Fundamental Performance Limitations in Tracking Sinusoidal Signals

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***Abstract*—This paper attempts to give a thorough treatment of the performance limitation of a linear time invariant multivariable system in tracking a reference signal which is a linear combination of a step signal and several sinusoids with different frequencies. The tracking performance is measured by an integral square error between the output of the plant and the reference signal. Our purpose is to find the fundamental limit for the attainable tracking performance, under any control structure and parameters, in terms of the characteristics and structural parameters of the given plant, as well as those of the reference signal under consideration. It is shown that this fundamental limit depends on the interaction between the reference signal and the nonminimum phase zeros of the plant and their frequency-dependent directional information.**

***Index Terms*—Linear system structure, nonminimum phase, optimal control, performance limitation, tracking.**

I. INTRODUCTION

THIS paper considers the performance limitations of a linear time-invariant (LTI) multivariable feedback control system in tracking a reference that is a linear combination of a step and several sinusoids of various frequencies. The setup is shown in Fig. 1. Here, $P(s)$ is the transfer matrix of a given plant whose measurement $y(t)$ and output $z(t)$ may not be the same, $K(s)$ is the transfer matrix of a two degree of freedom (2DOF) controller which is to be designed, $S(s)$ is the exosystem driven by an impulse which generates the reference. We assume that the controller has full information of the reference in the sense that it takes $v(t)$, the state of the exosystem $S(s)$, in addition to the measurement $y(t)$ of the plant, as its inputs. Whether or not the measurement $y(t)$ contains the full information of the plant, i.e., the state of the plant, is not important. The tracking problem is to design a controller $K(s)$ so that the closed loop system is internally stabilized and the plant output $z(t)$ asymptotically tracks a reference signal $r(t)$ of the form

$$r(t) = \sum_{k=-n}^{n} v_k e^{j\omega_k t} \tag{1}$$

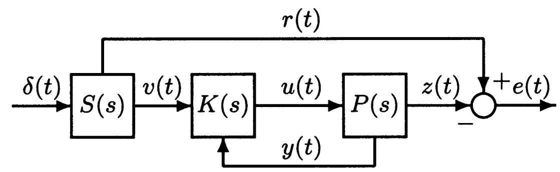

Fig. 1. Two-parameter control structure with reference full information.

where $\omega_k, k = 0, \pm 1, \ldots, \pm n$, are distinct real frequencies satisfying $\omega_{-k} = -\omega_k$ and $v_k, k = 0, \pm 1, \ldots, \pm n$, are complex vectors satisfying $v_{-k} = \bar{v}_k$. Implicitly, we have $\omega_0 = 0$ and v_0 is real. The reference defined in such a way is always a real valued signal. We use the vector

$$v = [v_{-n}^* \quad \cdots \quad v_{-1}^* \quad v_0^* \quad v_1^* \quad \cdots \quad v_n^*]^*$$

to capture the magnitude and phase information of all frequency components of the reference. The transient error is measured by its energy

$$J(v) = \int_0^\infty \|r(t) - z(t)\|^2 \, dt = \int_0^\infty \|e(t)\|^2 \, dt. \tag{2}$$

The tracking problem has a well-known solution, with well-known numerical methods to design controllers so that $J(v)$ is small. Nevertheless, it is desirable to have a deeper understanding of the smallest tracking error

$$J_{\rm opt}(v) = \inf_K J(v) \tag{3}$$

obtainable when the controller K is chosen among all possible stabilizing controllers. Such a smallest error then gives a fundamental limit in the transient performance of tracking. In this paper, we achieve this understanding in the form of an explicit, simple, and informative relationship between this fundamental limit and the plant characteristics.

The value $J_{\rm opt}(v)$ of course depends on v. If we are interested in an overall performance measure of the feedback system in tracking all references of the type (1), then we normally turn our attention to an averaged version of the tracking error, averaged over all possible v whose entries have zero mean, are mutually uncorrelated, and have a unit variance. Such an averaged performance measure is given by

$$E = \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} \tag{4}$$

where $\boldsymbol{E}$ is the expectation operator. In this case, the performance limit becomes

$$E_{\rm opt} = \inf_K E. \tag{5}$$

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It turns out that the averaged performance limit $E_{\rm opt}$ is simple enough to be presented as follows. Under some minor assumptions

$$E_{\rm opt} = 2\sum_{i=1}^{m}\sum_{k=-n}^{n}\frac{1}{z_i - j\omega_k} \tag{6}$$

where $z_i, i = 1, 2, \ldots, m$, are the nonminimum phase zeros of the transfer function from $u(t)$ to $z(t)$. The v dependent performance limit $J_{\rm opt}(v)$ need more elaborations but also turn out to be simple.

Results of this sort can be traced back for over a decade. For single-input–single-output (SISO) systems and constant references, vector v degenerates to a real scalar. Then, the linearity of the plant implies that $J_{\rm opt}(v) = v^2 J_{\rm opt}(1)$ and $E_{\rm opt} = J_{\rm opt}(1)$. It is obtained in [11] that

$$J_{\rm opt}(1) = 2\sum_{i=1}^{m}\frac{1}{z_i}.$$

For multivariable systems and for the case when the reference $r(t)$ is either a constant or a sinusoid with a single frequency ω, it was obtained in [14] that

$$E_{\rm opt} = 2\sum_{i=1}^{m}\frac{1}{z_i}$$

and

$$E_{\rm opt} = 2\sum_{i=1}^{m}\left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega}\right)$$

respectively. The study of the performance limit $J_{\rm opt}(v)$ for a fixed constant reference in the multivariable case started in [5]. It is shown there that the performance limit in this case depends on not only the locations of the nonminimun phase zeros but also their directional information. The study in [5] has since been extended to more general references [6], [7], and discrete-time systems [18]. There have also been generalizations to nonright-invertible plant [20], [3], to the cases where the controller has previewed information of [7], where the plant input is subject to saturation [12], and where the tracking performance measure includes the input energy [4], respectively. It has been shown that, consistent with common intuitions, the preview of the reference can reduce the best achievable tracking error and on the other hand any input saturation or any input energy constraint would likely increase the best achievable tracking error. Related issues for nonlinear systems and filtering problems are studied in [17], [1], and [15].

Although $E_{\rm opt}$ gives an overall quality measure for the plant as far as tracking is concerned, the reference direction dependent performance limit $J_{\rm opt}(v)$ gives more information and deeper insights. If we know $J_{\rm opt}(v)$ and if the optimal K which minimizes $J(v)$ is independent of v, then $E_{\rm opt}$ can be obtained after simple operations. This is why we adopt the thinking in [5] to place our main emphasis on $J_{\rm opt}(v)$. In the aforementioned formulation, the assumption that the state of the exosystem is available to the controller is crucial. This means that not only the reference but also all magnitude and phase information of its frequency components is known to the controller. This may look unrealistic in the first glance, but it does give a limitation more fundamental than any other one based only on the partial information of the reference. It is this assumption that makes it possible to find a uniformly optimal controller K to minimize $J(v)$ for all v. Note that when the reference only contains a constant term, the value of the reference already contains its full information. Therefore in this particular case, whether or not the controller can assess the state of the reference is not an issue.

This paper gives a rather complete picture for the tracking performance limitation problem for general reference signals containing several frequency components. We first give some new insight on linear system structure. We show that each nonminimum phase zero has associated frequency dependent directions. A key technical result in this paper is a relation among directions at different frequencies. Using this result, we derive an expression for $J_{\rm opt}(v)$ which elegantly exhibits the effect of the plant nonminimum phase zeros, as well as the interaction between each frequency component and the directions mentioned above, toward the performance limitations.

There has been a surge of activities in the study of performance limitations in feedback control. In addition to the type of performance limitations studied in this paper, which focus on system time responses and, hence, are called time domain performance limitations, there is a whole body of literature on design limitations on system frequency responses, known as frequency domain performance limitations. For the history and the recent progress on frequency domain performance limitations, see [2], [8], [16]. Some intriguing connections have been realized between the time domain limitations and the frequency domain ones [10].

The organization of this paper is as follows. In Section II, preliminary materials on linear system factorizations are presented. It is shown that a right-invertible system can be factorized as a cascade connection of a series of first-order inner factors and a minimum phase factor. The factorization is frequency dependent. The inner factors then contain all the information associated with the nonminimum phase zeros. In Section III, we formally formulate the problems studied and then state and discuss the main result and some of its consequences. Section IV extends the main result in Section III to the case where the plant contains time delays. Section V is the conclusion. Finally, the proof of the results in Sections III and IV are given in Appendices I and II, respectively.

The notation used throughout this paper is fairly standard. For any complex number, vector, and matrix, denote their conjugate, transpose, conjugate transpose, real and imaginary part by $\bar{(\cdot)}, (\cdot)', (\cdot)^*, {\rm Re}(\cdot)$, and ${\rm Im}(\cdot)$, respectively. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open-right and left-half plane be denoted by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. $\mathcal{L}_2$ is the standard frequency domain Lebesgue space. $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspaces of $\mathcal{L}_2$ containing functions that are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. It is well-known that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ constitute orthogonal complements in $\mathcal{L}_2$. $\mathcal{RH}_\infty$ is the set of all stable, rational transfer matrices. Finally, the inner product between two complex vectors u, v is defined by $\langle u, v\rangle := u^* v$.

II. Preliminaries

Let $G(s)$ be a real rational matrix representing the transfer function of a continuous time finite-dimensional, linear time invariant (FDLTI) system. Let us assume that $G(s)$ is right invertible. Its poles and zeros, including multiplicity, are defined according to its Smith–McMillan form. $G(s)$ is said to be minimum phase if all its zeros have nonpositive real parts; otherwise, it is said to be nonminimum phase.

Let $G(s) = N(s)M^{-1}(s)$, where $M(s), N(s) \in \mathcal{RH}_\infty$, be a right coprime factorization of $G(s)$. Let $z \in \mathbb{C}$ be a nonminimum phase zero of $G(s)$. Then z is also a nonminimum phase zero of $N(s)$ and there exists a unit vector η such that

$$\eta^* N(z) = 0.$$

We call the vector η a (left or output) zero vector of $G(s)$ corresponding to the nonminimum phase zero z.

Let us now order the nonminimum phase zeros of $G(s)$ (or $N(s)$ equivalently) as $z_1, z_2, \ldots, z_m$. Assume that each pair of complex conjugate zeros are ordered in adjacent order. Let us also fix a frequency $\omega_k \in \mathbb{R}$. We first find a unit zero vector $\eta_{\omega_k 1}$ of $G(s)$ corresponding to z_1 and define

$$\begin{aligned} G_{\omega_k 1}(s) &= I - \eta_{\omega_k 1} \frac{2\,\mathrm{Re}(z_1)}{z_1 - j\omega_k} \frac{s - j\omega_k}{z_1^* + s} \eta_{\omega_k 1}^* \\ &= U_{\omega_k 1} \begin{bmatrix} \frac{z_1^* + j\omega_k}{z_1 - j\omega_k} \frac{z_1 - s}{z_1^* + s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_k 1}^* \end{aligned}$$

where $U_{\omega_k 1}$ is a unitary matrix with the first column equal to $\eta_{\omega_k 1}$. Here, $G_{\omega_k 1}(s)$ is so constructed that it is inner, has the only zero at z_1 with $\eta_{\omega_k 1}$ as a zero vector corresponding to z_1, and $G_{\omega_k 1}(j\omega_k) = I$. Since $G_{\omega_k 1}(s)$ is a generalization of the standard scalar Blaschke factor, we call it a matrix Blaschke factor at the frequency w_k and $\eta_{\omega_k 1}$ a corresponding Blaschke vector. Also, notice that the choice of other columns in $U_{\omega_k 1}$ is immaterial. Now $G_{\omega_k 1}^{-1}(s)G(s)$ has zeros $z_2, z_3, \ldots, z_m$. Find a zero vector $\eta_{\omega_k 2}$ of $G_{\omega_k 1}^{-1}(s)G(s)$ corresponding to z_2 and define

$$\begin{aligned} G_{\omega_k 2}(s) &= I - \eta_{\omega_k 2} \frac{2\mathrm{Re}(z_2)}{z_2 - j\omega_k} \frac{s - j\omega_k}{z_2^* + z} \eta_{\omega_k 2}^* \\ &= U_{\omega_k 2} \begin{bmatrix} \frac{z_2^* + j\omega_k}{z_2 - j\omega_k} \frac{z_2 - s}{z_2^* + s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_k 2}^* \end{aligned}$$

where $U_{\omega_k 2}$ is a unitary matrix with the first column equal to $\eta_{\omega_k 2}$. Then $G_{\omega_k 2}^{-1}(s)G_{\omega_k 1}^{-1}(s)G(s)$ has zeros $z_3, z_4, \ldots, z_m$. Continue this process until Blaschke vectors $\eta_{\omega_k 1}, \ldots, \eta_{\omega_k m}$ and Blaschke factors $G_{\omega_k 1}(s), \ldots, G_{\omega_k m}(s)$ are all obtained. This procedure shows that $G(s)$ can be factorized as

$$G(s) = G_{\omega_k 1}(s) \ldots G_{\omega_k m}(s) G_{\omega_k 0}(s) \tag{7}$$

where

$$\begin{aligned} G_{\omega_k i}(s) &= I - \eta_{\omega_k i} \frac{2\,\mathrm{Re}(z_i)}{z_i - j\omega_k} \frac{s - j\omega_k}{z_i^* + s} \eta_{\omega_k i}^* \\ &= U_{\omega_k i} \begin{bmatrix} \frac{z_i^* + j\omega_k}{z_i - j\omega_k} \frac{z_i - s}{z_i^* + s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_k i}^* \end{aligned} \tag{8}$$

and $G_{\omega_k 0}(s)$ has no nonminimum phase zero. We call this factorization a cascade factorization at frequency ω_k, which is shown schematically in Fig. 2. In this factorization, each Blaschke vector and Blaschke factor correspond to one nonminimum phase zero. Keep in mind that these vectors and factors depend on the order of the nonminimum zeros, as well as on the frequency ω_k. The product

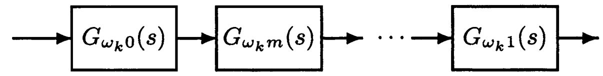

Fig. 2. Cascade factorization.

$$G_{\omega_k 1}(s) \ldots G_{\omega_k m}(s)$$

is called a matrix Blaschke product.

One should note that even when the order of $z_1, z_2, \ldots, z_m$ is fixed, the factorization at the frequency ω_k is not unique since $\eta_{\omega_k i}$ is not uniquely determined. Furthermore, if we have $2n+1$ different frequencies $\omega_k, k = 0, \pm 1, \ldots, \pm n$, then the factorizations at different frequencies are in general different. Nevertheless, they can be intimately related if we make the choices properly. For example, it is easy to see from the above construction that $\eta_{\omega_k 1}$, the first Blaschke vector, can be chosen independent of ω_k. The following lemma provides such relations and is the key technical vehicle that leads to the main result of this paper.

Lemma 1: Suppose that the order of $z_1, z_2, \ldots, z_m$ is fixed. Also suppose that we are given $2n+1$ different frequencies $\omega_k, k = 0, \pm 1, \ldots, \pm n$. Then the $2n+1$ cascade factorizations (7) can be chosen so that for all $k, l = 0, \pm 1, \ldots, \pm n$ and $i = 1, 2, \ldots, m$

$$\eta_{\omega_k i} = G_{\omega_l 1}(j\omega_k) G_{\omega_l 2}(j\omega_k) \ldots G_{\omega_l i-1}(j\omega_k) \eta_{\omega_l i}. \tag{9}$$

Proof: Let us first prove that the factorizations can be chosen so that for all $k, l = 0, \pm 1, \ldots, \pm n$ and $i = 1, 2, \ldots, m$

$$\eta_{\omega_k i} = W_{kl(i-1)} \eta_{\omega_l i} \tag{10}$$

and

$$G_{\omega_k i}^{-1}(s) \ldots G_{\omega_k 2}^{-1}(s) G_{\omega_k 1}^{-1}(s) G_{\omega_l 1}(s) G_{\omega_l 2}(s) \ldots G_{\omega_l i}(s) =: W_{kli} \tag{11}$$

are constant unitary matrices. Here, it is understood that $W_{kl0} = I$. The proof is based on an induction in i. Let us first consider the case when $i = 1$. We know that $\eta_{\omega_k 1}$ can be chosen independent of ω_k. Hence, $U_{\omega_k 1}$ can also be chosen independent of ω_k. Thus, rather trivially, for all $k = 0, \pm 1, \ldots, \pm n$

$$\eta_{\omega_k 1} = W_{kl0} \eta_{\omega_l 1}$$

and

$$\begin{aligned} &G_{\omega_k 1}^{-1}(s) G_{\omega_l 1}(s) \\ &= U_{\omega_k 1} \begin{bmatrix} \frac{z_1 - j\omega_k}{z_1^* + j\omega_k} \frac{z_1^* + s}{z_1 - s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_k 1}^* U_{\omega_l 1} \\ &\quad \times \begin{bmatrix} \frac{z_1^* + j\omega_l}{z_1 - j\omega_l} \frac{z_1 - s}{z_1^* + s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l 1}^* \\ &= U_{\omega_k 1} \begin{bmatrix} \frac{z_1 - j\omega_k}{z_1^* + j\omega_k} \frac{z_1^* + j\omega_l}{z_1 - j\omega_l} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l 1}^* \\ &=: W_{kl1} \end{aligned}$$

are constant unitary matrices.

Now, assume that $\eta_{\omega_k 1}, \ldots, \eta_{\omega_k i}$ and $G_{\omega_k 1}(s), \ldots, G_{\omega_k i}(s)$ have been chosen so that

$$\eta_{\omega_k i} = W_{kl(i-1)} \eta_{\omega_l i}, \qquad i = 1, 2, \ldots, m$$

and

$$G_{\omega_k i}^{-1}(s)\dots G_{\omega_k 2}^{-1}(s)G_{\omega_k 1}^{-1}(s)G_{\omega_l 1}(s) G_{\omega_l 2}(s)\dots G_{\omega_l i}(s) := W_{kli}$$

are constant unitary matrices for all $k, l = 0, \pm 1, \dots, \pm n$. For the frequency ω_0, choose a unit vector $\eta_{\omega_0(i+1)}$ such that

$$\eta_{\omega_0(i+1)}^* G_{\omega_0 i}^{-1}(z_{i+1})\dots G_{\omega_0 2}^{-1}(z_{i+1})\, G_{\omega_0 1}^{-1}(z_{i+1})N(z_{i+1}) = 0$$

and a unitary matrix $U_{\omega_0(i+1)}$ such that its first column is $\eta_{\omega_0(i+1)}$. Define

$$U_{\omega_k(i+1)} = W_{k0i}U_{\omega_0(i+1)}.$$

Then, the first column $\eta_{\omega_k(i+1)}$ of $U_{\omega_k(i+1)}$ satisfies

$$\begin{aligned}\eta_{\omega_k(i+1)}^* & G_{\omega_k i}^{-1}(z_{i+1})\dots G_{\omega_k 2}^{-1}(z_{i+1})G_{\omega_k 1}^{-1}(z_{i+1})N(z_{i+1}) \\ &= \eta_{\omega_0(i+1)}^* W_{k0i}^* G_{\omega_k i}^{-1}(z_{i+1})\dots G_{\omega_k 2}^{-1}(z_{i+1}) \\ &\quad \times G_{\omega_k 1}^{-1}(z_{i+1})N(z_{i+1}) \\ &= \eta_{\omega_0(i+1)}^* G_{\omega_0 i}^{-1}(z_{i+1})\dots G_{\omega_0 2}^{-1}(z_{i+1}) \\ &\quad \times G_{\omega_0 1}^{-1}(z_{i+1})N(z_{i+1}) \\ &= 0.\end{aligned}$$

This shows that $\eta_{\omega_k(i+1)}$ is indeed what we need. Define

$$G_{\omega_k(i+1)}(s) = U_{\omega_k(i+1)} \begin{bmatrix} \frac{z_{i+1}^*+j\omega_k}{z_{i+1}-j\omega_k}\frac{z_{i+1}-s}{z_{i+1}^*+s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_k(i+1)}^*$$

for all $k = 0, \pm 1, \dots, \pm n$. Then, we have

$$\begin{aligned}\eta_{\omega_k(i+1)} &= W_{k0i}\eta_{\omega_0(i+1)} = W_{k0i}W_{l0i}^*\eta_{\omega_l(i+1)} = W_{kli}\eta_{\omega_l(i+1)}\end{aligned}$$

and

$$\begin{aligned} G_{\omega_k(i+1)}^{-1}&(s)G_{\omega_k i}^{-1}(s)\dots G_{\omega_k 1}^{-1}(s)G_{\omega_l 1}(s)\dots \\ &\times G_{\omega_l i}(s)G_{\omega_l(i+1)}(s) \\ &= G_{\omega_k(i+1)}^{-1}(s)W_{kli}G_{\omega_l(i+1)}(s) \\ &= U_{\omega_k(i+1)} \begin{bmatrix} \frac{z_{i+1}-j\omega_k}{z_{i+1}^*+j\omega_k}\frac{z_{i+1}^*+s}{z_{i+1}-s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_k(i+1)}^* W_{kli} \\ &\quad \times U_{\omega_l(i+1)} \begin{bmatrix} \frac{z_{i+1}^*+j\omega_l}{z_{i+1}-j\omega_l}\frac{z_{i+1}-s}{z_{i+1}^*+s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l(i+1)}^*. \\ &= U_{\omega_k(i+1)} \begin{bmatrix} \frac{z_{i+1}-j\omega_k}{z_{i+1}^*+j\omega_k}\frac{z_{i+1}^*+s}{z_{i+1}-s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_k(i+1)}^* W_{k0i}W_{l0i}^* \\ &\quad \times U_{\omega_l(i+1)} \begin{bmatrix} \frac{z_{i+1}^*+j\omega_l}{z_{i+1}-j\omega_l}\frac{z_{i+1}-s}{z_{i+1}^*+s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l(i+1)}^* \\ &= U_{\omega_k(i+1)} \begin{bmatrix} \frac{z_{i+1}-j\omega_k}{z_{i+1}^*+j\omega_k}\frac{z_{i+1}^*+j\omega_l}{z_{i+1}-j\omega_l} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l(i+1)}^* \\ &=: W_{kl(i+1)}\end{aligned}$$

are unitary matrices for all $k, l = 0, \pm 1, \dots, \pm n$. This proves (10) and (11). Finally, (9) follows from substituting (11) into (10) with s replaced by $j\omega_k$. ∎

One may wonder what these Balschke vectors look like when $G(s)$ is SISO. In this case, proper choices lead to

$$\eta_{\omega_k i} = \frac{z_1^*}{z_1}\frac{z_1 - j\omega_k}{z_1^* + j\omega_k}\dots\frac{z_{i-1}^*}{z_{i-1}}\frac{z_{i-1}-j\omega_k}{z_{i-1}^*+j\omega_k} \tag{12}$$

for $k = 0, \pm 1, \dots, \pm n, i = 1, 2, \dots, m$.

III. Main Result

Let us go back to the setup shown in Fig. 1. The measurement output $y(t)$ of the plant might be different from the tracking output $z(t)$. We denote the transfer function from $u(t)$ to $z(t)$ by $G(s)$ and that from $u(t)$ to $y(t)$ by $H(s)$, i.e.,

$$P(s) = \begin{bmatrix} G(s) \\ H(s) \end{bmatrix}. \tag{13}$$

In order for the tracking problem to be meaningful and solvable, we make the following assumptions throughout the paper.

Assumption 1:

1) $P(s), G(s)$, and $H(s)$ have the same unstable poles.
2) $G(s)$ has no zero at $j\omega_k, k = 0, \pm 1, \dots, \pm n$.

The first item in the assumption means that the measurement can be used to stabilize the system and at the same time does not introduce any additional unstable modes. A more precise way of stating this is that if

$$P(s) = \begin{bmatrix} N(s) \\ L(s) \end{bmatrix} M^{-1}(s)$$

is a coprime factorization, then we assume that $N(s)M^{-1}(s)$ and $L(s)M^{-1}(s)$ are also coprime factorizations. The second item is necessary for the solvability of the tracking problem.

We now state our main result, whose proof will be given in Appendix I.

Theorem 1: Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \dots, z_m$ with corresponding Blaschke vectors $\eta_{\omega_k 1}, \dots, \eta_{\omega_k m}, k = 0, \pm 1, \dots, \pm n$, satisfying Lemma 1. Then

$$\begin{aligned} J_{\text{opt}}(v) &= \sum_{i=1}^m 2\,\text{Re}(z_i)\left|\sum_{k=-n}^n \frac{\langle \eta_{\omega_k i}, v_k\rangle}{z_i - j\omega_k}\right|^2 \\ &= \sum_{i=1}^m \sum_{k=-n}^n \sum_{l=-n}^n \frac{2\,\text{Re}(z_i)\,\langle v_k, \eta_{\omega_k i}\rangle\,\langle \eta_{\omega_l i}, v_l\rangle}{(z_i^* + j\omega_k)(z_i - j\omega_l)}.\end{aligned}$$

This formula shows that each nonminimum phase zero contributes additively to the performance limit. However, the contribution of each frequency component of the reference enters the performance limit in a quadratic form and the cross coupling of pairs of frequencies appears in the performance limit. It also shows that generically, perfect tracking is impossible when the plant is nonminimum phase. However, if it happens that the vector v is orthogonal to the vectors

$$\eta_i = \begin{bmatrix} \frac{\eta_{\omega_{-n}i}^*}{z_i - j\omega_{-n}} & \cdots & \frac{\eta_{\omega_0 i}^*}{z_i} & \cdots & \frac{\eta_{\omega_n i}^*}{z_i - j\omega_n} \end{bmatrix}^*, \quad i = 1, 2, \dots, m$$

then perfect tracking can be achieved. Here, v captures the magnitude and phase information of the reference and η_i captures the property of the plant at the nonminimum phase zero z_i. This orthogonality may happen in two ways. One is over the output channels: v_k is orthogonal to $\eta_{\omega_k i}$ for all $i = 1, 2, \dots, m, k = 0, \pm 1, \dots, \pm n$. This can only occur for multivariable systems. The other is over the frequencies: the orthogonality over output channels does not occur but v and η_i are orthogonal due to some special alignment of the magnitude and phase of the reference.

This can happen even for the SISO case. For example, in the case when $m = 1$ and $G(s)$ is SISO, if v_k happens to make $v_k/(z_1 - j\omega_k)$ imaginary for all $k = 0, \pm 1, \ldots, \pm n$, then the performance limit is zero.

In the case when $n = 0$, i.e., the reference only has a step component, we get

$$J_{\text{opt}}(v) = \sum_{i=1}^{m} \frac{2\,\text{Re}(z_i)}{|z_i|^2} |\langle \eta_{0i}, v\rangle|^2.$$

This is exactly the formula given in [5].

In the case when the system $G(s)$ is SISO, the performance limit becomes

$$J_{\text{opt}}(v) = \sum_{i=1}^{m}\sum_{k=-n}^{n}\sum_{l=-n}^{n} 2\,\text{Re}(z_i) \frac{\eta_{\omega_k i}\eta_{\omega_l i}^* v_k^* v_l}{(z_i^* + j\omega_k)(z_i - j\omega_l)} \tag{14}$$

where $\eta_{\omega_k i}, i = 1, \ldots, m$, are scalars with unit modulus given by (12).

The proof of Theorem 1, as given in Appendix I, shows that a controller or a sequence of controllers, independent of v, can be found to attain the performance limit $J_{\text{opt}}(v)$. Therefore

$$\begin{aligned} E_{\text{opt}} &= \inf_K \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} && (15)\\ &= \boldsymbol{E}\left\{\inf_K J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\right\} && (16)\\ &= \sum_{i=1}^{m}\sum_{k=-n}^{n}\sum_{l=-n}^{n} 2\,\text{Re}(z_i)\frac{\eta_{\omega_l i}^* \boldsymbol{E}(v_l v_k^*)\eta_{\omega_k i}}{(z_i^* + j\omega_k)(z_i - j\omega_l)}\\ &= \sum_{i=1}^{m}\sum_{k=-n}^{n} 2\,\text{Re}(z_i)\frac{\eta_{\omega_k i}^*\eta_{\omega_k i}}{(z_i^* + j\omega_k)(z_i - j\omega_k)}\\ &= \sum_{i=1}^{m}\sum_{k=-n}^{n} \frac{2\,\text{Re}(z_i)}{|z_i - j\omega_k|^2}. \end{aligned}$$

This immediately leads to the following theorem.

Theorem 2: Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$. Then

$$E_{\text{opt}} = \sum_{i=1}^{m}\sum_{k=-n}^{n} \frac{2\,\text{Re}(z_i)}{|z_i - j\omega_k|^2} = 2\sum_{i=1}^{m}\sum_{k=-n}^{n} \frac{1}{z_i - j\omega_k}.$$

From Theorem 2 it is seen that the average performance limit has a strikingly simple form; it is the simple sum of the contributions of all nonminimum phase zeros at all frequencies; each of such contributions is the reciprocal of the distance between a nonminimum phase zero and a mode of the reference.

IV. Performance Limitation for Systems With Time Delays

In this section, we generalize the previous result a bit further to systems with time delays. We assume that $G(s)$ admits a factorization of the form

$$G(s) = L_1(s)\ldots L_d(s)G_1(s)\ldots G_m(s)G_0(s). \tag{17}$$

Here, $L_i(s)$ is assumed to have the form

$$L_i(s) = I - \zeta_i(1 - e^{-\tau_i s})\zeta_i^* = V_i \begin{bmatrix} e^{-\tau_i s} & 0 \\ 0 & I\end{bmatrix} V_i^*$$

where ζ_i is a real unit vector characterizing the directional information of this delay and V_i is a real orthogonal matrix whose first column is ζ_i. $G_i(s)$ is assumed to be a Blaschke factor with zero z_i. The last factor $G_0(s)$, not necessarily rational, is assumed to have a coprime factorization $N_0(s)M_0^{-1}(s)$ with an outer $N_0(s)$. It is easy to see that a multivariable FDLTI system with independent delays in all output channels can be written in the form of (17). However, at this moment, it is not clear what is the general class of transfer matrices that admit this type of factorizations. It is not even clear how we can write a multivariable system with independent time delays in the input channels in the form of (17). It would be interesting to clarify these issues.

For systems given in the form of (17), we have a generalized version of Lemma 1.

Lemma 2: Suppose that we are given $2n+1$ different frequencies $\omega_k, k = 0, \pm 1, \ldots, \pm n$. Then, there exist $2n+1$ cascade factorizations

$$G(s) = L_{\omega_k 1}(s)\ldots L_{\omega_k d}(s)G_{\omega_k 1}(s)\ldots G_{\omega_k m}(s)G_{\omega_k 0}(s)$$

where for $i = 1, \ldots, d$

$$L_{\omega_k i}(s) = I - \zeta_{\omega_k i}\left(1 - e^{-\tau_i(s - j\omega_k)}\right)\zeta_{\omega_k i}^*$$

and for $i = 1, \ldots, m$

$$G_{\omega_k i}(s) = I - \eta_{\omega_k i}\frac{2\text{Re}(z_i)}{z_i - j\omega_k}\frac{s - j\omega_k}{z_i^* + s}\eta_{\omega_k i}^*.$$

The factorizations can be chosen such that for all $k, l = 0, \pm 1, \ldots, \pm n$

$$\begin{aligned} \zeta_{\omega_k i} &= L_{\omega_l 1}(j\omega_k)\ldots L_{\omega_l i-1}(j\omega_k)\zeta_{\omega_l i}, && i = 1, \ldots, d\\ \eta_{\omega_k i} &= L_{\omega_l 1}(j\omega_k)\ldots L_{\omega_l d}(j\omega_k)\\ &\quad \times G_{\omega_l 1}(j\omega_k)\ldots G_{\omega_l i}(j\omega_k)\eta_{\omega_l i}, && i = 1, \ldots, m. \end{aligned}$$

The proof of this lemma, in a constructive way, is similar to that of Lemma 1 and is omitted.

Now, again we consider the setup shown in Fig. 1, with reference signal $r(t)$ given in (1) and the performance limits $J_{\text{opt}}(v)$ and E_{opt} defined in (3) and (5). Assume that Assumption 1 holds. Before stating the result, we note that when $l = k$, the fraction $(e^{j(\omega_l - \omega_k)\tau_i} - 1)/(j(\omega_l - \omega_k))$ should be interpreted as τ_i, the limit of the fraction as ω_l goes to ω_k.

Theorem 3: Let $G(s)$ be a system with factorizations satisfying Lemma 2. Then

$$\begin{aligned} &J_{\text{opt}}(v)\\ &\quad = \sum_{i=1}^{d}\sum_{k=-n}^{n}\sum_{l=-n}^{n} \frac{e^{j(\omega_l - \omega_k)\tau_i} - 1}{j(\omega_l - \omega_k)} \langle v_k, \zeta_{\omega_k i}\rangle \langle \zeta_{\omega_l i}, v_l\rangle\\ &\qquad + \sum_{i=1}^{m}\sum_{k=-n}^{n}\sum_{l=-n}^{n} \frac{2\text{Re}(z_i)\langle v_k, \eta_{\omega_k i}\rangle\langle \eta_{\omega_l i}, v_l\rangle}{(z_i^* + j\omega_k)(z_i - j\omega_l)} \end{aligned} \tag{18}$$

and

$$E_{\text{opt}} = \sum_{i=1}^{d}(2n+1)\tau_i + 2\sum_{i=1}^{m}\sum_{k=-n}^{n}\frac{1}{z_i - j\omega_k}. \tag{19}$$

The proof of this theorem is given in Appendix II.

V. Conclusion

In this paper, we have accomplished the following.

1) A formula is obtained for the best tracking performance when the reference is a given linear combination of step and sinusoidal signals. This is given in Theorem 1. This formula clearly reveals the role that each nonminimum phase zero, as well as its corresponding frequency-dependent directions, plays toward the performance limitation.
2) A formula is obtained for the best average tracking performance over all references with the same frequency components. This is given in Theorem 2.
3) The formulas are extended beyond FDLTI systems to systems with time delays. This is done in Theorem 3.

In our derivation, great emphasis has been placed on the simplicity and the elegance of the formulas obtained. We believe that these results are significant in the further understanding of linear system structures and their effects on the best achievable performance by feedback control.

We have used 2DOF controllers in our study of tracking performance limitations in this paper. Since such controllers are the most general controllers with given plant measurement and reference information, the performance limits obtained herein are the most fundamental regardless of what controller structure may be used. A pleasant consequence of using 2DOF controllers is that the performance limits only depend on the nonminimum phase zeros, together with their directional properties, but not on the poles and other zeros. One may also notice that the tracking performance when using 2DOF controllers depends on only one degree of freedom among the two available. In other words, the other degree of freedom in the controller is completely irrelevant as far as the tracking error is concerned. This gives us an opportunity to use this extra degree of freedom to achieve other performance specifications, such as disturbance rejection and robustness. We are currently trying to propose a meaningful performance specification which requires the proper utilization of both degrees of freedom in the controller and will then study the limitation in achieving such a performance specification.

In the setup of this paper, we assumed that the controller has full information of the reference. What will happen if the full information of the reference is unavailable, in particular if only the value of the reference is available, to the controller? Recently, we have shown that in this case the best achievable performance will suffer deterioration compared to the performance limitations reported in this paper. In some special cases, the increment in the minimum achievable cost due to the partial information of the reference can be exactly characterized, in a rather simple way, in terms of the nonminimum phase zeros and the reference frequencies. Such results will be reported in a follow-up paper.

Another possible extension is to the case when the controller has previewed information of the reference, which occurs in many practical tracking problems. It is easy to conclude that the preview can help to reduce the performance limitation in general, but a simply and exact characterization on the amount of reduction seems technically difficult.

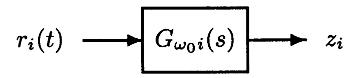

Fig. 3. Matrix Blaschke factor.

APPENDIX I
PROOF OF THE THEOREM 1

We start with a system shown in Fig. 3. Here, $G_{\omega_0 i}(s)$ is a matrix Blaschke factor of the form

$$G_{\omega_0 i}(s) = I - \eta_{\omega_0 i} \frac{2\,\mathrm{Re}(z_i)}{z_i - j\omega_0} \frac{s - j\omega_0}{z_i^* + s} \eta_{\omega_0 i}^* = U_{\omega_0 i} \begin{bmatrix} \frac{z_i^* + j\omega_0}{z_i - j\omega_0} \frac{z_i - s}{z_i^* + s} & 0 \\ 0 & I \end{bmatrix} U_{\omega_0 i}^*. \quad (20)$$

The input and output of the system are $r_i(t)$ and $z_i(t)$ respectively. Let us first consider the following problem. Given a reference signal

$$r_{i-1}(t) = \sum_{k=-n}^{n} a_{(i-1)k} e^{j\omega_k t}$$

which is parameterized by the vector shown in the equation at the bottom of the page, find a bounded input $r_i(t)$ to minimize

$$J_i(a_{i-1}) = \int_0^\infty \|r_{i-1}(t) - z_i(t)\|_2^2 \, dt.$$

Applying the Parseval's identity and denoting the Laplace transform of $r_i(t)$ by $R_i(s)$, we have

$$\begin{aligned} J_i(a_{i-1}) &= \left\| \sum_{k=-n}^{n} \frac{a_{(i-1)k}}{s - j\omega_k} - G_{\omega_0 i}(s) R_i(s) \right\|_2^2 \\ &= \left\| G_{\omega_0 i}^{-1}(s) \sum_{k=-n}^{n} \frac{a_{(i-1)k}}{s - j\omega_k} - R_i(s) \right\|_2^2 \\ &= \left\| \eta_{\omega_0 i} \frac{z_i - j\omega_0}{z_i^* + j\omega_0} \frac{2\mathrm{Re}(z_i)}{z_i - s} \eta_{\omega_0 i}^* \sum_{k=-n}^{n} \frac{a_{(i-1)k}}{z_i - j\omega_k} \right. \\ &\quad \left. + \sum_{k=-n}^{n} \frac{G_{\omega_0 i}^{-1}(j\omega_k) a_{(i-1)k}}{s - j\omega_k} - R_i(s) \right\|_2^2. \end{aligned}$$

Obviously, the optimal $R_i(s)$ which minimizes $J_i(a_{i-1})$ is

$$R_i(s) = \sum_{k=-n}^{n} \frac{G_{\omega_0 i}^{-1}(j\omega_k) a_{(i-1)k}}{s - j\omega_k}$$

and the minimum value of $J_i(a_{i-1})$ is

$$\begin{aligned} J_{i,\mathrm{opt}}(a_{i-1}) &= \left\| \eta_{\omega_0 i} \frac{z_i - j\omega_0}{z_i^* + j\omega_0} \frac{2\,\mathrm{Re}(z_i)}{z_i - s} \eta_{\omega_0 i}^* \sum_{k=-n}^{n} \frac{a_{(i-1)k}}{z_i - j\omega_k} \right\|_2^2 \\ &= 2\,\mathrm{Re}(z_i) \left| \sum_{k=-n}^{n} \frac{\langle \eta_{\omega_0 i}, a_{(i-1)k} \rangle}{z_i - j\omega_k} \right|^2. \quad (21) \end{aligned}$$

$$a_{i-1} = \begin{bmatrix} a_{(i-1)(-n)}^* & \cdots & a_{(i-1)(-1)}^* & a_{(i-1)0}^* & a_{(i-1)1}^* & \cdots & a_{(i-1)n}^* \end{bmatrix}^*$$

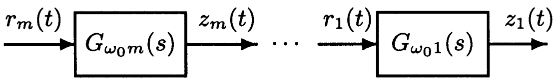

Fig. 4. Matrix Blaschke product.

Next, we consider a matrix Blaschke product $G_{\omega_0 1}(s)\dots G_{\omega_0 m}(s)$ shown in Fig. 4. The signals $r_i(t)$ and $z_i(t), i = 1,\dots,m$, are the inputs and outputs of $G_{\omega_0 i}(s), i = 1,\dots,m$, respectively. Suppose that a reference signal $r_0(t) = \sum_{k=-n}^{n} a_{0k} e^{j\omega_k t}$ is given and we wish to find a bounded input $r_m(t)$ so that

$$J(a_0) = \int_0^\infty \|r_0(t) - z_1(t)\|_2^2 \, dt$$

is minimized. If we denote the Laplace transform of $r_m(t)$ by $R_m(s)$, then $J(a_0)$ can be rewritten as

$$\begin{aligned} J(a_0) &= \left\| \sum_{k=-n}^{n} \frac{a_{0k}}{s - j\omega_k} - G_{\omega_0 1}(s)\, G_{\omega_0 2}(s) \dots \right. \\ &\qquad \left. G_{\omega_0 m}(s) R_m(s) \right\|_2^2 \\ &= \left\| G_{\omega_0 1}^{-1}(s) \sum_{k=-n}^{n} \frac{a_{0k}}{s - j\omega_k} - G_{\omega_0 2}(s) \dots \right. \\ &\qquad \left. G_{\omega_0 m}(s) R_m(s) \right\|_2^2 \\ &= \left\| \eta_{\omega_0 1} \frac{z_1 - j\omega_0}{z_1^* + j\omega_0} \frac{2\mathrm{Re}(z_1)}{z_1 - s} \eta_{\omega_0 1}^* \sum_{k=-n}^{n} \frac{a_{0k}}{z_i - j\omega_k} \right. \\ &\qquad + \sum_{k=-n}^{n} \frac{a_{1k}}{s - j\omega_k} - G_{\omega_0 2}(s) \dots \\ &\qquad \left. \times\, G_{\omega_0 m}(s) R_m(s) \right\|_2^2 \end{aligned}$$

where $a_{1k} = G_{\omega_0 1}^{-1}(j\omega_k) a_{0k}$ and $\sum_{k=-n}^{n} (a_{1k}/(s - j\omega_k))$ is exactly $R_1(s)$ which minimizes $J_1 = \|R_0(s) - G_{\omega_0 1} R_1(s)\|_2^2$. It follows from the orthogonality that

$$\begin{aligned} J(a_0) &= \left\| \eta_{\omega_0 1} \frac{z_1 - j\omega_0}{z_1^* + j\omega_0} \frac{2\mathrm{Re}(z_1)}{z_1 - s} \eta_{\omega_0 1}^* \sum_{k=-n}^{n} \frac{a_{0k}}{z_i - j\omega_k} \right\|_2^2 \\ &\qquad + \left\| \sum_{k=-n}^{n} \frac{a_{1k}}{s - j\omega_k} - G_{\omega_0 2}(s) \dots \right. \\ &\qquad \left. G_{\omega_0 m}(s) R_m(s) \right\|_2^2 \\ &= J_{1,\mathrm{opt}}(a_0) + \left\| \sum_{k=-n}^{n} \frac{a_{1k}}{s - j\omega_k} - G_{\omega_0 2}(s) \dots \right. \\ &\qquad \left. G_{\omega_0 m}(s) R_m(s) \right\|_2^2 . \end{aligned}$$

By repeatedly applying this procedure, we get

$$\begin{aligned} J_{\mathrm{opt}}(a_0) &= \sum_{i=1}^{m} J_{i,\mathrm{opt}}(a_{i-1}) \\ &= \sum_{i=1}^{m} 2\,\mathrm{Re}(z_i) \left| \sum_{k=-n}^{n} \frac{\langle \eta_{\omega_0 i}, a_{(i-1)k} \rangle}{z_i - j\omega_k} \right|^2 \end{aligned} \tag{22}$$

where $a_{ik} = G_{\omega_0 i}^{-1}(j\omega_k) a_{(i-1)k}, i = 1,\dots,m$, and the optimal $r_m(t)$ is given by

$$R_m(s) = \sum_{k=-n}^{n} \frac{a_{mk}}{s - j\omega_k}. \tag{23}$$

The procedure in deriving (22) shows that the problem of finding an optimal input to minimize the tracking error of a Blaschke product can be decomposed into a series of such problems for its factors. The reference signal for a particular factor is the optimal input of the subsequent factor. The synthesis of the optimal input is carried out in an opposite direction to that of the signal flow.

Now, let us consider our original tracking problem as shown in Fig. 1. Let $G(s) = N(s)M^{-1}(s)$ be a coprime factorization. Using the parametrization of all stabilizing 2DOF controllers [19], we find that, under Assumption 1, all possible transfer functions from $v(t)$ to $z(t)$ are given by $N(s)Q(s)$, where $Q(s)$ is an arbitrary transfer function in $\mathcal{H}_\infty$. Let us denote the Laplace transform of $r(t)$ by $R(s)$ and that of $v(t)$ by $V(s)$. Then, the integral square error (2) becomes

$$J(v) = \|R(s) - N(s)Q(s)V(s)\|_2^2.$$

Notice that $N(s)$ is stable and its nonminimum phase zeros are the same as those of $G(s)$. If $G(s)$ is factored as

$$G(s) = G_{\omega_0 1}(s) \dots G_{\omega_0 m}(s) G_{\omega_0 0}(s)$$

where $G_{\omega_0 i}(s)$ is a Blaschke factor of the form of (20) and $G_{\omega_0 0}(s)$ is minimum phase. Then $N(s)$ has the inner–outer factorization

$$\begin{aligned} N(s) &= G_{\mathrm{in}}(s) N_{\mathrm{out}}(s) \\ &= [G_{\omega_0 1}(s) \dots G_{\omega_0 m}(s)][G_{\omega_0 0}(s) M(s)]. \end{aligned}$$

The tracking performance $J(v)$ can be rewritten as

$$\begin{aligned} J(v) &= \left\| \sum_{k=-n}^{n} \frac{v_k}{s - j\omega_k} - G_{\mathrm{in}}(s) N_{\mathrm{out}}(s) Q(s) V(s) \right\|_2^2 \\ &= \left\| G_{\mathrm{in}}^{-1}(s) \sum_{k=-n}^{n} \frac{v_k}{s - j\omega_k} - N_{\mathrm{out}}(s) Q(s) V(s) \right\|_2^2 \\ &= \left\| \left[G_{\mathrm{in}}^{-1}(s) \sum_{k=-n}^{n} \frac{v_k}{s - j\omega_k} - \sum_{k=-n}^{n} \frac{G_{\mathrm{in}}^{-1}(j\omega_k) v_k}{s - j\omega_k} \right] \right. \\ &\qquad \left. + \left[\sum_{k=-n}^{n} \frac{G_{in}^{-1}(j\omega_k) v_k}{s - j\omega_k} - N_{\mathrm{out}}(s) Q(s) V(s) \right] \right\|_2^2 . \end{aligned}$$

It is easy to see that

$$G_{\mathrm{in}}^{-1}(s) \sum_{k=-n}^{n} \frac{v_k}{s - j\omega_k} - \sum_{k=-n}^{n} \frac{G_{\mathrm{in}}^{-1}(j\omega_k) v_k}{s - j\omega_k} \in \mathcal{H}_2^\perp$$

and

$$\sum_{k=-n}^{n} \frac{G_{\rm in}^{-1}(j\omega_k)v_k}{s-j\omega_k} - N_{\rm out}(s)Q(s)V(s)$$

can be made to belong to $\mathcal{H}_2$ by properly choosing $Q(s)$. It then follows that

$$J(v) = \left\| G_{\rm in}^{-1}(s)\sum_{k=-n}^{n}\frac{v_k}{s-j\omega_k} - \sum_{k=-n}^{n}\frac{G_{\rm in}^{-1}(j\omega_k)v_k}{s-j\omega_k}\right\|_2^2 + \left\|\sum_{k=-n}^{n}\frac{G_{\rm in}^{-1}(j\omega_k)v_k}{s-j\omega_k} - N_{\rm out}(s)Q(s)V(s)\right\|_2^2. \quad (24)$$

Without loss of generality, we can assume

$$V(s) = \begin{bmatrix} \frac{v_{-n}^*}{s+j\omega_{-n}} & \cdots & \frac{v_0^*}{s+j\omega_0} & \cdots & \frac{v_n^*}{s+j\omega_n}\end{bmatrix}^*.$$

Partition $Q(s)$ consistently as

$$Q(s) = [Q_{-n}(s) \quad \cdots \quad Q_0(s) \quad \cdots \quad Q_n(s)].$$

Then, we have

$$\begin{aligned}&\left\|\sum_{k=-n}^{n}\frac{G_{\rm in}^{-1}(j\omega_k)v_k}{s-j\omega_k} - N_{\rm out}(s)Q(s)V(s)\right\|_2^2 \\ &= \Bigg\| \Bigg\{ \begin{bmatrix}\frac{G_{\rm in}^{-1}(j\omega_{-n})}{s-j\omega_{-n}} & \cdots & \frac{G_{\rm in}^{-1}(j\omega_n)}{s-j\omega_n}\end{bmatrix} \\ &\quad - N_{\rm out}(s)\begin{bmatrix}\frac{Q_{-n}(s)}{s-j\omega_{-n}} & \cdots & \frac{Q_n(s)}{s-j\omega_n}\end{bmatrix}\Bigg\} \\ &\quad \times \begin{bmatrix} v_{-n} \\ \vdots \\ v_n\end{bmatrix}\Bigg\|_2^2.\end{aligned}$$

Let

$$Q_k(s) = N^\dagger(j\omega_k) + \frac{s-j\omega_k}{s+1}\tilde{Q}_k(s)$$

where $N^\dagger(j\omega_k)$ is a right inverse of $N(j\omega_k)$. Also, denote

$$\tilde{N}_k(s) = \frac{G_{\rm in}^{-1}(j\omega_k)}{s-j\omega_k} - N_{\rm out}(s)\frac{N^\dagger(j\omega_k)}{s-j\omega_k}$$

and notice that it is an $\mathcal{H}_2$ function. Then

$$\begin{aligned}&\Bigg\|\begin{bmatrix}\frac{G_{\rm in}^{-1}(j\omega_{-n})}{s-j\omega_{-n}} & \cdots & \frac{G_{\rm in}^{-1}(j\omega_n)}{s-j\omega_n}\end{bmatrix} \\ &\quad - N_{\rm out}(s)\begin{bmatrix}\frac{Q_{-n}(s)}{s-j\omega_{-n}} & \cdots & \frac{Q_n(s)}{s-j\omega_n}\end{bmatrix}\Bigg\|_2^2 \\ &= \Bigg\|[\tilde{N}_{-n}(s) \quad \cdots \quad \tilde{N}_n(s)] \\ &\quad - N_{\rm out}(s)\frac{1}{s+1}[\tilde{Q}_{-n}(s) \quad \cdots \quad \tilde{Q}_n(s)]\Bigg\|_2^2.\end{aligned}$$

Since $N_{\rm out}(s)(1/(s+1))$ is outer, we can always find $\tilde{Q}_k(s), k = 0, \pm 1, \ldots, \pm n$, such that the above expression is arbitrarily small. This shows that the second term of (24) can be made arbitrarily small by choosing $Q(s)$, independent of v. Consequently

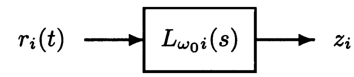

Fig. 5. Delay factor.

$$J_{\rm opt}(v) = \left\|\sum_{k=-n}^{n}\frac{v_k}{s-j\omega_k} - G_{\rm in}(s)\sum_{k=-n}^{n}\frac{G_{\rm in}^{-1}(j\omega_k)v_k}{s-j\omega_k}\right\|_2^2.$$

If we let $v = a_0$, then $\sum_{k=-n}^{n}(G_{\rm in}^{-1}(j\omega_k)v_k)/(s-j\omega_k)$ is exactly the optimal input $R_m(s)$ defined in (23). Therefore

$$J_{\rm opt}(v) = \sum_{i=1}^{m} 2{\rm Re}(z_i)\left|\sum_{k=-n}^{n}\frac{\langle \eta_{\omega_0 i}, a_{(i-1)k}\rangle}{z_i - j\omega_k}\right|^2$$

where $a_0 = v$ and $a_{ik} = G_{\omega_0 i}^{-1}(j\omega_k)a_{(i-1)k}$. Plugging a_i into the expression, we get

$$J_{\rm opt}(v) = \sum_{i=1}^{m} 2{\rm Re}(z_i) \times \left|\sum_{k=-n}^{n}\frac{\left\langle \eta_{\omega_0 i}, G_{\omega_0(i-1)}^{-1}(j\omega_k)\ldots G_{\omega_0 1}^{-1}(j\omega_k)v_k\right\rangle}{z_i - j\omega_k}\right|^2.$$

Finally, Lemma 1 immediately yields

$$J_{\rm opt}(v) = \sum_{i=1}^{m} 2{\rm Re}(z_i)\left|\sum_{k=-n}^{n}\frac{\langle \eta_{\omega_k i}, v_k\rangle}{z_i - j\omega_k}\right|.$$

This completes the proof.

Appendix II
Proof of Theorem 3

We will only sketch the proof since it follows the same idea as that for the delay-free case. Let us first consider the following problem: Given $r_{i-1}(t) = \sum_{k=-n}^{n} a_{(i-1)k}e^{j\omega_k t}$ where $\omega_{-k} = -\omega_k$ and $a_{(i-1)(-k)} = \bar{a}_{(i-1)k}$, choose a bounded $r_i(t)$ to minimize

$$J_i(a_{i-1}) = \int_0^\infty \|r_{i-1}(t) - z_i(t)\|_2^2\, dt$$

where the transfer function between $r_i(t)$ and $z_i(t)$ is given by $L_{\omega_0 i}(s)$ as shown in Fig. 5. The Laplace transform of $r_{i-1}(t)$ is

$$R_{i-1}(s) = \sum_{k=-n}^{n}\frac{a_{(i-1)k}}{s-j\omega_k}.$$

Denote the Laplace transform of $r_i(t)$ by $R_i(s)$. Apparently

$$\begin{aligned} J_i(a_{i-1}) &= \|R_{i-1}(s) - L_{\omega_0 i}(s)R_i(s)\|_2^2 \\ &= \left\|L_{\omega_0 i}^{-1}(s)R_{i-1}(s) - R_i(s)\right\|_2^2 \\ &= \Bigg\|\left[L_{\omega_0 i}^{-1}(s)R_{i-1}(s) - \sum_{k=-n}^{n}\frac{L_{\omega_0 i}^{-1}(j\omega_k)a_{(i-1)k}}{s-j\omega_k}\right] \\ &\quad + \left[\sum_{k=-n}^{n}\frac{L_{\omega_0 i}^{-1}(j\omega_k)a_{(i-1)k}}{s-j\omega_k} - R_i(s)\right]\Bigg\|_2^2.\end{aligned}$$

$$J_{\rm opt}(v) = \sum_{i=1}^{d}\sum_{k=-n}^{n}\sum_{l=-n}^{n}\left[e^{j(\omega_l-\omega_k)\tau_i}-1\right]\frac{\left\langle a_{(i-1)k},\zeta_{\omega_0 1}\right\rangle\left\langle \zeta_{\omega_0 1},a_{(i-1)l}\right\rangle}{j(\omega_l-\omega_k)} + \sum_{i=1}^{d}\sum_{k=-n}^{n}\sum_{l=-n}^{n}\frac{2\,{\rm Re}(z_i)\left\langle b_{(i-1)k},\eta_{\omega_0 i}\right\rangle\left\langle \eta_{\omega_0 i},b_{(i-1)l}\right\rangle}{(z_i^*+j\omega_k)(z_i-j\omega_l)}$$

Using the orthogonality, we see that the optimal $r_i(t)$ is

$$r_i(t) = \sum_{k=-n}^{n} L_{\omega_0 i}^{-1}(j\omega_k)a_{(i-1)k}e^{j\omega_k t}$$

and the optimal $J_i(a_{i-1})$ is

$$\begin{aligned} J_{i,\rm opt}(a_{i-1}) &= \left\|L_{\omega_0 i}^{-1}(s)R_{i-1}(s) - \sum_{k=-n}^{n}\frac{L_{\omega_0 i}^{-1}(j\omega_k)a_{(i-1)k}}{s-j\omega_k}\right\|_2^2 \\ &= \left\|\sum_{k=-n}^{n}\frac{\left[L_{\omega_0 i}^{-1}(s)-L_{\omega_0 i}^{-1}(j\omega_k)\right]a_{(i-1)k}}{s-j\omega_i}\right\|_2^2 \\ &= \left\|\sum_{k=-n}^{n}\frac{\left(e^{\tau_i s}-e^{\tau_i j\omega_k}\right)\left\langle \zeta_{\omega_o i},a_{(i-1)k}\right\rangle}{s-j\omega_i}\right\|_2^2. \end{aligned}$$

Rewriting the previous expression in an inner product form, we have

$$\begin{aligned} J_{i,\rm opt}(a_{i-1}) = \sum_{k=-n}^{n}\sum_{l=-n}^{n}\Bigg\langle & e^{\tau_i s}\left[1-e^{-\tau_i(s-j\omega_k)}\right] \\ &\times\frac{\left\langle \zeta_{\omega_0 1},a_{(i-1)k}\right\rangle}{s-j\omega_k}, e^{\tau_i s}\left[1-e^{-\tau_i(s-j\omega_l)}\right] \\ &\times\frac{\left\langle \zeta_{\omega_0 1},a_{(i-1)l}\right\rangle}{s-j\omega_l}\Bigg\rangle. \end{aligned}$$

The Parseval's identity leads to

$$\begin{aligned} &\Bigg\langle e^{\tau_i s}\left[1-e^{-\tau_i(s-j\omega_k)}\right]\frac{\left\langle \zeta_{\omega_0 1},a_{(i-1)k}\right\rangle}{s-j\omega_k}, \\ &\quad e^{\tau_i s}\left[1-e^{-\tau_i(s-j\omega_l)}\right]\frac{\left\langle \zeta_{\omega_0 1},a_{(i-1)l}\right\rangle}{s-j\omega_l}\Bigg\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left[1-e^{j\tau_i(\omega-\omega_k)}\right]\left\langle a_{(i-1)k},\zeta_{\omega_0 1}\right\rangle}{-j(\omega-\omega_k)} \\ &\quad\times\frac{\left[1-e^{-j\tau_i(\omega-\omega_l)}\right]\left\langle \zeta_{\omega_0 1},a_{(l-1)k}\right\rangle}{j(\omega-\omega_l)}\,d\omega \\ &= \int_0^{\tau_i} e^{-j\omega_k t}e^{j\omega_l t}\left\langle a_{(i-1)k},\zeta_{\omega_0 1}\right\rangle\left\langle \zeta_{\omega_0 1},a_{(i-1)l}\right\rangle dt \\ &= \left[e^{j(\omega_l-\omega_k)\tau_1}-1\right]\frac{\left\langle a_{(i-1)k},\zeta_{\omega_0 1}\right\rangle\left\langle \zeta_{\omega_0 1},a_{(i-1)l}\right\rangle}{j(\omega_l-\omega_k)}. \end{aligned}$$

By using the same idea as in the delay-free case, we can show that the equation at the top of the page holds, where $a_i = [a_{(i-1)(-n)}^*,\ldots,a_{(i-1)n}^*]^*$ is defined as $a_0 = v$ and $a_{ik} = L_{\omega_0 i}^{-1}(j\omega_k)a_{(i-1)k}$, and b_i is defined as $b_0 = a_d$ and $b_{ik} = G_{\omega_0 i}^{-1}(j\omega_k)b_{(i-1)k}$. In this expression, the first term gives the performance limit due to the delay factors and the second term gives that due to the nonminimum phase zeros. Plugging a_i and b_i into the previous expression and, using Lemma 2, we then obtain

$$\begin{aligned} J_{\rm opt}(v) = \sum_{i=1}^{d}\sum_{k=-n}^{n}\sum_{l=-n}^{n}&\left[e^{j(\omega_l-\omega_k)\tau_i}-1\right] \\ &\times\frac{\left\langle v_k,\zeta_{\omega_k 1}\right\rangle\left\langle \zeta_{\omega_k 1},v_l\right\rangle}{j(\omega_l-\omega_k)} \\ &+\sum_{i=1}^{d}\sum_{k=-n}^{n}\sum_{l=-n}^{n}\frac{2\,{\rm Re}(z_i)\left\langle v_k,\eta_{\omega_k i}\right\rangle\left\langle \eta_{\omega_l i},v_l\right\rangle}{(z_i^*+j\omega_k)(z_i-j\omega_l)}. \end{aligned}$$

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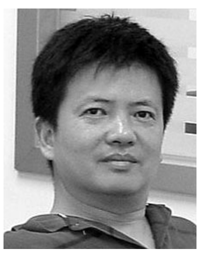

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