)$ associated with q_1, and $U_{\omega_l 1}$ is a unitary matrix whose first column is $\eta_{\omega_l 1}$ and remaining columns are arbitrary as long as $U_{\omega_l 1}$ is unitary. Being so constructed, $G_{\omega_l 1}(\lambda)$ is inner, has only one zero at q_1 with zero vector $\eta_{\omega_l 1}$, and satisfies the property $G_{\omega_l 1}(e^{j\omega_l}) = I$. We call $G_{\omega_l 1}(\lambda)$ a matrix *Blaschke factor* at the frequency ω_l and $\eta_{\omega_l 1}$ the corresponding *Blaschke vector*; indeed, $G_{\omega_l 1}(\lambda)$ generalizes the notion of scalar Blaschke factor to matrix-valued functions. It is evident that $G_{\omega_l 1}^{-1}(\lambda)G(\lambda)$ has nonminimum phase zeros $q_2, q_3, \ldots, q_m$. Performing the same construction for the zero q_2, we find

$$G_{\omega_l 2}(\lambda) = U_{\omega_l 2} \begin{bmatrix} \frac{e^{j\omega_l} q_2^* - 1}{e^{j\omega_l} - q_2} \frac{\lambda - q_2}{\lambda q_2^* - 1} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l 2}^*,$$

where the first column of $U_{\omega_l 2}$ is the zero vector of $G_{\omega_l 1}^{-1}(\lambda)G(\lambda)$ associated with its zero q_2 and $U_{\omega_l 2}$ a unitary matrix defined accordingly. We may continue this process to construct all the Blaschke factors $G_{\omega_l 1}(\lambda), \ldots, G_{\omega_l m}(\lambda)$. Consequently $G(\lambda)$ can be factorized as

$$G(\lambda) = G_{\omega_l 1}(\lambda) \cdots G_{\omega_l m}(\lambda) G_{\omega_l 0}(\lambda), \quad (3)$$

with $G_{\omega_l 0}(\lambda)$ being a minimum phase transfer matrix and

$$G_{\omega_l i}(\lambda) = U_{\omega_l i} \begin{bmatrix} \frac{e^{j\omega_l} q_i^* - 1}{e^{j\omega_l} - q_i} \frac{\lambda - q_i}{\lambda q_i^* - 1} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l i}^*. \quad (4)$$

This factorization is referred to as a cascade factorization at the frequency ω_l. The product $G_{\omega_l 1}(\lambda) \cdots G_{\omega_l m}(\lambda)$ is called a matrix Blaschke product. One should note that this factorization is in general nonunique despite that the frequency ω_l and the order of $q_1, q_2, \ldots, q_m$ are both fixed, since $\eta_{\omega_l i}$ need not be uniquely determined. Furthermore, the Blaschke vectors and factors depend on the order of the nonminimum zeros, as well as on the frequency ω_l, and thus for different frequencies ω_l, $l = 0, \pm 1, \ldots, \pm n$, different factorizations will result. Nevertheless, our construction is carried out specifically in such a way to ensure that for different frequencies ω_l, $l = 0, \pm 1, \ldots, \pm n$, the factorizations are closely related. The following lemma exhibits such relations and they play a pivotal role leading to our main result.

Lemma 1. *Suppose that the order of $q_1, q_2, \ldots, q_m$ is fixed. Also suppose that we are given $2n+1$ different frequencies ω_l, $l = 0, \pm 1, \ldots, \pm n$. Then the $2n+1$ cascade factorizations (3) can be chosen so that for all $l, l' = 0, \pm 1, \ldots, \pm n$ and $i = 1, 2, \ldots, m$,*

$$\eta_{\omega_l i} = G_{\omega_{l'} 1}(e^{j\omega_l}) G_{\omega_{l'} 2}(e^{j\omega_l}) \cdots G_{\omega_{l'} i-1}(e^{j\omega_l}) \eta_{\omega_{l'} i}.$$

Proof. See Appendix A. □

4. Main results: performance limits

Consider now the system shown in Fig. 1. We partition the plant transfer matrix function compatibly with the tracking output $z(k)$ and the measurement $y(k)$, that is,

$$P(\lambda) = \begin{bmatrix} G(\lambda) \\ H(\lambda) \end{bmatrix}.$$

For the tracking problem to be meaningful, we make the following assumptions throughout the remainder of the paper.

Assumption 1.
(1) $P(\lambda)$, $G(\lambda)$ and $H(\lambda)$ have the same unstable poles.
(2) $G(\lambda)$ has no zero at $e^{j\omega_l}$, $l = 0, \pm 1, \ldots, \pm n$.

Assumption 1.1 means that the plant is stabilizable via output feedback of the measurement $y(k)$, a premise to achieve the tracking, while the measurement channel does not introduce more anti-stable poles. A simple interpretation of this assumption is that if $P(\lambda) = \left[\begin{smallmatrix} N(\lambda) \\ L(\lambda) \end{smallmatrix}\right] M^{-1}(\lambda)$ is a coprime factorization, then $N(\lambda)M^{-1}(\lambda)$ and $L(\lambda)M^{-1}(\lambda)$ are also coprime factorizations of $G(\lambda)$ and $H(\lambda)$.

We are now ready to state our main results.

4.1. Full-information tracking

Our first result concerns the directional tracking performance limit under the full-information tracking scheme.

Theorem 1. *Let $G(\lambda)$ have nonminimum phase zeros $q_1, \ldots, q_m$ with Blaschke vectors $\eta_{\omega_l 1}, \ldots, \eta_{\omega_l m}$, $l = 0, \pm 1, \ldots, \pm n$, so that they are constructed to satisfy Lemma 1. Assume that $w(k) = v(k)$. Then*

$$J_{\mathrm{opt}}(v) = \sum_{i=1}^{m} (1 - |q_i|^2) \left| \sum_{l=-n}^{n} \frac{\langle \eta_{-\omega_l i}, v_l \rangle}{1 - q_i e^{j\omega_l}} \right|^2$$
$$= \sum_{i=1}^{m} \sum_{l,l'=-n}^{n} \frac{\langle v_l, \eta_{-\omega_l i} \rangle \langle \eta_{-\omega_{l'} i}, v_{l'} \rangle (1 - |q_i|^2)}{(1 - q_i^* e^{-j\omega_l})(1 - q_i e^{j\omega_{l'}})}. \quad (5)$$

Proof. See Appendix B. □

The above formula shows that the performance limit $J_{\mathrm{opt}}(v)$ depends on the nonminimum phase zeros in an additive way. It also demonstrates how the frequency components of the multi-tone reference signal may affect the performance limit, and this effect is characterized by the inner products of their directional vectors and the Blaschke vectors at the corresponding frequencies. It is clear that when the zeros q_i are near the reference modes $e^{j\omega_l}$, $J_{opt}(v)$ is likely to become very large unless the directional vector v_l of the reference component and the zero vectors $\eta_{-\omega_l i}$ are nearly orthogonal. Note also that when $n=0$, i.e., the reference contains only a step component, then we get

$$J_{opt}(v)=\sum_{i=1}^{m}\frac{(1-|q_i|^2)}{|1-q_i|^2}|\langle\eta_{0i},v\rangle|^2.$$

This expression replicates the formula given by Toker et al. (2002), concerning the tracking of pure step reference signals.

The effect of the reference directional vectors on $J_{opt}(v)$ can be further illustrated by examining the following extreme scenarios. Define

$$Z^*=[z^*_{\omega_n}\ \cdots\ z^*_{\omega_0}\ \cdots\ z^*_{-\omega_n}]$$

and, for $i=0,\pm1,\ldots,\pm n$,

$$z^*_{\omega_i}=\begin{bmatrix}\dfrac{1-q_1}{1-q_1^*}\dfrac{\sqrt{1-|q_1|^2}\eta^*_{\omega_i 1}}{1-q_1e^{-j\omega_i}}\\ \dfrac{1-q_2}{1-q_2^*}\dfrac{\sqrt{1-|q_2|^2}\eta^*_{\omega_i 2}}{1-q_2e^{-j\omega_i}}\\ \vdots\\ \dfrac{1-q_m}{1-q_m^*}\dfrac{\sqrt{1-|q_m|^2}\eta^*_{\omega_i m}}{1-q_me^{-j\omega_i}}\end{bmatrix}.$$

Then, the performance limit $J_{opt}(v)$ in (5) can be rewritten as

$$J_{opt}(v)=v^*ZZ^*v.$$

Every row of the matrix Z characterizes the directional information of the plant associated with one of its nonminimum phase zeros at all frequencies of the reference. It is clear that

$$\sigma^2_{min}(Z)\|v\|^2\leqslant J_{opt}(v)\leqslant\sigma^2_{max}(Z)\|v\|^2,$$

where $\sigma_{min}(Z)$ and $\sigma_{max}(Z)$ are the smallest and largest singular values of the matrix Z. In particular, in the limiting case, the lower and upper bounds can be attained by the reference input vector v, when it coincides with the singular vector of Z corresponding to $\sigma_{min}(Z)$ and $\sigma_{max}(Z)$, respectively. These two cases give rise to the best and the worst-case tracking performance, corresponding to the most and least desired reference input.

Our next result concerns the average tracking performance E_{opt}, averaged over all reference inputs with the designated statistical specifications. We also assume that the full reference information is available to the tracking controller. In light of the proof for Theorem 1, it follows that a controller, or a sequence of controllers, independent of v can be found to attain the performance limit given in (5). This feature provides a shortcut to the following expression of E_{opt}. The result illustrates in a simple way the essential role of the plant nonminimum phase zeros, showing how in an average sense, the zeros and the reference modes may interact to limit the best performance achievable.

Theorem 2. *Let $G(\lambda)$ have nonminimum phase zeros $q_1,\ldots,q_m$. Assume that $w(k)=v(k)$. Then,*

$$E_{opt}=\sum_{i=1}^{m}\sum_{l=-n}^{n}\frac{1-|q_i|^2}{|1-q_ie^{j\omega_l}|^2}.$$

Proof. It follows from the definition of E_{opt} that

$$\begin{aligned}E_{opt}&=\inf_{K\in\mathscr{K}}\boldsymbol{E}\{J(v):\boldsymbol{E}(v)=0,\boldsymbol{E}(vv^*)=I\}\\&=\boldsymbol{E}\left\{\inf_{K\in\mathscr{K}}J(v):\boldsymbol{E}(v)=0,\boldsymbol{E}(vv^*)=I\right\}\\&=\sum_{i=1}^{m}\sum_{l,l'=-n}^{n}\frac{\eta^*_{-\omega_{l'}i}\boldsymbol{E}(v_{l'}v_l^*)\eta_{-\omega_l i}(1-|q_i|^2)}{(1-q_i^*e^{-j\omega_l})(1-q_ie^{j\omega_{l'}})}\\&=\sum_{i=1}^{m}\sum_{l=-n}^{n}\frac{\eta^*_{-\omega_l i}\eta_{-\omega_l i}(1-|q_i|^2)}{(1-q_i^*e^{-j\omega_l})(1-q_ie^{j\omega_l})}\\&=\sum_{i=1}^{m}\sum_{l=-n}^{n}\frac{1-|q_i|^2}{|1-q_ie^{j\omega_l}|^2}.\end{aligned}\quad(6)$$

The crucial equation (6) is due to the independence of the optimal controller or nearly optimal controller K on the coefficient vector v. The proof is completed. □

4.2. Partial-information tracking

We now study the tracking performance in the partial-information case. This case, unlike that of the full-information tracking, proves to be considerably more intertwined; it appears rather difficult to derive a general analytical result of desired simplicity. For this reason, we consider single-tone sinusoidal reference signals of the form

$$r(k)=v_{-1}e^{-j\omega k}+v_1e^{j\omega k},\quad \omega\in[0,\pi].\quad(7)$$

Furthermore, we address the average tracking performance E_{opt}. The results should serve as a useful anecdote in how the lack of reference state information may further impact the tracking performance.

Let $G(s)=G_{in}N_{out}(s)$ be an inner–outer factorization of the MIMO plant $G(s)$. We give the following result for the system.

Theorem 3. *Let $G(\lambda)$ have nonminimum phase zeros $q_1, \ldots, q_m$. Assume that $w(k) = r(k)$. Then,*

$$E_{\text{opt}} = \sum_{i=1}^{m} \left(\frac{1-|q_i|^2}{|1-q_i e^{-j\omega}|^2} + \frac{1-|q_i|^2}{|1-q_i e^{j\omega}|^2} \right) + E_a,$$

where

$$E_a = \frac{\operatorname{tr}\{\Delta^* G_{\text{in}}(e^{j\omega}) \Delta G_{\text{in}}(e^{j\omega})\}}{2 \sin \omega}$$

and

$$\Delta G_{\text{in}}(e^{j\omega}) = G_{\text{in}}(e^{j\omega}) - G_{\text{in}}(e^{-j\omega}).$$

Proof. See Appendix C. □

When restricted to the following more specialized cases, more explicit expression can be obtained, as summarized in Corollaries 1 and 2.

Corollary 1. *Suppose that $G(\lambda)$ is a SISO transfer function and that it has nonminimum phase zeros $q_1, \ldots, q_m$. Assume that $w(k) = r(k)$. Then,*

$$E_{\text{opt}} = \sum_{i=1}^{m} \left(\frac{1-|q_i|^2}{|1-q_i e^{-j\omega}|^2} + \frac{1-|q_i|^2}{|1-q_i e^{j\omega}|^2} \right) + E_a,$$

with

$$E_a = \frac{2}{\sin \omega} \sin^2 \left[2 \sum_{i=1}^{m} \angle(1 - q_i e^{-j\omega}) + m\omega \right].$$

Proof. For the SISO plant $G(\lambda)$, one of its inner factor $G_{\text{in}}(\lambda)$ is given by

$$G_{\text{in}}(\lambda) = \frac{\lambda - q_1}{1 - \lambda q_1^*} \cdots \frac{\lambda - q_m}{1 - \lambda q_m^*}.$$

Denote

$$\phi_i = 2\angle(1 - q_i e^{-j\omega}) + \omega, \quad i = 1, \ldots, m.$$

Then,

$$G_{\text{in}}(e^{j\omega}) = e^{j(\phi_1 + \cdots + \phi_m)}$$

and

$$G_{\text{in}}(e^{-j\omega}) = e^{-j(\phi_1 + \cdots + \phi_m)}.$$

From

$$\phi_1 + \cdots + \phi_m = 2 \sum_{i=1}^{m} \angle(1 - q_i e^{-j\omega}) + m\omega,$$

it follows that

$$\Delta^* G_{\text{in}}(e^{j\omega}) \Delta G_{\text{in}}(e^{j\omega}) = 4\sin^2 \left[2 \sum_{i=1}^{m} \angle(1 - q_i e^{-j\omega}) + m\omega \right]. \tag{8}$$

The proof is then completed by invoking Theorem 3. □

Corollary 2. *Let $G(\lambda)$ have only one real nonminimum phase zero q. Assume that $w(k) = r(k)$. Then,*

$$E_{\text{opt}} = \frac{1-q^2}{|1-q e^{-j\omega}|^2} + \frac{1-q^2}{|1-q e^{j\omega}|^2} + 2 \sin \omega \left(\frac{1-q^2}{1 - 2q \cos \omega + q^2} \right)^2. \tag{9}$$

Proof. Since $G(\lambda)$ has only one real nonminimum phase zero q with Blaschke vector η_ω, its inner factor is given by

$$G_{\text{in}}(\lambda) = I - \eta_\omega \eta_\omega^* \frac{(1-q^2)(\lambda - e^{j\omega})}{(e^{j\omega} - q)(\lambda q - 1)}.$$

Then we have

$$\Delta G_{\text{in}}(e^{j\omega}) = \eta_\omega \eta_\omega^* \frac{(1-q^2)(e^{-j\omega} - e^{j\omega})}{(e^{j\omega} - q)(e^{-j\omega} q - 1)}.$$

The rest of the proof follows from Theorem 3 and a straightforward calculation. □

In Theorem 2, it is stated that if the full state information of the reference signal is available, the best tracking performance with respect to the single-tone reference (7) is

$$E_{\text{opt}} = \sum_{i=1}^{m} \left(\frac{1-|q_i|^2}{|1-q_i e^{-j\omega}|^2} + \frac{1-|q_i|^2}{|1-q_i e^{j\omega}|^2} \right).$$

Theorem 3, along with Corollaries 1 and 2, reveals that due to the additional nonnegative term E_a, an additional limit is incurred when such information becomes inaccessible. This additional limit may be interpreted as the cost for the controller to estimate the reference state, whereas the estimated state is then used by the controller to track the reference, resulting in the same performance as that of full-information tracking. In this vein, Theorem 3 exhibits a trade-off between the tracking performance and the information of the reference available for tracking.

It is instructive to inspect the term E_a in further depth. Toward this end, one first notes that $E_a \to 0$ as $\omega \to 0$. This can be readily verified, and it shows that in the limit when the reference signal becomes the unit step, the term vanishes, thus replicating the best achievable performance in tracking the latter. One also notes that $E_a \to 0$ as $\omega \to \pi$. In these two cases, the exogenous system degenerates from an order twice the output dimension to an order equal to the output dimension, and hence no effort is needed in estimating its state from the output. Consider next the case where $G(\lambda)$ has only one real nonminimum phase zero q. With no loss of generality, assume that $\omega > 0$. Then, it follows from (9) that

$$E_{\mathrm{a}} = 2\sin\omega\left(\frac{1-q^2}{1-2q\cos\omega+q^2}\right)^2.$$

As a function of $\omega > 0$, E_{a} attains its minimum $E_{\mathrm{a}} = 0$ at $\omega = 0$ or π, and maximum at

$$\omega = \cos^{-1}\left(\frac{2q}{1+q^2}\right).$$

At this frequency,

$$E_{\mathrm{a}} = 2\frac{1+q^2}{1-q^2} = \frac{1}{2}\,E_{\mathrm{opt}}.$$

It is then evident that the contribution of E_{a} to E_{opt} can be rather substantial.

5. Optimal cheap controller

Having obtained the performance limits in the preceding section, we now examine how a control law may be devised to attain the limits. We show that in the full-information case the best tracking performance can be attained in the limit using a cheap control strategy. For continuous-time systems, it is known that in tracking a step or a single-tone sinusoid, cheap control can be employed to asymptotically achieve the best, limiting performance (see, e.g., Qiu & Davison, 1993); we extend this result herein to multi-tone sinusoidal signals in the discrete-time setting, by constructing explicitly a cheap control law that is asymptotically optimal. This controller being so designed consists of a measurement feedback and an exosystem state feedforward. More specifically, the feedback part is an observer state feedback with a cheap control gain, and the feedforward part is designed to provide the required steady-state input. The development indicates, as an important byproduct, that nonlinear time-varying controllers have no advantage for the tracking problem under consideration.

We begin by first providing a brief review of the cheap linear quadratic control problem; for more details, see, e.g., Kwakernaak and Sivan (1972). Let a stabilizable and detectable system be given by

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0,$$

$$y(k) = Cx(k) + Du(k). \tag{10}$$

Consider the quadratic cost

$$J_\varepsilon = \sum_{k=0}^{\infty}(\|y(k)\|^2 + \varepsilon^2\|u(k)\|^2).$$

It is well-known that the optimal input that minimizes J_ε is given by

$$u(k) = F_\varepsilon x(k),$$

where

$$F_\varepsilon = -(\varepsilon^2 I + B^* P_\varepsilon B)^{-1} B^* P_\varepsilon A,$$

and P_ε is the unique stabilizing solution of the ARE associated with the system (10), that is, P_ε is the positive definite solution of

$$P_\varepsilon - A^*P_\varepsilon A + (A^*P_\varepsilon B + C^*D)(D^*D + \varepsilon^2 I + B^*P_\varepsilon B)^{-1}(B^*P_\varepsilon A + CD^*) - C^*C = 0. \tag{11}$$

With this control input applied, the optimal cost is given by

$$J_{\varepsilon,\mathrm{opt}} = \inf_u J_\varepsilon(x_0) = x_0^* P_\varepsilon x_0.$$

The cheap control problem is to find the linear optimal control law to minimize the cost function in the limit as $\varepsilon \to 0$. A remarkable result on the cheap control was obtained by Kwakernaak and Sivan (1972) for continuous-time LTI systems, who showed that for a right-invertible minimum phase system the minimal cost approaches zero as $\varepsilon \to 0$. Likewise, based on Kwakernaak and Sivan (1972), it is straightforward to show that for right-invertible minimum phase discrete-time LTI systems, the solution P_ε of the ARE (11) approaches zero as $\varepsilon \to 0$, i.e., $\lim_{\varepsilon\to 0} P_\varepsilon = 0$.

Consider next a nonminimum phase plant $G(\lambda)$. Let $G(\lambda)$ be factorized as $G(\lambda) = G_{\mathrm{in}}(\lambda)G_0(\lambda)$, where $G_{\mathrm{in}}(\lambda)$ is an inner factor of $G(\lambda)$ and $G_0(\lambda)$ is minimum phase. Let

$$G_{\mathrm{in}}(\lambda) = \left[\begin{array}{c|c} A_{\mathrm{in}} & B_{\mathrm{in}} \\ \hline C_{\mathrm{in}} & D_{\mathrm{in}} \end{array}\right]$$

be a minimum *balanced realization* (Zhou, Doyle, & Glover, 1995), and

$$G_0(\lambda) = \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right]$$

be a stabilizable and detectable realization. Then $G(s)$ has a stabilizable and detectable realization

$$G(\lambda) = G_{\mathrm{in}}(\lambda)G_0(\lambda) = = \left[\begin{array}{cc|c} A_{\mathrm{in}} & B_{\mathrm{in}}C_0 & B_{\mathrm{in}}D_0 \\ 0 & A_0 & B_0 \\ \hline C_{\mathrm{in}} & D_{\mathrm{in}}C_0 & D_{\mathrm{in}}D_0 \end{array}\right] =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{12}$$

The following lemma provides a characterization of the ARE solution P_ε when $\varepsilon \to 0$. Its proof mimicks that of Qiu and Davison (1993) for continuous-time systems and hence is omitted.

Lemma 2. *Let* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *be the realization of a nonminimum phase transfer matrix* $G(\lambda)$, *as given in* (12). *Let* P_ε *be the unique solution of the ARE associated with the realization. The matrix* P_ε *is given by*

$$P_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & P_{0\varepsilon} \end{bmatrix}.$$

The identity matrix I is the solution of the ARE associated with the minimum balanced realization of the inner factor $G_{\rm in}(\lambda)$, *i.e.*,

$$I - A_{\rm in}^* A_{\rm in} + (A_{\rm in}^* B_{\rm in} + C_{\rm in}^* D_{\rm in})(D_{\rm in}^* D_{\rm in} + \varepsilon^2 I + B_{\rm in}^* B_{\rm in})^{-1}(B_{\rm in}^* A_{\rm in} + C_{\rm in} D_{\rm in}^*) - C_{\rm in}^* C_{\rm in} = 0,$$

and the matrix $P_{0\varepsilon}$ *is the solution of the ARE associated with the outer factor* $G_0(\lambda)$, *i.e.*,

$$P_{0\varepsilon} - A_0^* P_{0\varepsilon} A_0 + (A_0^* P_{0\varepsilon} B_0 + C_0^* D_0)(D_0^* D_0 + \varepsilon^2 I + B_0^* P_{0\varepsilon} B_0)^{-1}(B_0^* P_{0\varepsilon} A_0 + C_0 D_0^*) - C_0^* C_0 = 0.$$

Furthermore, it holds

$$\lim_{\varepsilon\to 0} P_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

This characterization is instrumental in analyzing the asymptotic behavior of the optimal cheap controller, to be constructed below for the tracking problem.

Suppose that a stabilizable and detectable realization of the plant $P(\lambda) = \begin{bmatrix} G(\lambda) \\ H(\lambda) \end{bmatrix}$ is given by

$$x(k+1) = Ax(k) + Bu(k),$$

$$z(k) = Cx(k) + Du(k),$$

$$y(k) = Ex(k).$$

It follows from Assumption 1.1 that both (C, A) and (E, A) are detectable.

For the problem of tracking the sinusoidal signal (1), let the steady-state parts of the plant input, plant state, and plant output be denoted by $u_{\rm ss}(k)$, $x_{\rm ss}(k)$, and $y_{\rm ss}(k)$, respectively. Then a simple steady-state analysis shows that the asymptotic tracking requires that

$$y_{\rm ss}(k) = r(k) = \sum_{l=-n}^{n} v_l e^{j\omega_l k},$$

$$u_{\rm ss}(k) = \sum_{l=-n}^{n} G^\dagger\left(e^{-j\omega_l}\right) v_l e^{j\omega_l k},$$

$$x_{\rm ss}(k) = \sum_{l=-n}^{n} (e^{j\omega_l} I - A)^{-1} B G^\dagger (e^{-j\omega_l}) v_l e^{j\omega_l k},$$

where $G^\dagger(e^{-j\omega_l})$ is a right inverse of $G(e^{-j\omega_l})$. Denote the transient parts of $x(k)$ and $u(k)$ by $x_t(k)$ and $u_t(k)$, respectively, i.e.,

$$x_t(k) = x(k) - x_{\rm ss}(k), \quad u_t(k) = u(k) - u_{\rm ss}(k).$$

Then it follows from algebraic manipulations that

$$x_t(k+1) = Ax_t(k) + Bu_t(k),$$

$$e(k) = Cx_t(k) + Du_t(k),$$

$$u_t(k) = Fx_t(k), \tag{13}$$

with

$$x_t(0) = -x_{\rm ss}(0) = -\sum_{l=-n}^{n} (e^{j\omega_l} I - A)^{-1} B G^\dagger (e^{-j\omega_l}) v_l.$$

It follows from the discussion above that the cost function of the cheap optimal control problem for system (13) is given by

$$J_\varepsilon(v) = \sum_{k=0}^{\infty} (\|e(k)\|^2 + \varepsilon^2 \|u_t(k)\|^2).$$

Lemma 2 gives the optimal solution for this cheap optimal control problem. Note the fact that

$$J(v) = \lim_{\varepsilon\to 0} J_\varepsilon(v).$$

The optimal tracking problem, which amounts to minimizing the cost function $J(v)$, is equivalent to the optimal cheap control problem for system (13), in the limit as $\varepsilon \to 0$. In light of this equivalence, the optimal tracking problem may be solved as one of the optimal cheap control.

Theorem 4. *Let* P_ε *be the stabilizing solution of ARE* (11). *Let*

$$F_\varepsilon = -(\varepsilon^2 I + B^* P_\varepsilon B)^{-1} B^* P_\varepsilon A,$$

and L be any matrix such that $A + LE$ *is stable. Construct the controller as*

$$\tilde{x}(k+1) = (A + BF_\varepsilon + LE)\tilde{x}(k) - Ly(k),$$

$$u(k) = F_\varepsilon \tilde{x}(k) + \sum_{l=-n}^{n} [I - F_\varepsilon (e^{j\omega_l} I - A)^{-1} B] G^\dagger (e^{-j\omega_l}) v_l e^{j\omega_l k}.$$

Then,

$$\lim_{\varepsilon\to 0} J_\varepsilon(v) = J_{\rm opt}(v).$$

Proof. See Appendix D. □

We note that the controller in the theorem is simply an observer-based state feedback plus a reference state feedforward, where the latter is seen to provide the required steady-state input to the plant. The feedforward term requires the knowledge of $v_l e^{j\omega_l k}$, $l = 0, \pm 1, \ldots, \pm n$, which can be considered as the state variables of the exosystem. Theorem 4 states that whenever such information is available, it can be embedded in the control law, so that the tracking performance limit $J_{\rm opt}(v)$ is attained in the full-information case by the optimal cheap controller asymptotically when $\varepsilon \to 0$.

6. An example

In this section, we use a numerical example to illustrate the results in this paper. The plant considered is an FIR system with two nonminimum phase zeros, whose transfer function is given by

$$G(\lambda) = (\lambda - 0.5)(\lambda + 0.5).$$

The reference under consideration is the sinusoidal signal of the form

$$r(t) = \bar{v}\mathrm{e}^{-\mathrm{j}\omega t} + v\mathrm{e}^{\mathrm{j}\omega t} = 2|v|\cos(\omega t + \angle v),$$

where ω is the frequency of the sinusoid and v is the coefficient that determines its magnitude and phase.

We first consider performance limits under the full-information control structure when the coefficient v is known to the controller. Set $\omega = \pi/2$ and assume that the magnitude $|v|$ is normalized to 1 and the phase $\angle v$ varies in $[0, \pi]$. The v-dependent performance limit $J_{\mathrm{opt}}(v)$ is plotted in Fig. 2. For comparison purpose, we also draw the averaged performance limit E_{opt} in Fig. 2. One can see from Fig. 2 that in the full-information case J_{opt} can be much greater and less than its averaged counterpart depending on how the phase of the reference and the nonminimum phase zeros are aligned.

We next compare the averaged performance limits E_{opt} in the full-information and partial-information cases. Fig. 3 shows the performance limits E_{opt} in both cases, as functions of the reference frequency ω varying in the interval $[0, \pi]$.

From Fig. 3, we see that the difference of E_{opt} in the full-information case and the partial information case can be quite significant, possibly beyond 100% of the performance limit in the full-information case. We also see

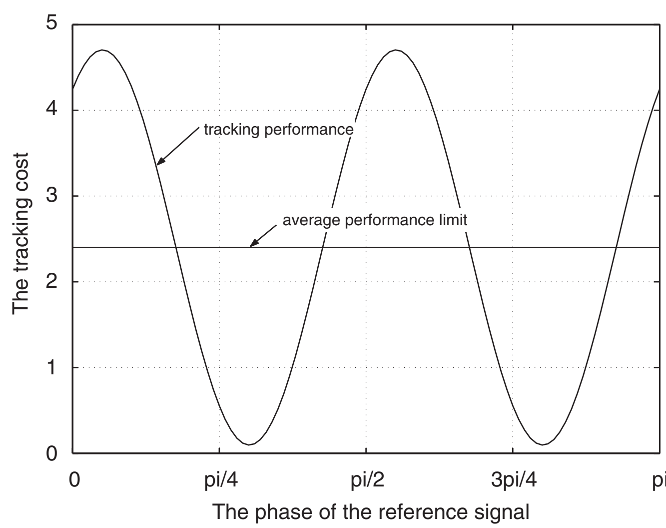

Fig. 2. The plot of $J_{\mathrm{opt}}(v)$ when $|v| = 1$ and $\angle v$ varies in $[0, \pi]$.

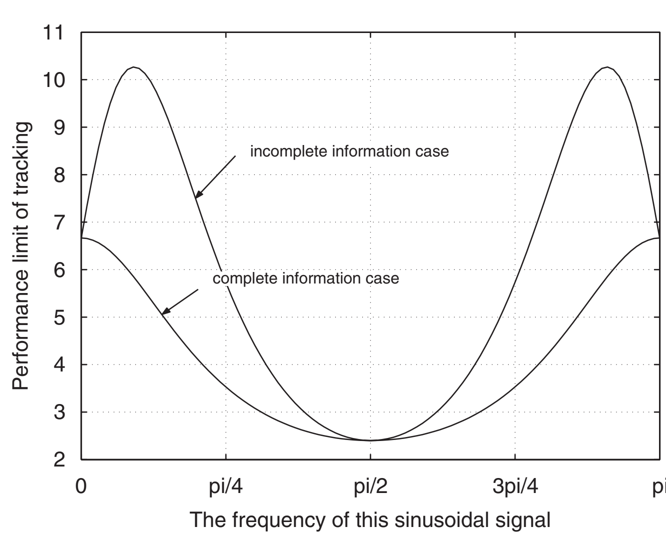

Fig. 3. Plots of E_{opt} in the full-information and the partial information cases as functions of the reference frequency ω.

that the performance limits $E_{\rm opt}$ in both cases are the same when $\omega = 0, \pi$, and $\pi/2$. Whereas this phenomenon is no surprising and is always the case when $\omega = 0$ or $\omega = \pi$ due to the reasons discussed in Section 4, why this also occurs when $\omega = \pi/2$ is due to the special symmetric zero pattern in this particular example.

7. Conclusion

In this paper, analytical expressions were obtained for the best achievable performance of LTI MIMO discrete-time systems in tracking multi-tone sinusoidal signals, which consist of linear combinations of step and sinusoidal signals. The full-information tracking problem was studied in its full generality. The results exhibit how plant nonminimum phase zeros may fundamentally constrain a system's tracking ability, and characterize explicitly the roles of such zeros and their corresponding zero vectors. Both the directional and average tracking performance measures demonstrated the effects. It is seen that the fundamental performance limits can be approached asymptotically by a linear optimal cheap control law, which is explicitly constructed in the paper.

The partial-information tracking problem proved far more difficult, for which we provided an analytical result for a more specialized reference with one single-tone sinusoid. The result shows that when the full state information of the reference is unavailable, an additional limit to the tracking performance results. This extra cost is seen as a result of the trade-off between the tracking performance objective and the information available to the controller, and it can be interpreted as a penalty for the system to estimate the reference state needed to perform tracking.

Acknowledgments

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Appendix A. Proof of Lemma 1

We shall first show that the factorizations can be constructed so that for all $l, l' = 0, \pm 1, \ldots, \pm n$ and $i = 1, 2, \ldots, m$,

$$\eta_{\omega_l i} = W_{ll'(i-1)} \eta_{\omega_{l'} i}, \tag{14}$$

where

$$W_{ll'i} = G_{\omega_l i}^{-1}(\lambda) \cdots G_{\omega_l 2}^{-1}(\lambda) G_{\omega_l 1}^{-1}(\lambda) \times G_{\omega_{l'} 1}(\lambda) G_{\omega_{l'} 2}(\lambda) \cdots G_{\omega_{l'} i}(\lambda) \tag{15}$$

are constant unitary matrices. We begin by adopting the convention that $W_{ll'0} = I$, and next carrying out induction in i. Consider $i = 1$. The fact that $\eta_{\omega_l 1}$ can be chosen independent of ω_l leads to

$$\eta_{\omega_l 1} = W_{ll'0} \eta_{\omega_{l'} 1} = \eta_{\omega_{l'} 1}, \quad l, l' = 0, \pm 1, \ldots, \pm n.$$

Hence U_{ω_l} can be selected independent of ω_l so that

$$U_{\omega_l 1} = U_{\omega_{l'} 1}, \quad l, l' = 0, \pm 1, \ldots, \pm n.$$

These lead to

$$G_{\omega_l 1}^{-1}(\lambda) G_{\omega_{l'} 1}(\lambda) = U_{\omega_l 1} \begin{bmatrix} \frac{e^{j\omega_l} - q_1}{e^{j\omega_l} q_1^* - 1} \frac{e^{j\omega_{l'}} q_1^* - 1}{e^{j\omega_{l'}} - q_1} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l 1}^*,$$

that is, $W_{ll'1}$ is a constant unitary matrix. Suppose then that $\eta_{\omega_l 1}, \ldots, \eta_{\omega_l i}$ and $G_{\omega_l 1}(\lambda), \ldots, G_{\omega_l i}(\lambda)$ have been chosen so that

$$\eta_{\omega_l i} = W_{ll'(i-1)} \eta_{\omega_{l'} i}, \quad i = 1, 2, \ldots, m$$

and

$$W_{ll'i} = G_{\omega_l i}^{-1}(\lambda) \cdots G_{\omega_l 2}^{-1}(\lambda) G_{\omega_l 1}^{-1}(\lambda) \times G_{\omega_{l'} 1}(\lambda) G_{\omega_{l'} 2}(\lambda) \cdots G_{\omega_{l'} i}(\lambda)$$

are constant unitary matrices for all $l, l' = 0, \pm 1, \ldots \pm n$. For the frequency ω_0, choose a unit vector $\eta_{\omega_0 (i+1)}$ such that

$$\eta_{\omega_0 (i+1)}^* G_{\omega_0 i}^{-1}(q_{i+1}) \cdots G_{\omega_0 2}^{-1}(q_{i+1}) G_{\omega_0 1}^{-1}(q_{i+1}) N(q_{i+1}) = 0.$$

Select also a unitary matrix $U_{\omega_0 (i+1)}$ such that its first column is $\eta_{\omega_0 (i+1)}$. Define

$$U_{\omega_l (i+1)} = W_{l0i} U_{\omega_0 (i+1)}.$$

Then the first column $\eta_{\omega_l (i+1)}$ of $U_{\omega_l (i+1)}$ satisfies the property

$$\begin{aligned} &\eta_{\omega_l (i+1)}^* G_{\omega_l i}^{-1}(q_{i+1}) \cdots G_{\omega_l 2}^{-1}(q_{i+1}) G_{\omega_l 1}^{-1}(q_{i+1}) N(q_{i+1}) \\ &= \eta_{\omega_0 (i+1)}^* W_{l0i}^* G_{\omega_l i}^{-1}(q_{i+1}) \cdots G_{\omega_l 2}^{-1}(q_{i+1}) G_{\omega_l 1}^{-1}(q_{i+1}) \\ &\quad \times N(q_{i+1}) \\ &= \eta_{\omega_0 (i+1)}^* G_{\omega_0 i}^{-1}(q_{i+1}) \cdots G_{\omega_0 2}^{-1}(q_{i+1}) G_{\omega_0 1}^{-1}(q_{i+1}) N(q_{i+1}) \\ &= 0. \end{aligned}$$

Define further

$$G_{\omega_l (i+1)}(\lambda) = U_{\omega_l (i+1)} \begin{bmatrix} \frac{e^{j\omega_l} q_{i+1}^* - 1}{e^{j\omega_l} - q_{i+1}} \frac{\lambda - q_{i+1}}{\lambda q_{i+1}^* - 1} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l (i+1)}^*$$

for all $l = 0, \pm 1, \ldots \pm n$. It follows that

$$\eta_{\omega_l (i+1)} = W_{l0i} \eta_{\omega_0 (i+1)} = W_{l0i} W_{l'0i}^* \eta_{\omega_{l'} (i+1)}.$$

Due to the fact that $W_{ll'i} = W_{l0i} W^*_{l'0i}$, we have

$$\eta_{\omega_l(i+1)} = W_{ll'i}\eta_{\omega_{l'}(i+1)}.$$

Moreover,

$$\begin{aligned} W_{ll'(i+1)} &= G^{-1}_{\omega_l(i+1)}(\lambda)G^{-1}_{\omega_l i}(\lambda)\cdots G^{-1}_{\omega_l 1}(\lambda) \\ &\quad\times G_{\omega_{l'}1}(\lambda)\cdots G_{\omega_{l'}i}(\lambda)G_{\omega_{l'}(i+1)}(\lambda) \\ &= G^{-1}_{\omega_l(i+1)}(\lambda)W_{ll'i}G_{\omega_{l'}(i+1)}(\lambda). \end{aligned}$$

Note that

$$U^*_{\omega_l(i+1)}W_{ll'i}U_{\omega_{l'}(i+1)} = U^*_{\omega_l(i+1)}W_{l0i}W^*_{l'0i}U_{\omega_{l'}(i+1)} = I.$$

Consequently, it holds

$$W_{ll'(i+1)} = U_{\omega_l(i+1)}\begin{bmatrix} \frac{e^{j\omega_l}-q_{i+1}}{e^{j\omega_l}q^*_{i+1}-1}\frac{e^{j\omega_{l'}}q^*_{i+1}-1}{e^{j\omega_{l'}}-q_{i+1}} & 0 \\ 0 & I \end{bmatrix}U^*_{\omega_{l'}(i+1)}.$$

This shows that $W_{ll'(i+1)}$ are constant unitary matrices for all $l, l' = 0, \pm 1, \ldots, \pm n$. As such, we have established (14) and the fact that $W_{ll'i}$ in (15) are constant unitary matrices. The Proof of Lemma 1 is then completed by substituting (15) into (14), with λ replaced by $e^{j\omega_l}$.

Appendix B. Proof of Theorem 1

Denote the λ-transforms of $r(k)$, $v(k)$, $w(k)$, $z(k)$ by $R(\lambda)$, $V(\lambda)$, $W(\lambda)$, and $Z(\lambda)$, respectively. Introducing the parameterization of all stabilizing two-degree-of-freedom controllers (Vidyasagar, 1985), it is easy to find that $Z(\lambda) = N(\lambda)Q(\lambda)W(\lambda)$, where $Q(\lambda) \in \mathscr{H}_\infty$ is an arbitrary transfer matrix at the designer's choice. Under the full-information tracking scheme, $W(\lambda) = V(\lambda)$. Hence, according to Parseval identity, we have

$$J(v) = \|R(\lambda) - N(\lambda)Q(\lambda)V(\lambda)\|_2^2.$$

Since $N(\lambda)$ is stable and its nonminimum phase zeros coincide with those of $G(\lambda)$, an inner–outer factorization of $N(\lambda)$ at the frequency ω_0 can be obtained as

$$\begin{aligned} N(\lambda) &= G_{\omega_0 1}(\lambda)\cdots G_{\omega_0 m}(\lambda)N_{\omega_0\text{out}}(\lambda) \\ &= G_{\omega_0\text{in}}(\lambda)N_{\omega_0\text{out}}(\lambda) \end{aligned} \tag{16}$$

where $G_{\omega_0 i}(\lambda)$ is a Blaschke factor of q_i in the form of (2) and $N_{\omega_0\text{out}}(\lambda)$ is outer. For simplicity, denote $G_{\omega_0\text{in}}(\lambda)$ and $N_{\omega_0\text{out}}(\lambda)$ by $G_{\text{in}}(\lambda)$ and $N_{\text{out}}(\lambda)$, respectively. With the factorization (16) and $R(\lambda)$, the tracking performance $J(v)$ can be written as

$$J(v) = \left\|\sum_{l=-n}^{n}\frac{v_l}{1-\lambda e^{j\omega_l}} - G_{\text{in}}(\lambda)N_{\text{out}}(\lambda)Q(\lambda)V(\lambda)\right\|_2^2.$$

Furthermore, noting the fact that $G_{\text{in}}(\lambda)$ is an inner function, we have

$$\begin{aligned} J(v) &= \left\|\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(\lambda)v_l}{1-\lambda e^{j\omega_l}} - N_{\text{out}}(\lambda)Q(\lambda)V(\lambda)\right\|_2^2 \\ &= \left\|\left[\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(\lambda)v_l}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}}\right]\right. \\ &\quad \left. + \left[\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{\text{out}}(\lambda)Q(\lambda)V(\lambda)\right]\right\|_2^2. \end{aligned}$$

One can see that

$$\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(\lambda)v_l}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} \in \mathscr{H}_2^{\perp},$$

and that by a proper choice of $Q(\lambda) \in \mathscr{H}_\infty$,

$$\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{\text{out}}(\lambda)Q(\lambda)V(\lambda) \in \mathscr{H}_2.$$

That $\mathscr{H}_2$ and $\mathscr{H}_2^{\perp}$ form an orthogonal complement suggests that

$$\begin{aligned} J(v) &= \left\|\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(\lambda)v_l}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}}\right\|_2^2 \\ &\quad + \left\|\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{\text{out}}(\lambda)Q(\lambda)V(\lambda)\right\|_2^2. \end{aligned} \tag{17}$$

Without loss of generality, we may assume that

$$V(\lambda) = \begin{bmatrix}\frac{v^T_{-n}}{1-\lambda e^{j\omega_{-n}}} & \cdots & \frac{v^T_0}{1-\lambda e^{j\omega_0}} & \cdots & \frac{v^T_n}{1-\lambda e^{j\omega_n}}\end{bmatrix}^T.$$

Partition $Q(\lambda)$ compatibly as

$$Q(\lambda) = [Q_{-n}(\lambda) \;\; \cdots \;\; Q_0(\lambda) \;\; \cdots \;\; Q_n(\lambda)]$$

and construct

$$Q_l(\lambda) = N^{\dagger}_{\text{out}}(e^{-j\omega_l})G^{-1}_{\text{in}}(e^{-j\omega_l}) + (1-\lambda e^{j\omega_l})\tilde{Q}_l(\lambda),$$

where $N^{\dagger}_{\text{out}}(e^{-j\omega_l})$ is a right inverse of $N_{\text{out}}(e^{-j\omega_l})$, and $\tilde{Q}_l(\lambda) \in \mathscr{H}_\infty$. It follows that

$$\begin{aligned} &\sum_{l=-n}^{n}\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{\text{out}}(\lambda)Q(\lambda)V(\lambda) \\ &\quad = \sum_{l=-n}^{n}\left\{[I - N_{\text{out}}(\lambda)N^{\dagger}_{\text{out}}(e^{-j\omega_l})]\frac{G^{-1}_{\text{in}}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}}\right. \\ &\quad\quad \left. - N_{\text{out}}(\lambda)\tilde{Q}_l(\lambda)v_l\right\} \in \mathscr{H}_2. \end{aligned}$$

Evidently, for $l=0,\pm 1,\ldots,\pm n$,

$$[I-N_{\mathrm{out}}(\lambda)N_{\mathrm{out}}^{\dagger}(\mathrm{e}^{-\mathrm{j}\omega_l})]\,\frac{G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\in\mathscr{H}_2.$$

It is therefore possible to find $\tilde{Q}_l(\lambda)$, which is independent of v_l, such that

$$\left\{[I-N_{\mathrm{out}}(\lambda)N_{\mathrm{out}}^{\dagger}(\mathrm{e}^{-\mathrm{j}\omega_l})]\,\frac{G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}} - N_{\mathrm{out}}(\lambda)\tilde{Q}_l(\lambda)\right\}v_l\longrightarrow 0.$$

Equivalently, we may find $Q(\lambda)\in\mathscr{H}_\infty$ independent of v, such that

$$\left\|\sum_{l=-n}^{n}\frac{G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-N_{\mathrm{out}}(\lambda)Q(\lambda)V(\lambda)\right\|_2^2\longrightarrow 0.$$

As a result, the second term of (17) can be made arbitrarily small by choosing $Q(\lambda)$ independent of v, hence leading to

$$J_{\mathrm{opt}}(v)=\left\|\sum_{l=-n}^{n}\left[\frac{v_l}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-G_{\mathrm{in}}(\lambda)\,\frac{G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\right]\right\|_2^2. \quad (18)$$

With $\omega_0=0$ and $G_{\mathrm{in}}(\lambda)$ given by (16), we have

$$\begin{aligned}J_{\mathrm{opt}}(v)=\Bigg\|\sum_{l=-n}^{n}\Bigg\{&\left[\frac{G_{01}^{-1}(\lambda)}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-\frac{G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\right]v_l\\&+\left[\frac{G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-G_{02}(\lambda)\cdots G_{0m}(\lambda)\\&\times\frac{G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\right]v_l\Bigg\}\Bigg\|_2^2.\end{aligned} \quad (19)$$

It can be readily observed that

$$\frac{G_{01}^{-1}(\lambda)}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-\frac{G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\in\mathscr{H}_2^{\perp}$$

and

$$\frac{G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-G_{02}(\lambda)\cdots G_{0m}(\lambda)\,\frac{G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\in\mathscr{H}_2.$$

Hence (19) can be rewritten as

$$\begin{aligned}J_{\mathrm{opt}}(v)=&\left\|\sum_{l=-n}^{n}\left[\frac{G_{01}^{-1}(\lambda)}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-\frac{G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\right]v_l\right\|_2^2\\&+\Bigg\|\sum_{l=-n}^{n}\Bigg\{\frac{G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-G_{02}(\lambda)\cdots G_{0m}(\lambda)\\&\times\frac{[G_{0m}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})\cdots G_{02}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})][G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l]}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\Bigg\}\Bigg\|_2^2.\end{aligned}$$

Repeating this procedure gives rise to

$$\begin{aligned}J_{\mathrm{opt}}(v)=&\left\|\sum_{l=-n}^{n}\left[\frac{G_{01}^{-1}(\lambda)}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-\frac{G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\right]v_l\right\|_2^2\\&+\sum_{i=2}^{m}\Bigg\|\sum_{l=-n}^{n}\left[\frac{G_{0i}^{-1}(\lambda)}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-\frac{G_{0i}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\right]\\&\times G_{0i-1}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})\cdots G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l\Bigg\|_2^2.\end{aligned} \quad (20)$$

In light of (4),

$$\begin{aligned}&\sum_{l=-n}^{n}\frac{G_{0i}^{-1}(\lambda)}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}-\sum_{l=-n}^{n}\frac{G_{0i}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})}{1-\lambda \mathrm{e}^{\mathrm{j}\omega_l}}\\&=\frac{1-q_i}{1-q_i^*}\left[\sum_{l=-n}^{n}\frac{1-|q_i|^2}{1-\mathrm{e}^{\mathrm{j}\omega_l}q_i}\eta_{0i}\eta_{0i}^*\right]\frac{1}{\lambda-q_i}.\end{aligned} \quad (21)$$

It thus follows by substituting (21) into (20) that

$$\begin{aligned}J_{\mathrm{opt}}(v)&=\sum_{i=1}^{m}\Bigg\|\left[\sum_{l=-n}^{n}\frac{1-|q_i|^2}{1-q_i\mathrm{e}^{\mathrm{j}\omega_l}}\eta_{0i}\eta_{0i}^*\right]\frac{1}{\lambda q_i^*-1}\\&\qquad\times G_{0i-1}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})\cdots G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l\Bigg\|_2^2\\&=\sum_{i=1}^{m}(1-|q_i|^2)\\&\qquad\times\left|\sum_{l=-n}^{n}\frac{\langle\eta_{0i},G_{0(i-1)}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})\cdots G_{01}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l\rangle}{1-q_i\mathrm{e}^{\mathrm{j}\omega_l}}\right|^2.\end{aligned}$$

The proof is then completed by invoking Lemma 1, yielding

$$J_{\mathrm{opt}}(v)=\sum_{i=1}^{m}(1-|q_i|^2)\left|\sum_{l=-n}^{n}\frac{\langle\eta_{-\omega_l i},v_l\rangle}{1-q_i\mathrm{e}^{\mathrm{j}\omega_l}}\right|^2.$$

Appendix C. Proof of Theorem 3

It follows from the proof of Theorem 1 that, the output of the closed-loop system shown in Fig. 1 with a two-degree-of-freedom controller is given by

$$Z(\lambda)=N(\lambda)Q(\lambda)W(\lambda)$$

for some $Q(\lambda)\in\mathscr{H}_\infty$. Hence, the energy of the transient tracking error response is written as

$$J(v)=\|R(\lambda)-N(\lambda)Q(\lambda)W(\lambda)\|_2^2.$$

In the partial-information case under consideration, the information from the signal generator is the reference, i.e., $W(\lambda)=R(\lambda)$. Furthermore the reference is assumed to be a signal-tone sinusoidal signal

$$R(\lambda)=\frac{v_{-1}}{1-\lambda e^{-j\omega}}+\frac{v_1}{1-\lambda e^{j\omega}}$$

and the coefficients of the signal have zero-mean, unit variance and are mutually uncorrelated, i.e.,

$$\boldsymbol{E}(v)=0, \quad \boldsymbol{E}(vv^*)=I,$$

where $v^*=[v_{-1}^* \ v_1^*]$.

The averaged tracking performance becomes

$$\begin{aligned}E&=\boldsymbol{E}\{J(v):\boldsymbol{E}(v)=0 \text{ and } \boldsymbol{E}(vv^*)=I\}\\&=\left\|[I-N(\lambda)Q(\lambda)]\left[\frac{I}{1-\lambda e^{-j\omega}}\ \frac{I}{1-\lambda e^{j\omega}}\right]\right\|_2^2.\end{aligned}$$

Conduct a *spectral factorization* (Vidyasagar, 1985) to find $\hat{R}(\lambda)$ such that

$$\begin{aligned}&\hat{R}(\lambda)\hat{R}^{\sim}(\lambda)I\\&=\left[\frac{I}{1-\lambda e^{-j\omega}}\ \frac{I}{1-\lambda e^{j\omega}}\right]\left[\frac{I}{1-\lambda e^{-j\omega}}\quad \frac{I}{1-\lambda e^{j\omega}}\right]^{\sim}\end{aligned}$$

where $\hat{R}^{\sim}(\lambda)=\hat{R}(1/\lambda)$. Then E can be rewritten as

$$E=\|[I-N(\lambda)Q(\lambda)]\hat{R}(\lambda)\|_2^2. \tag{22}$$

Note that the spectral factor $\hat{R}(\lambda)$ need not be unique. One particular spectral factor that will simplify our analysis is given by

$$\hat{R}(\lambda)=\frac{e^{j(\omega/2-(\pi/4))}}{1-\lambda e^{j\omega}}+\frac{e^{-j(\omega/2-(\pi/4))}}{1-\lambda e^{-j\omega}}.$$

Denote an inner–outer factorization of $N(\lambda)$ by

$$N(\lambda)=G_{\rm in}(\lambda)N_{\rm out}(\lambda).$$

Denote the dimension of the output by n and pth column of identify matrix I by e_p, $p=1,\ldots,n$. Then, (22) is rewritten to

$$E=\sum_{p=1}^{n}[\|\hat{R}(\lambda)e_p-G_{\rm in}(\lambda)N_{\rm out}(\lambda)Q(\lambda)\hat{R}(\lambda)e_p\|_2^2].$$

Applying the same argument as that used to establish (17) in the proof of Theorem 1, we have

$$\begin{aligned}E=&\sum_{p=1}^{n}\left\|G_{\rm in}^{-1}(\lambda)\hat{R}(\lambda)e_p-\hat{R}_m(\lambda)e_p\right\|_2^2\\&+\|\hat{R}_m(\lambda)-N_{\rm out}(\lambda)Q(\lambda)\hat{R}(\lambda)\|_2^2,\end{aligned} \tag{23}$$

where

$$\hat{R}_m(\lambda)=G_{\rm in}^{-1}(e^{-j\omega})\frac{e^{j(\omega/2-(\pi/4))}}{1-\lambda e^{j\omega}}+G_{\rm in}^{-1}(e^{j\omega})\frac{e^{-j(\omega/2-(\pi/4))}}{1-\lambda e^{-j\omega}}.$$

Due to the fact that $\hat{R}(\lambda)$ is minimum phase, the optimal $Q(\lambda)$ which minimizes E is given by

$$Q(\lambda)=\hat{Q}(\lambda):=N_{\rm out}^{\dagger}(\lambda)\frac{\hat{R}_m(\lambda)}{\hat{R}(\lambda)}. \tag{24}$$

Consequently, it holds that

$$\begin{aligned}E_{\rm opt}&=\|[I-N(\lambda)\hat{Q}(\lambda)]\hat{R}(\lambda)\|_2^2\\&=\boldsymbol{E}\{\|[I-N(\lambda)\hat{Q}(\lambda)]R(\lambda)\|_2^2:\\&\qquad \boldsymbol{E}(v)=0,\ \boldsymbol{E}(vv^*)=I\}.\end{aligned} \tag{25}$$

Denote the performance function $J(v)$ under the optimal $\hat{Q}(\lambda)$ by $\hat{J}(v)$, i.e.,

$$\hat{J}(v)=\|[I-N(\lambda)\hat{Q}(\lambda)]R(\lambda)\|_2^2. \tag{26}$$

With the same argument as that used to establish (23), Eq. (26) is written as

$$\begin{aligned}\hat{J}(v)=&\left\|G_{\rm in}^{-1}(\lambda)R(\lambda)-R_{\rm opt}(\lambda)\right\|_2^2\\&+\|R_{\rm opt}(\lambda)-N_{\rm out}(\lambda)\hat{Q}(\lambda)R(\lambda)\|_2^2,\end{aligned} \tag{27}$$

where

$$R_{\rm opt}(\lambda)=\frac{G_{\rm in}^{-1}(e^{j\omega})v_{-1}}{1-\lambda e^{-j\omega}}-\frac{G_{\rm in}^{-1}(e^{-j\omega})v_1}{1-\lambda e^{j\omega}}.$$

It is easily verified from (18) that

$$J_{\rm opt}(v)=\|G_{\rm in}^{-1}(\lambda)R(\lambda)-R_{\rm opt}(\lambda)\|_2^2.$$

Substituting (24) into (27) leads to

$$\hat{J}(v)=J_{\rm opt}(v)+\left\|R_{\rm opt}(\lambda)-\frac{\hat{R}_m(\lambda)}{\hat{R}(\lambda)}R(\lambda)\right\|_2^2. \tag{28}$$

It follows from some further algebra manipulation that

$$\begin{aligned}&R_{\rm opt}(\lambda)-\frac{\hat{R}_m(\lambda)}{\hat{R}(\lambda)}R(\lambda)\\&=[G_{\rm in}^{-1}(e^{-j\omega})-G_{\rm in}^{-1}(e^{j\omega})]\\&\quad\times\frac{(e^{-j(\omega/2-(\pi/4))}v_{-1}-e^{j(\omega/2-(\pi/4))}v_1)}{2\cos(\frac{\omega}{2}-\frac{\pi}{4})-2\lambda\cos(\frac{\omega}{2}+\frac{\pi}{4})}.\end{aligned} \tag{29}$$

Substituting (29) into (28) results in

$$\hat{J}(v) = J_{\text{opt}}(v) + \left\| [G_{\text{in}}^{-1}(e^{-j\omega}) - G_{\text{in}}^{-1}(e^{j\omega})] \times \frac{(e^{-j(\omega/2-(\pi/4))}v_{-1} - e^{j(\omega/2-(\pi/4))}v_1)}{2\cos(\frac{\omega}{2}-\frac{\pi}{4}) - 2\lambda\cos(\frac{\omega}{2}+\frac{\pi}{4})} \right\|_2^2. \quad (30)$$

Following (25), (26) and taking average at both sides of (30), we have

$$E_{\text{opt}} = J_{\text{opt}} + 2\left\| \frac{G_{\text{in}}^{-1}(e^{-j\omega}) - G_{\text{in}}^{-1}(e^{j\omega})}{2\cos(\frac{\omega}{2}-\frac{\pi}{4}) - 2\lambda\cos(\frac{\omega}{2}+\frac{\pi}{4})} \right\|_2^2 = J_{\text{opt}} + 2\left\| \frac{G_{\text{in}}(e^{j\omega}) - G_{\text{in}}(e^{-j\omega})}{2\cos(\frac{\omega}{2}-\frac{\pi}{4}) - 2\lambda\cos(\frac{\omega}{2}+\frac{\pi}{4})} \right\|_2^2,$$

where the last equality is from the fact that $G_{\text{in}}(s)$ is an inner factor. Note that

$$\left\| \frac{1}{2\cos(\frac{\omega}{2}-\frac{\pi}{4}) - 2\lambda\cos(\frac{\omega}{2}+\frac{\pi}{4})} \right\|_2^2 = \frac{1}{4[\cos^2(\frac{\omega}{2}-\frac{\pi}{4}) - \cos^2(\frac{\omega}{2}+\frac{\pi}{4})]}$$

and

$$\cos^2\left(\frac{\omega}{2}-\frac{\pi}{4}\right) - \cos^2\left(\frac{\omega}{2}+\frac{\pi}{4}\right) = \sin\omega.$$

We have

$$E_{\text{opt}} = J_{\text{opt}} + \frac{\text{tr}[G_{\text{in}}(e^{j\omega}) - G_{\text{in}}(e^{-j\omega})]^*[G_{\text{in}}(e^{j\omega}) - G_{\text{in}}(e^{-j\omega})]}{2\sin\omega}.$$

Appendix D. Proof of Theorem 4

The following lemma will be used in the subsequent proof of Theorem 4. Its proof mimicks that of a continuous-time counterpart (see, e.g., Chen, 2000; Qiu & Davison, 1993) and is thus omitted.

Lemma 3. *Let* $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ *and* $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ *be the balanced realizations of the inner transfer matrices* $G_1(\lambda)$ *and* $G_2(\lambda)$*, respectively. Then*

$$\left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

is a balanced realization of $G_1(\lambda)G_2(\lambda)$.

Now we proceed to prove Theorem 4. Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a realization of $G(\lambda)$ given as in (12), i.e.,

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_{\text{in}} & B_{\text{in}}C_0 & B_{\text{in}}D_0 \\ 0 & A_0 & B_0 \\ \hline C_{\text{in}} & D_{\text{in}}C_0 & D_{\text{in}}D_0 \end{array}\right].$$

Then

$$x_t(0) = -\sum_{k=-n}^{n} \begin{bmatrix} e^{j\omega_k}I - A_{\text{in}} & -B_{\text{in}}C_0 \\ 0 & e^{j\omega_k}I - A_0 \end{bmatrix}^{-1} \begin{bmatrix} B_{\text{in}}D_0 \\ B_0 \end{bmatrix} \times G_0^{\dagger}(e^{-j\omega_k})G_{\text{in}}^{-1}(e^{-j\omega_k})v_k = -\sum_{k=-n}^{n} \begin{bmatrix} (e^{j\omega_k}I - A_{\text{in}})^{-1}B_{\text{in}}G_{\text{in}}^{-1}(e^{-j\omega_k})v_k \\ ? \end{bmatrix}. \quad (31)$$

Here the question mark stands for an irrelevant quantity. Let $G(\lambda)$ be factorized in the cascade form at frequency $\omega_0 = 0$, i.e., $G(\lambda) = G_{\omega_0 1}(\lambda)\cdots G_{\omega_0 m}(\lambda)G_{\omega_0 0}(\lambda)$. A balanced realization of $G_{\omega_0 i}(\lambda)$ can be found to be (see, e.g., Chen, 2000)

$$\left[\begin{array}{c|c} q_i^* & \sqrt{\Delta_i}\eta_{\omega_0 i}^* \\ \hline -\sqrt{\Delta_i}\eta_{\omega_0 i}\dfrac{e^{j\omega_0}q_i^* - 1}{e^{j\omega_0} - q_i} & I + \eta_{\omega_0 i}\dfrac{\Delta_i e^{j\omega_0}}{e^{j\omega_0} - q_i}\eta_{\omega_0 i}^* \end{array}\right]$$

where $\Delta_i = 1 - |q_i|^2$. Denote the state in this realization by $x_i(k)$. Then the vector $[x_1(k)\ \cdots\ x_m(k)]'$ forms the state of $G_{\text{in}}(\lambda) = G_{\omega_0 1}(\lambda)\cdots G_{\omega_0 m}(\lambda)$, with the realization $\left[\begin{array}{c|c} A_{\text{in}} & B_{\text{in}} \\ \hline C_{\text{in}} & D_{\text{in}} \end{array}\right]$, which, by virtue of Lemma 3, is a balanced realization. Furthermore, the transfer matrix from the input to the state is

$$(\lambda^{-1}I - A_{\text{in}})^{-1}B_{\text{in}} = \begin{bmatrix} \dfrac{\sqrt{1-|q_1|^2}\lambda}{1-q_1^*\lambda}\eta_{\omega_0 1}^* G_{\omega_0 2}(\lambda)\cdots G_{\omega_0 m}(\lambda) \\ \dfrac{\sqrt{1-|q_2|^2}\lambda}{1-q_2^*\lambda}\eta_{\omega_0 2}^* G_{\omega_0 3}(\lambda)\cdots G_{\omega_0 m}(\lambda) \\ \vdots \\ \dfrac{\sqrt{1-|q_m|^2}\lambda}{1-q_m^*\lambda}\eta_{\omega_0 m}^* \end{bmatrix}.$$

Note that

$$\eta_{\omega_0 i}^* G_{\omega_0 i}^{-1}(\lambda) = \frac{1-q_i}{q_i^*-1}\,\frac{q_i^*\lambda - 1}{\lambda - q_i}\,\eta_{\omega_0 i}^*.$$

Then

$$(\lambda^{-1}I - A_{\mathrm{in}})^{-1}B_{\mathrm{in}}G_{\mathrm{in}}^{-1}(\lambda) = \begin{bmatrix} \dfrac{1-q_1}{1-q_1^*}\dfrac{\sqrt{1-|q_1|^2}\lambda}{\lambda-q_1}\eta_{\omega_0 1}^* \\ \dfrac{1-q_2}{1-q_2^*}\dfrac{\sqrt{1-|q_2|^2}\lambda}{\lambda-q_2}\eta_{\omega_0 2}^* G_{\omega_0 1}^{-1}(\lambda) \\ \vdots \\ \dfrac{1-q_m}{1-q_m^*}\dfrac{\sqrt{1-|q_m|^2}\lambda}{\lambda-q_m}\eta_{\omega_0 m}^* G_{\omega_0(m-1)}^{-1}(\lambda)\cdots G_{\omega_0 1}^{-1}(\lambda) \end{bmatrix}.$$

This leads to

$$(\mathrm{e}^{\mathrm{j}\omega_l}I - A_{\mathrm{in}})^{-1}B_{\mathrm{in}}G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l}) = z_{-\omega_l}^*.$$

Here the last equality follows from Lemma 1. Finally, it follows from Lemma 2 and (31) that

$$\begin{aligned}\lim_{\varepsilon\to 0} J_\varepsilon(v) &= \left[\sum_{l=-n}^{n}(\mathrm{e}^{\mathrm{j}\omega_l}I - A_{\mathrm{in}})^{-1}B_{\mathrm{in}}G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l\right]^* \\ &\quad\times\left[\sum_{l=-n}^{n}(\mathrm{e}^{\mathrm{j}\omega_l}I - A_{\mathrm{in}})^{-1}B_{\mathrm{in}}G_{\mathrm{in}}^{-1}(\mathrm{e}^{-\mathrm{j}\omega_l})v_l\right] \\ &= \sum_{i=1}^{m}\sum_{l,l'=-n}^{n}(1-|q_i|^2)\frac{\langle v_l, \eta_{-\omega_l i}\rangle\langle \eta_{-\omega_{l'} i}, v_{l'}\rangle}{(1-q_i^*\mathrm{e}^{-\mathrm{j}\omega_l})(1-q_i\mathrm{e}^{\mathrm{j}\omega_{l'}})}.\end{aligned}$$

This completes the proof.

References

Chen, J. (2000). Logarithmic integrals, interpolation bounds, and performance limitations in MIMO feedback systems. *IEEE Transactions on Automatic Control*, *AC-45*, 1098–1115.

Chen, J., Qiu, L., & Toker, O. (2000). Limitations on maximal tracking accuracy. *IEEE Transactions on Automatic Control*, *AC-45*, 326–331.

Davison, E. J., & Scherzinger, B. M. (1987). Perfect control of the robust servomechanism problem. *IEEE Transactions on Automatic Control*, *32*, 689–702.

Francis, B. A. (1979). The optimal linear–quadratic time-invariant regulator with cheap control. *IEEE Transactions on Automatic Control*, *24*, 616–621.

Jemaa, L. B., & Davison, E. J. (2003). Performance limitations in the robust servomechanism problem for discrete-time LTI systems. *IEEE Transactions on Automatic Control*, *48*, 1299–1311.

Kwakernaak, H., & Sivan, R. (1972). *Linear optimal control systems*. New York: Wiley-Interscience.

Morari, M., & Zafiriou, E. (1989). *Robust process control*. Englewood Cliffs, NJ: Prentice-Hall.

Qiu, L., & Chen, J. (1999). Time domain characterizations of performance limitations of feedback control. In: Y. Yamamoto, & S. Hara (Eds.), *Learning, control and hybrid systems* (pp. 397–415). London: Springer.

Qiu, L., & Davison, E. J. (1993). Performance limitations of nonminimum phase systems in the servomechanism problem. *Automatica*, *29*, 337–349.

Su, W., Qiu, L., & Chen, J. (2003). Fundamental performance limitations in tracking sinusoidal signals. *IEEE Transactions on Automatic Control*, *48*, 1371–1380.

Toker, O., Chen, J., & Qiu, L. (2002). Tracking performance limitations in LTI multivariable discrete-time systems. *IEEE Transactions on Circuits and Systems, Part I*, *49*, 657–670.

Vidyasagar, M. (1985). *Control system synthesis: A factorization approach*. Cambridge, MA: MIT Press.

Zhou, K., Doyle, J. C., & Glover, K. (1995). *Robust and optimal control*. Upper Saddle River, NJ: Prentice-Hall.

Weizhou Su received the B.Eng. and M.Eng. degrees in automatic control engineering from the Southeast University, Nanjing, Jiangsu, China, in 1983 and 1986, respectively, and the Ph.D. degree in electrical engineering from the University of Newcastle, Newcastle, NSW, Australia, in 2000.

From 2000 to 2004, he was with the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Hong Kong, China; the School of QMMS, University of Western Sydney, Sydney, Australia. He joined the School of Automation Science and Engineering, South China University of Technology, Guangzhou, China, where he is currently a professor. His research interests include robust control, fundamental performance limitation of feedback control, and signal processing.

Li Qiu received the B.Eng degree in electrical engineering from Hunan University, Changsha, Hunan, China, in 1981, and the M.A.Sc. and Ph.D. degrees in electrical engineering from the University of Toronto, Toronto, Ont., Canada, in 1987 and 1990, respectively.

Since 1993, he has been with the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong SAR, China. He has also held visiting teaching and research positions in the University of Toronto, Canadian Space Agency, University of Waterloo, University of Minnesota, Zhejiang University, Australia Defence Force Academy, Harbin Institute of Technology. His current research interests include systems control theory, information theory, signal processing, applied and numerical linear algebra. He served as an associate editor of the *IEEE Transactions on Automatic Control and* an associate editor of *Automatica*. He is now an associate editor of *Journal of Control Theory and Applications* and a member of the editorial board of the *International Journal of Control*.

Jie Chen teaches at The University of California, Riverside, CA in the field of systems and control, and signal processing. He was born in Yichun, Jiangxi Province, The People's Republic of China on 14 January 1963. He received the B.S. degree in aerospace engineering from Northwestern Polytechnic University, Xian, China in 1982, the M.S.E. degree in electrical engineering, the M.A. degree in mathematics, and the Ph.D. degree in electrical engineering, all from The University of Michigan, Ann Arbor, Michigan, in 1985, 1987, and 1990, respectively.

From 1990 to 1993, he was with School of Aerospace Engineering and School of Electrical and Computer Engineering at Georgia Institute of Technology, Atlanta, Georgia. He joined University of California, Riverside, California as an Assistant Professor in 1994, where he has been a Professor since 1999, and served as the Department Chair for the Department of Electrical Engineering from 2001 to 2006. He has held guest positions and visiting appointments with Northwestern Polytechnic University, Xian, Zhejiang University, Hangzhou, Dalian University of Technology, Dalian, Harbin Institute of Technology-Shenzhen Graduate School, Shenzhen, Hong Kong University of Science and Technology, Hong Kong, China; Tokyo Institute of Technology, Tokyo, Japan; and The University of Newcastle, Callaghan, The University of Western Sydney, Penrith, Australia. His main research interests are in the areas of linear multivariable systems theory, system identification,

robust control, optimization, and networked control. He is the author of two books, respectively, (with G. Gu) Control-Oriented System Identification: An H-infinity Approach (Wiley-Interscience, 2000), and (with K. Gu and V.L. Kharitonov) Stability of Time-Delay Systems (Birkhauser, 2003).

Dr. Chen is a recipient of 1996 US National Science Foundation CAREER Award and of 2004 SICE International Award. He was a past Associate Editor for the IEEE Transactions on Automatic Control, Automatica, the Journal of Control Theory and Applications, and a past Guest Editor for the IEEE Transactions on Automatic Control. He currently serves as the founding Editor-in-Chief for the Journal of Control Science and Engineering, and a Guest Editor for the IEEE Control Systems Magazine.