Fundamental Performance Limitations in Tracking Sinusoidal Signals[1]

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Abstract

This paper attempts to give a thorough treatment of the performance limitation of a linear time invariant multivariable system in tracking a reference signal which is a linear combination of a step signal and several sinusoids with different frequencies. The tracking performance is measured by an integral square error between the output of the plant and the reference signal. Our purpose is to find the fundamental limitation for the attainable tracking performance, under any control structure and parameters, in terms of the characteristics and structural parameters of the given plant, as well as those of the reference signal under consideration. It is shown that this fundamental limitation depends on the interaction between the reference signal and the nonminimum phase zeros of the plant and their frequency-dependent directional information. The main results of this paper are based on the assumption that the controller accesses the full information of the reference. However, when the full information of the reference is unavailable, we also obtain a simple performance limit, for an important special case, which clearly shows the extra cost one has to pay due to the information restriction.

Keywords: Linear system, Performance limitation, Optimal Control, Tracking, Nonminimum phase.

1 Introduction

This paper considers the performance limitations of an LTI multivariable feedback control system in tracking a reference that is a linear combination of a step and several sinusoids of various frequencies. The initial setup is shown in Figure 1. Here $P(s)$ is the transfer matrix of a given plant whose measurement $y(t)$ and output $z(t)$ may not be the same, $K(s)$ is the transfer matrix of a two degree of freedom (2DOF) controller which is to be designed, $S(s)$ is the exosystem driven by an impulse which generates the reference. Initially, we assume that the controller have full information of the reference in the sense that it takes $v(t)$, the state of the exosystem $S(s)$, in addition to the measurement $y(t)$ of the plant, as its inputs. Whether or not the measurement $y(t)$ contains the full information of the plant, i.e., the state of the plant, is not important. The tracking problem is to design a controller $K(s)$ so that the closed loop system is internally stabilized and the plant output $z(t)$ asymptotically tracks a reference signal $r(t)$ of the form:

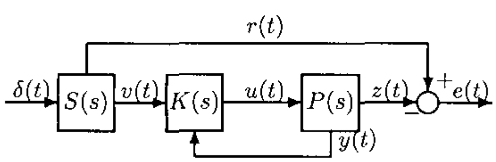

Figure 1: A two-parameter control structure with reference full information

$$r(t) = \sum_{k=-n}^{n} v_k e^{j\omega_k t} \qquad (1)$$

where ω_k, $k = 0, \pm 1, \ldots, \pm n$, are distinct real frequencies satisfying $\omega_{-k} = -\omega_k$ and v_k, $k = 0, \pm 1, \ldots, \pm n$, are complex vectors satisfying $v_{-k} = \bar{v}_k$. Implicitly, we have $\omega_0 = 0$ and v_0 is real. The reference defined in such a way is always a real valued signal. We use the vector $v = [v_{-n}^* \ \cdots \ v_{-1}^* \ v_0^* \ v_1^* \cdots \ v_n^*]^*$ to capture the magnitude and phase information of all frequency components of the reference. The transient error is measured by its energy:

$$J(v) = \int_0^\infty \|r(t) - z(t)\|^2 dt = \int_0^\infty \|e(t)\|^2 dt. \qquad (2)$$

The tracking problem has a well-known solution, with well-known numerical methods to design controllers so that $J(v)$ is small. Nevertheless, it is desirable to have a deeper understanding of the smallest tracking error

$$J_{opt}(v) = \inf_K J(v) \qquad (3)$$

obtainable when the controller K is chosen among all possible stabilizing controllers. Such a smallest error then gives a fundamental limit in the transient performance of tracking. In this paper, we achieve this understanding in the form of an explicit, simple, and informative relationship between this fundamental limit and the plant characteristics.

The value $J_{opt}(v)$ of course depends on v. If we are interested in an overall performance measure of the feedback system in tracking all references of the type (1), then we normally turn our attention to an averaged version of the tracking error, averaged over all possible v whose entries have zero mean, are mutually uncorrelated, and have a unit variance. Such an averaged performance measure is given by

$$J = \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\}. \qquad (4)$$

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In this case, the performance limit becomes

$$J_{opt} = \inf_K J. \quad (5)$$

It turns out that the averaged performance limit J_{opt} is simple enough to be presented as follows: Under some minor assumptions,

$$J_{opt} = 2\sum_{i=1}^{m}\sum_{k=-n}^{n}\frac{1}{z_i - j\omega_k} \quad (6)$$

where $z_i, i = 1, 2, \ldots, m$, are the nonminimum phase zeros of the transfer function from $u(t)$ to $z(t)$. The v dependent performance limit $J_{opt}(v)$ need more elaborations but also turn out to be simple.

Results of this sort can be traced back for over a decade. For SISO systems and constant references, vector v degenerates to a real scalar. Then the linearity of the plant implies that $J_{opt}(v) = v^2 J_{opt}(1)$ and $J_{opt} = J_{opt}(1)$. It is obtained in [8] that

$$J_{opt}(1) = 2\sum_{i=1}^{m}\frac{1}{z_i}.$$

For multivariable systems and for the case when the reference $r(t)$ is either a constant or a sinusoid with a single frequency ω, it was obtained in [11] that

$$J_{opt} = 2\sum_{i=1}^{m}\frac{1}{z_i} \text{ and } J_{opt} = 2\sum_{i=1}^{m}\left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega}\right),$$

respectively. The study of the performance limit $J_{opt}(v)$ for a fixed reference in the multivariable case started in [3]. It is shown there that the performance limit in this case depends on not only the locations of the nonminimun phase zeros but also their directional information. The study in [3] has since been extended to more general references [4, 5], and discrete time systems [16]. There have also been generalizations to non-right-invertible plant [1], to the cases where the controller has previewed information of the reference [5], where the plant input is subject to saturation [9], and where the tracking performance measure includes the input energy [2], respectively. It has been shown that, consistent with common intuitions, the preview of the reference can reduce the best achievable tracking error and on the other hand any input saturation or any input energy constraint would likely increase the best achievable tracking error. Related issues for nonlinear systems and filtering problems are studied in [14, 12].

Although J_{opt} gives an overall quality measure for the plant as far as tracking is concerned, the reference direction dependent performance limit $J_{opt}(v)$ gives more information and deeper insights. If we know $J_{opt}(v)$ and if the optimal K which minimizes $J(v)$ is independent of v, then J_{opt} can be obtained after simple operations. This is why we adopt the thinking in [3] to place our main emphasis on $J_{opt}(v)$. In the above formulation, the assumption that the state of the exosystem is available to the controller is crucial. This means that not only the reference but also all magnitude and phase information of its frequency components is known to the controller. This may look unrealistic in the first glance, but it does give a limitation more fundamental than any other one based only on the partial information of the reference. It is this assumption that makes it possible to find a uniformly optimal controller K to minimize $J(v)$ for all v. Note that when the reference only contains a constant term, the value of the reference already contains its full information. Therefore in this particular case, whether or not the controller can assess the state of the reference is not an issue.

This paper gives a rather complete picture for the tracking performance limitation problem for general reference signals containing several frequency components. We first give some new insight on linear system structure. We show that each nonminimum phase zero has associated frequency dependent directions. A key technical result in this paper is a relation among directions at different frequencies. Using this result, we derive an expression for $J_{opt}(v)$ which elegantly exhibits the effect of the plant nonminimum phase zeros, as well as the interaction between each frequency component and the directions mentioned above, towards the performance limitations.

What will happen if the full information of the reference is unavailable, in particular if only the value of the reference is available, to the controller? This issue motivates us to consider the performance limitation for the setup shown in Figure 2. Mathematically, it still

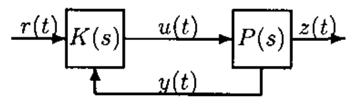

Figure 2: A two-parameter control structure with only reference information

makes sense to consider the v dependent performance limit $J_{opt}(v)$. We will show that in this case the expression of $J_{opt}(v)$ remains the same as that in the full information case. What becomes different is that the optimal controller K is no longer independent of the initial state v of the reference. If such a controller is used, Figure 2 is essentially Figure 1 in disguise from the viewpoint of information flow. We believe that in the case when only the reference is accessable to the controller, it is more meaningful to consider the averaged performance limit J_{opt} defined by (4) and (5). It turns out that deriving a simple expression for J_{opt} is hard for the general reference of the form (1). We will consider instead a special case when $r(t)$ is a scalar signal containing a single sinusoid:

$$r(t) = \bar{\alpha}e^{-j\omega t} + \alpha e^{j\omega t}.$$

With our notation convention, we have $v = [\bar{\alpha} \quad \alpha]'$ here. Under some mild assumptions, we are able to find that

$$J_{opt} = 2\sum_{i=1}^{m}\left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega}\right) + \frac{2}{\omega}\sin^2\left[2\sum_{i=1}^{m}\angle(z_i - j\omega)\right]$$

where $\angle(z_i - j\omega)$ stands for the phase or argument of the complex number $z_i - j\omega$. Comparing this with the

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performance limit in the full reference information case, which is

$$J_{opt} = 2\sum_{i=1}^{m}\left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega}\right),$$

we are able to pinpoint exactly the performance deterioration due to the limited information.

There has been a surge of activities in the study of performance limitations in feedback control. In addition to the type of performance limitations studied in this paper, which focus on system time responses and hence are called time domain performance limitations, there is a whole body of literature on design limitations on system frequency responses, known as frequency domain performance limitations. For the history and the recent progress on frequency domain performance limitations, see [13]. Some intriguing connections have been realized between the time domain limitations and the frequency domain ones [7].

The organization of this paper is as follows. In Section 2, preliminary materials on linear system factorizations are presented. It is shown that a right-invertible system can be factorized as a cascade connection of series of first order inner factors and a minimum phase factor. The factorization is frequency dependent. The inner factors then contain all the information associated to the nonminimum phase zeros. In Section 3, we formally formulate the problems studied and then state and discuss the main result and some of its consequences. The proof of the main result is given in Section 4. Section 5 extends the main results in the previous sections to the case in which the linear system contains delays. Section 6 considered the case when only the reference signal, not the state of the reference signal, is assessable to the controller. Section 7 is the conclusion. Due to space limitation, all proofs are omitted in this conference paper. They can be found in [15].

The notation used throughout this paper is fairly standard. For any complex number, vector and matrix, denote their conjugate, transpose, conjugate transpose, real and imaginary part by $\bar{(\cdot)}$, $(\cdot)'$, $(\cdot)^*$, $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$, respectively. The phase or argument of a nonzero complex number is denoted by $\angle(\cdot)$. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open right and left half plane be denoted by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. $\mathcal{L}_2$ is the standard frequency domain Lebesgue space. $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspaces of $\mathcal{L}_2$ containing functions that are analytical in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. It is well-known that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ constitute orthogonal complements in $\mathcal{L}_2$. $\mathcal{RH}_\infty$ is the set of all stable, rational transfer matrices. Finally, the inner product between two complex vectors u, v is defined by $\langle u, v\rangle := u^*v$.

2 Preliminaries

Let $G(s)$ be a real rational matrix representing the transfer function of a continuous time finite-dimensional, linear time invariant (FDLTI) system. Let us assume that $G(s)$ is right invertible. Its poles and zeros, including multiplicity, are defined according to its Smith-McMillan form. $G(s)$ is said to be minimum phase if all its zeros have nonpositive real parts; otherwise it is said to be nonminimum phase.

Let $G(s) = N(s)M^{-1}(s)$, where $M(s), N(s) \in \mathcal{RH}_\infty$, be a right coprime factorization of $G(s)$. Let $z \in \mathbb{C}$ be a nonminimum phase zero of $G(s)$. Then z is also a nonminimum phase zero of $N(s)$ and there exists a unit vector η such that $\eta^* N(z) = 0$. We call the vector η a (left or output) zero vector of $G(s)$ corresponding to the nonminimum phase zero z.

Let us now order the nonminimum phase zeros of $G(s)$ (or $N(s)$ equivalently) as $z_1, z_2, \ldots, z_m$. Assume that each pair of complex conjugate zeros are ordered in adjacent order. Let us also fix a frequency $\omega_k \in \mathbb{R}$. We first find a unit zero vector $\eta_{\omega_k 1}$ of $G(s)$ corresponding to z_1 and define

$$G_{\omega_k 1}(s) = U_{\omega_k 1}\begin{bmatrix} \frac{z_1^* + j\omega_k}{z_1 - j\omega_k}\frac{z_1 - s}{z_1^* + s} & 0 \\ 0 & I\end{bmatrix} U_{\omega_k 1}^*$$

where $U_{\omega_{k1}}$ is a unitary matrix with the first column equal to $\eta_{\omega_k 1}$. Here $G_{\omega_k 1}(s)$ is so constructed that it is inner, has the only zero at z_1 with $\eta_{\omega_k 1}$ as a zero vector corresponding to z_1, and $G_{\omega_k 1}(j\omega_k) = I$. Since $G_{\omega_k 1}(s)$ is a generalization of the standard scalar Blaschke factor, we call it a matrix Blaschke factor at the frequency w_k and $\eta_{\omega_k 1}$ a corresponding Blaschke vector. Also notice that the choice of other columns in $U_{\omega_k 1}$ is immaterial. Now $G_{\omega_k 1}^{-1}(s)G(s)$ has zeros $z_2, z_3, \ldots, z_m$. Find a zero vector $\eta_{\omega_k 2}$ of $G_{\omega_k 1}^{-1}(s)G(s)$ corresponding to z_2 and define $G_{\omega_k 2}(s)$ in the same form as that of $G_{\omega_k 1}(s)$. Then $G_{\omega_k 2}^{-1}(s)G_{\omega_k 1}^{-1}(s)G(s)$ has zeros $z_3, z_4, \ldots, z_m$. Continue this process until Blaschke vectors $\eta_{\omega_k 1}, \ldots, \eta_{\omega_k m}$ and Blaschke factors $G_{\omega_k 1}(s), \ldots, G_{\omega_k m}(s)$ are all obtained. This procedure shows that $G(s)$ can be factorized as

$$G(s) = G_{\omega_k 1}(s)\cdots G_{\omega_k m}(s) G_{\omega_k 0}(s) \qquad (7)$$

where

$$G_{\omega_k i}(s) = U_{\omega_k i}\begin{bmatrix} \frac{z_i^* + j\omega_k}{z_i - j\omega_k}\frac{z_i - s}{z_i^* + s} & 0 \\ 0 & I\end{bmatrix} U_{\omega_k i}^* \qquad (8)$$

and $G_{\omega_k 0}(s)$ has no nonminimum phase zero. We call this factorization a cascade factorization at frequency ω_k, which is shown schematically in Figure 3. In this factorization, each Blaschke vector and Blaschke factor correspond to one nonminimum phase zero. Keep in mind that these vectors and factors depend on the order of the nonminimum zeros, as well as on the frequency ω_k. The product

$$G_{\omega_k 1}(s)\cdots G_{\omega_k m}(s)$$

is called a matrix Blaschke product. One should note

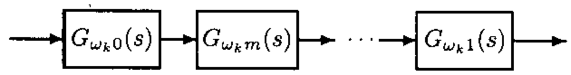

Figure 3: Cascade factorization

that even when the order of $z_1, z_2, \ldots, z_m$ is fixed, the factorization at the frequency ω_k is not unique since $\eta_{\omega_k i}$ is not uniquely determined. Furthermore, if we have $2n+1$ different frequencies $\omega_k, k = 0, \pm 1, \ldots, \pm n$, then the factorizations at different frequencies are in general different. Nevertheless, they can be intimately related if we make the choices properly. For example, it is easy to see from the above construction that $\eta_{\omega_k 1}$, the first Blaschke vector, can be chosen independent of ω_k. The following lemma provides such relations and is the key technical vehicle that leads to the main result of this paper.

Lemma 2.1 *Suppose that the order of $z_1, z_2, \ldots, z_m$ is fixed. Also suppose that we are given $2n+1$ different frequencies $\omega_k, k = 0, \pm 1, \ldots, \pm n$. Then the $2n+1$ cascade factorizations (7) can be chosen so that for all $k, l = 0, \pm 1, \cdots, \pm n$ and $i = 1, 2, \ldots, m$,*

$$\eta_{\omega_k i} = G_{\omega_l 1}(j\omega_k) G_{\omega_l 2}(j\omega_k) \cdots G_{\omega_l i-1}(j\omega_k) \eta_{\omega_l i}. \quad (9)$$

One may wonder what these Balschke vectors look like when $G(s)$ is SISO. In this case, proper choices lead to

$$\eta_{\omega_k i} = \frac{z_1^*}{z_1} \frac{z_1 - j\omega_k}{z_1^* + j\omega_k} \cdots \frac{z_{i-1}^*}{z_{i-1}} \frac{z_{i-1} - j\omega_k}{z_{i-1}^* + j\omega_k} \quad (10)$$

for $k = 0, \pm 1, \cdots, \pm n, i = 1, 2, \ldots, m$.

3 The Main Result

Let us go back to the setup shown in Figure 1. The measurement output $y(t)$ of the plant might be different from the tracking output $z(t)$. We denote the transfer function from $u(t)$ to $z(t)$ by $G(s)$ and that from $u(t)$ to $y(t)$ by $H(s)$, i.e., $P(s) = \begin{bmatrix} G(s) \\ H(s) \end{bmatrix}$. In order for the tracking problem to be meaningful and solvable, we make the following assumptions throughout the paper.

Assumption 3.1

1. *$P(s)$, $G(s)$ and $H(s)$ have the same unstable poles.*
2. *$G(s)$ has no zero at $j\omega_k, k = 0, \pm 1, \cdots, \pm n$.*

The first item in the assumption means that the measurement can be used to stabilize the system and at the same time does not introduce any additional unstable modes. A more precise way of stating this is that if $P(s) = \begin{bmatrix} N(s) \\ L(s) \end{bmatrix} M^{-1}(s)$ is a coprime factorization, then we assume that $N(s)M^{-1}(s)$ and $L(s)M^{-1}(s)$ are also coprime factorizations. The second item is necessary for the solvability of the tracking problem.

We now state our main result.

Theorem 3.1 *Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$ with corresponding Blaschke vectors $\eta_{\omega_k 1}, \ldots, \eta_{\omega_k m}$, $k = 0, \pm 1, \ldots, \pm n$, satisfying Lemma 2.1. Then*

$$J_{opt}(v) = \sum_{i=1}^{m} 2\mathrm{Re}(z_i) \left| \sum_{k=-n}^{n} \frac{\langle \eta_{\omega_k i}, v_k \rangle}{z_i - j\omega_k} \right|^2$$
$$= \sum_{i=1}^{m} \sum_{k=-n}^{n} \sum_{l=-n}^{n} \frac{2\mathrm{Re}(z_i) \langle v_k, \eta_{\omega_k i} \rangle \langle \eta_{\omega_l i}, v_l \rangle}{(z_i^* + j\omega_k)(z_i - j\omega_l)}. \quad (11)$$

This formula shows that each nonminimum phase zero contributes additively to the performance limit. However the contribution of each frequency components of the reference enters the performance limit in a quadratic form and the cross coupling of pairs of frequencies appears in the performance limit. It also shows that generically, perfect tracking is impossible when the plant is nonminimum phase. However, if it happens that the vector v is orthogonal to the vectors

$$\eta_i = \left[\frac{\eta_{\omega_{-n} i}^*}{z_i - j\omega_{-n}} \cdots \frac{\eta_{\omega_0 i}^*}{z_i} \cdots \frac{\eta_{\omega_n i}^*}{z_i - j\omega_n} \right]^*, \quad i = 1, 2, \ldots, m,$$

then perfect tracking can be achieved. Here v captures the magnitude and phase information of the reference and η_i captures the property of the plant at the nonminimum phase zero z_i. This orthogonality may happen in two ways. One is over the output channels: v_k is orthogonal to $\eta_{\omega_k i}$ for all $i = 1, 2, \ldots, m$, $k = 0, \pm 1, \ldots, \pm n$. This can only occur for multivariable systems. The other is over the frequencies: the orthogonality over output channels does not occur but v and η_i are orthogonal due to some special alignment of the magnitude and phase of the reference. This can happen even for the SISO case. For example, in the case when $m = 1$ and $G(s)$ is SISO, if v_k happens to make $\frac{v_k}{z_1 - j\omega_k}$ imaginary for all $k = 0, \pm 1, \ldots, \pm n$, then the performance limit is zero.

In the case when $n = 0$, i.e., the reference only has a step component, we get

$$J_{opt}(v) = \sum_{i=1}^{m} \frac{2\,\mathrm{Re}(z_i)}{|z_i|^2} |\langle \eta_{0i}, v \rangle|^2.$$

This is exactly the formula given in [3].

In the case when the system $G(s)$ is SISO, the performance limit becomes

$$J_{opt}(v) = \sum_{i=1}^{m} \sum_{k=-n}^{n} \sum_{l=-n}^{n} 2\,\mathrm{Re}(z_i) \frac{\eta_{\omega_k i} \eta_{\omega_l i}^* v_k^* v_l}{(z_i^* + j\omega_k)(z_i - j\omega_l)} \quad (12)$$

where $\eta_{\omega_k i}, i = 1, \ldots, m$, are scalars with unit modulus given by (10).

From Assumption 3.1, it can be seen that a controller or a sequence of controllers, independent of v, can be found to attain the performance limit $J_{opt}(v)$ (see [15]). Therefore

$$\begin{aligned} J_{opt} &= \inf_K \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} \quad (13) \\ &= \boldsymbol{E}\{\inf_K J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} \quad (14) \\ &= \sum_{i=1}^{m} \sum_{k=-n}^{n} \sum_{l=-n}^{n} 2\mathrm{Re}(z_i) \frac{\eta_{\omega_l i}^* \boldsymbol{E}(v_l v_k^*) \eta_{\omega_k i}}{(z_i^* + j\omega_k)(z_i - j\omega_l)} \\ &= \sum_{i=1}^{m} \sum_{k=-n}^{n} 2\mathrm{Re}(z_i) \frac{\eta_{\omega_k i}^* \eta_{\omega_k i}}{(z_i^* + j\omega_k)(z_i - j\omega_k)} \\ &= \sum_{i=1}^{m} \sum_{k=-n}^{n} \frac{2\mathrm{Re}(z_i)}{|z_i - j\omega_k|^2}. \end{aligned}$$

This immediately leads to the following theorem.

Theorem 3.2 *Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$. Then*

$$J_{opt} = \sum_{i=1}^{m} \sum_{k=-n}^{n} \frac{2\,\mathrm{Re}(z_i)}{|z_i - j\omega_k|^2} = 2 \sum_{i=1}^{m} \sum_{k=-n}^{n} \frac{1}{z_i - j\omega_k}.$$

From Theorem 3.2 it is seen that the average performance limit has a strikingly simple form; it is the simple sum of the contributions of all nonminimum phase zeros at all frequencies; each of such contributions is the reciprocal of the distance between a nonminimum phase zero and a mode of the reference.

4 Performance Limitation for Systems with Time Delays

In this section, we generalize the previous result a bit further to systems with time delays. We assume that $G(s)$ admits a factorization of the form

$$G(s) = L_1(s) \cdots L_d(s) G_1(s) \cdots G_m(s) G_0(s). \quad (15)$$

Here $L_i(s)$ is assumed to have the form

$$L_i(s) = I - \zeta_i(1 - e^{-\tau_i s})\zeta_i^* = V_i \begin{bmatrix} e^{-\tau_i s} & 0 \\ 0 & I \end{bmatrix} V_i^*$$

where ζ_i is a real unit vector characterizing the directional information of this delay and V_i is a real orthogonal matrix whose first column is ζ_i. $G_i(s)$ is assumed to be a Blaschke factor with zero z_i. The last factor $G_0(s)$, not necessarily rational, is assumed to have a coprime factorization $N_0(s)M_0^{-1}(s)$ with an outer $N_0(s)$. It is easy to see that a multivariable FDLTI system with independent delays in all output channels can be written in the form of (15). However, at this moment, it is not clear what is the general class of transfer matrices that admit this type of factorizations. It is not even clear how we can write a multivariable system with independent time delays in the input channels in the form of (15). It would be interesting to clarify these issues.

For systems given in the form of (15), we have a generalized version of Lemma 2.1.

Lemma 4.1 *Suppose that we are given $2n+1$ different frequencies $\omega_k, k = 0, \pm 1, \ldots, \pm n$. Then there exist $2n+1$ cascade factorizations*

$$G(s) = L_{\omega_k 1}(s) \cdots L_{\omega_k d}(s) G_{\omega_k 1}(s) \cdots G_{\omega_k m}(s) G_{\omega_k 0}(s) \quad (16)$$

where for $i = 1, \ldots, d$

$$L_{\omega_k i}(s) = I - \zeta_{\omega_k i}(1 - e^{-\tau_i(s - j\omega_k)})\zeta_{\omega_k i}^*$$

and for $i = 1, \ldots, m$

$$G_{\omega_k i}(s) = I - \eta_{\omega_k i} \frac{2\,\mathrm{Re}\,(z_i)}{z_i - j\omega_k} \frac{s - j\omega_k}{z_i^* + s} \eta_{\omega_k i}^*.$$

The factorizations can be chosen such that for all $k, l = 0, \pm 1, \ldots, \pm n$,

$$\begin{aligned} \zeta_{\omega_k i} &= L_{\omega_l 1}(j\omega_k) \cdots L_{\omega_l i-1}(j\omega_k)\zeta_{\omega_l i}, \; i = 1, \ldots, d \\ \eta_{\omega_k i} &= L_{\omega_l 1}(j\omega_k) \cdots L_{\omega_l d}(j\omega_k) G_{\omega_l 1}(j\omega_k) \cdots G_{\omega_l i}(j\omega_k)\eta_{\omega_l i} \\ & \qquad i = 1, \ldots, m. \end{aligned}$$

Now again we consider the setup shown in Figure 1, with reference signal $r(t)$ given in (1) and the performance limits $J_{opt}(v)$ and J_{opt} defined in (3) and (5). Assume that Assumption 3.1 holds. Before stating the result, we note that when $l = k$, the fraction $\frac{e^{j(\omega_l - \omega_k)\tau_i} - 1}{j(\omega_l - \omega_k)}$ should be interpreted as τ_i, the limit of the fraction as ω_l goes to ω_k.

Theorem 4.1 *Let $G(s)$ be a system with factorizations satisfying Lemma 4.1. Then*

$$\begin{aligned} J_{opt}(v) = & \sum_{i=1}^{d} \sum_{k=-n}^{n} \sum_{l=-n}^{n} \frac{e^{j(\omega_l - \omega_k)\tau_i} - 1}{j(\omega_l - \omega_k)} \langle v_k, \zeta_{\omega_k i}\rangle \langle \zeta_{\omega_l i}, v_l \rangle \\ & + \sum_{i=1}^{m} \sum_{k=-n}^{n} \sum_{l=-n}^{n} \frac{2\mathrm{Re}(z_i)\langle v_k, \eta_{\omega_k i}\rangle \langle \eta_{\omega_l i}, v_l\rangle}{(z_i^* + j\omega_k)(z_i - j\omega_l)} \end{aligned} \quad (17)$$

and

$$J_{opt} = \sum_{i=1}^{d} (2n+1)\tau_i + 2 \sum_{i=1}^{m} \sum_{k=-n}^{n} \frac{1}{z_i - j\omega_k}. \quad (18)$$

5 Limitation without Full Information of the Reference

In this section, we discuss the tracking performance limitation when only the reference, not the state of the reference, is available to the controller. Recall the setup shown in Figure 2. Again we denote $P(s) = \begin{bmatrix} G(s) \\ H(s) \end{bmatrix}$ and assume Assumptions 3.1 holds. First we would like to point out that in this setup Theorem 3.1 and Theorem 4.1 can be restated. This can be shown by modifying the proof in Section 4 in a rather simple way. However, in this setup, the optimal or near optimal controller in general depends on the magnitude and phase information of all frequency components of the reference. Consequently, Theorem 3.2 is no longer true because the exchange of the infimum and expectation in the step from (13) to (14) is no longer valid. In the rest of this section, we will study J_{opt} for the special case when $G(s)$ is the transfer function of a SISO system and the reference $r(t)$ is a single sinusoid:

$$r(t) = \bar{\alpha} e^{-j\omega t} + \alpha e^{j\omega t}. \quad (19)$$

In this case, the magnitude and phase information of the reference is given by vector $v = [\bar{\alpha} \quad \alpha]'$.

Theorem 5.1 *Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$. Then*

$$\begin{aligned} J_{opt} = & 2 \sum_{i=1}^{m} \left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega} \right) \\ & + \frac{2}{\omega} \sin^2 \left[2 \sum_{i=1}^{m} \angle(z_i - j\omega) \right]. \end{aligned} \quad (20)$$

Notice that when the state of the reference is available, we have the following performance limit, as stated in Theorem 3.2

$$J_{opt} = 2 \sum_{i=1}^{m} \left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega} \right).$$

Theorem 5.1 gives an exact picture on how the unavailability of the reference state affects the best tracking performance. In this case, the controller attempts to estimate, in effect, the state of the reference first and then track the reference based on the estimated information. The cost of tracking is thus duely increased, with an extra term devoted to estimation.

Finally, we extend Theorem 5.1 to the case when $G(s)$ contains a time delay.

Theorem 5.2 *Let $G(s) = e^{\tau s}G_r(s)$ where $G_r(s)$ is a rational transfer function with nonminimum phase zeros $z_1, \cdots, z_m$. Then*

$$J_{opt} = 2\tau + 2\sum_{i=1}^{m}\left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega}\right) + \frac{2}{\omega}\sin^2\left[-\omega\tau + 2\sum_{i=1}^{m}\angle(z_i - j\omega)\right]. \quad (21)$$

6 Conclusion

In this paper, we have accomplished the following:

1. A formula is obtained for the best tracking performance when the reference is a given linear combination of step and sinusoidal signals in Theorem 3.1. This formula clearly reveals the role that each nonminimum phase zero, as well as its corresponding frequency-dependent directions, plays towards the performance limitation.
2. Theorem 3.2 presents a formula for the average tracking performance over all references with the same frequency components.
3. Theorem 4.1 gives the formulas which are extended FDLTI systems with time delays.
4. The tracking performance limitation with a restricted information availability is studied for the case when the reference is a scalar sinusoidal signal with a single frequency in Theorem 5.1 and 5.2. The performance degradation due to this information restriction is clearly shown.

In our derivation, great emphasis has been placed on the simplicity and the elegance of the formulas obtained. We believe that these results are significant in the further understanding of linear system structures and their effects on the best achievable performance by feedback control.

We have used 2DOF controllers in our study of tracking performance limitations in this paper. Since such controllers are most general controllers with given plant measurement and reference information, the performance limits obtained therein are the most fundamental regardless of what controller structure may be used. A pleasant consequence of using 2DOF controllers is that the performance limits only depend on the nonminimum phase zeros, together with their directional properties, but not on the poles and other zeros. One may also notice that the tracking performance when using 2DOF controllers depends on only one degree of freedom among the two available. In other words, the other degree of freedom in the controller is completely irrelevant as far as the tracking error is concerned. This gives us an opportunity to use this extra degree of freedom to achieve other performance specifications, such as disturbance rejection and robustness. We are currently trying to propose a meaningful performance specification which requires the proper utilization of both degrees of freedom in the controller and will then study the limitation in achieving such a performance specification.

We are trying to extend Theorem 5.1 and Theorem 5.2 to the case when the plant is an MIMO system and/or when multiple frequencies present in the reference. The MIMO extension appears to be possible but a simple formula in the spirit of (20) for the multiple frequency case is out of reach at this moment.

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