GENERAL MULTIRATE BUILDING BLOCKS AND THEIR APPLICATION IN NONUNIFORM FILTER BANKS

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ABSTRACT

This paper proposes a general linear dual-rate structure for multirate signal processing. Such general dual-rate systems are implementable at finite cost; in particular, they are equivalent to the cascade connections of linear periodically time-varying (LPTV) systems and block expanders and decimators. Using the general building blocks in nonuniform multirate filter banks, one can achieve what are otherwise impossible, thus paving the way for optimal design of synthesis systems.

1. INTRODUCTION

Traditional multirate building blocks in digital signal processing [13] are decimators, expanders, and LTI filters, with possibly some summing junctions. An example is the fractional sample-rate changer shown in Figure 1, where $\uparrow m$ is

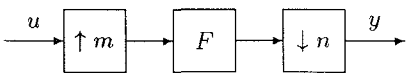

Figure 1. A sample-rate changer.

the expander by a factor m, $\downarrow n$ the decimator by n, and F a suitable LTI filter. The output sample rate is m/n times the input sample rate.

Such rate changers are not only useful in their own right [12], e.g., sample-rate conversion for bandlimited signals, they are also fundamental building blocks for multirate filter banks with uniform [13] or nonuniform [10, 6] bands.

The rate changer in Figure 1 is in fact a dual-rate system with the input-output property that shifting the input (u) by n samples results in shifting the output (y) by m samples. Such a property is defined as (m, n)-shift invariance, which is a generalization of time invariance for single-rate systems.

Consider two classes of dual-rate systems: the class associated with the structure in Figure 1, namely, the systems which are characterized by $\uparrow m$, LTI filter F, and $\downarrow n$, and the class of linear, causal, dual-rate systems shown in Figure 2, in which $G : (m, n)$ means that G is (m, n)-shift-invariant. Is the latter more general than the former? Or,

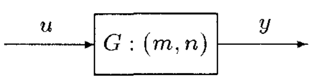

Figure 2. A general dual-rate system.

is it always possible to realize a causal, dual-rate system in Figure 2 by the structure in Figure 1? The answer is positive if the two integers m and n are coprime and *negative* otherwise [11], in this case m and n have some non-trivial common factor.

The class of dual-rate systems in Figure 2 are more general than the structure in Figure 1 and are especially interesting when m and n have common factors. In this paper, we study such (m, n)-shift-invariant systems and use them as the fundamental building blocks for multirate systems. One contribution of this paper is to show that any linear (m, n)-shift-invariant system is realizable by cascade combination of a block expander, an LPTV system, and a block decimator.

Advantages of the more general setup have been already observed: Khansari and Leon-Garcia [5] and Nayebi *et. al.* [9] used LPTV filters and block decimators and expanders in filter-bank systems; in particular, Khansari and Leon-Garcia [5] showed that using general synthesis systems, perfect reconstruction is possible if and only if the analysis filters have no common zero (note that this condition is necessary but not sufficient for perfect reconstruction [14] using traditional building blocks); Shenoy [11] showed that using general structures one can achieve what is otherwise impossible in design of fractional rate changers and also pointed out that the structural dependency in designing nonuniform filter banks [6, 1] disappears when general structures are used for analysis and synthesis.

As an application of the general dual-rate systems, consider the three-channel nonuniform filter bank shown in Figure 3, where the analysis and synthesis filters H_i and F_i are all LTI and causal. Hoang and Vaidyanathan [4] showed

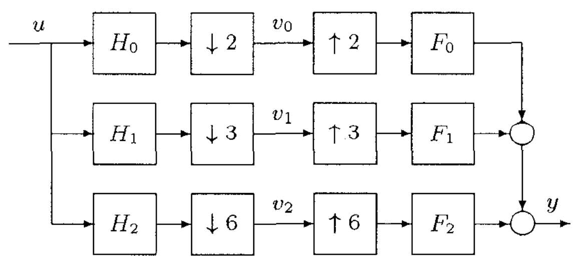

Figure 3. A nonuniform filter bank.

that the decimation integers, $\{2, 3, 6\}$, form an incompatible set; i.e., it is impossible to achieve alias cancellation

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by designing the six filters, let alone perfect reconstruction. Also, a design difficulty called structural dependency arises in this setup [6, 1].

However, if appropriate dual-rate systems are used for analysis and synthesis as shown in Figure 4, the difficulties encountered using the structure in Figure 3 no longer exist. In this case, optimal design based on model-matching the-

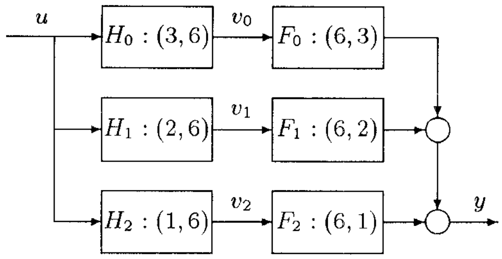

Figure 4. A filter bank with general structures.

ory, which was advocated in [12] for multirate filter design and in [2] for uniform filter-bank design, can be accomplished with relative ease.

2. GENERAL DUAL-RATE SYSTEMS

In this section we study basic concepts of linear dual-rate systems such as shift invariance, causality, and their various representations.

Let ℓ be the space of discrete-time signals defined on the set of all integers. A linear system G is regarded as a linear transformation mapping ℓ to itself, written $y = Gu$. It is a *dual-rate* system if the output and input have different sample rates – in this paper we shall make the assumption that the ratio of the two rates is a rational number, say, m/n; i.e., the output rate is m/n times the input rate.

A linear, dual-rate system G can always be represented by a kernel function $g(k,l)$:

$$y = Gu \;\Leftrightarrow\; y(k) = \sum_l g(k,l)u(l), \quad \forall k \tag{1}$$

Shift Invariance

To define shift invariance precisely, let U be the unit time delay on ℓ with transfer function z^{-1}. For a linear, dual-rate system G with output and input sample-rate ratio m/n, we define G to be (m,n)*-shift-invariant* if $GU^n = U^m G$. (The two integers m and n need not be coprime.) This means that shifting the input by n samples results in shifting the output by m samples. In terms of the kernel functions, (m,n)-shift invariance is characterized by the following relation

$$g(k+m, l+n) = g(k,l), \quad \forall k, l.$$

Figure 2 represents an (m,n)-shift-invariant G using block diagrams.

This shift invariance guarantees that appropriate blocking of the input and output gives rise to a multi-input, multi-output (MIMO), LTI system [7].

For an integer $p > 0$, define the p-fold *blocking operator*, L_p, via $\underline{x} = L_p x$ (underlining denotes blocking):

$$\underline{x}(k) = [\; x(kp) \quad x(kp+1) \quad \cdots \quad x(kp+p-1) \;]'.$$

L_p maps ℓ to ℓ^p, the external direct sum of p copies of ℓ. The inverse L_p^{-1} maps ℓ^p to ℓ.

Let G be linear, dual-rate, and SISO (single-input, single-output). Block the input and output appropriately to get the blocked system $\underline{G} := L_m G L_n^{-1}$, which has n inputs and m outputs. It is a well-known fact that $\underline{G}$ is LTI iff G is (m,n)-shift-invariant. Hence if G is (m,n)-shift-invariant, $\underline{G}$ has an $m \times n$ transfer matrix:

$$\hat{\underline{G}}(z) = \begin{bmatrix} \hat{G}_{00}(z) & \hat{G}_{01}(z) & \cdots & \hat{G}_{0,n-1}(z) \\ \hat{G}_{10}(z) & \hat{G}_{11}(z) & \cdots & \hat{G}_{1,n-1}(z) \\ \vdots & \vdots & & \vdots \\ \hat{G}_{m-1,0}(z) & \hat{G}_{m-1,1}(z) & \cdots & \hat{G}_{m-1,n-1}(z) \end{bmatrix}.$$

The entries in this matrix relate to the kernel function $g(k,l)$ of G as follows:

$$\hat{G}_{ij}(z) = \sum_k g(i+km, j)z^{-k}. \tag{2}$$

Causality

Causality of a dual-rate system G reflects implementability of the system in real time. Let G have input u and output y. Because the ratio of the sample rates of y and u is m/n, we can take the sample periods of u and y to be mh and nh, respectively, where h is some real number. Assuming both u and y are synchronized at time $t = 0$, we have that $u(k)$ occurs at time $t = k(mh)$ and $y(k)$ at $t = k(nh)$. Thus G is *causal* if for any k, the output $y(k)$ depends only on inputs occurred at $t \le k(nh)$, or on $u(l)$ for all l satisfying $lm \le kn$. Similarly, G is *strictly causal* if $y(k)$ depends only on $u(l)$ for all l such that $lm < kn$.

In terms of the kernel function in (1), G is causal iff

$$g(k,l) = 0 \quad \text{whenever} \quad kn < lm, \tag{3}$$

and is strictly causal iff

$$g(k,l) = 0 \quad \text{whenever} \quad kn \le lm.$$

If the dual-rate system G is both (m,n)-shift-invariant and causal, the blocked system $\underline{G}$ is LTI and causal. Moreover, the direct feedthrough matrix

$$\hat{\underline{G}}(\infty) = \begin{bmatrix} D_{00} & D_{01} & \cdots & D_{0,n-1} \\ D_{10} & D_{11} & \cdots & D_{1,n-1} \\ \vdots & \vdots & & \vdots \\ D_{m-1,0} & D_{m-1,1} & \cdots & D_{m-1,n-1} \end{bmatrix}$$

must have a (block) lower-triangular structure due to causality. This is called causality constraint [3] and can be derived from (3) and (2):

$$D_{ij} = 0 \quad \text{whenever} \quad in < jm.$$

From now on, we shall restrict our attention to linear dual-rate systems which are shift-invariant, causal, and moreover, whose blocked transfer matrices have proper, real-rational functions of z as their entries. All such transfer matrices have state-space models. Real-time implementation of these dual-rate systems is possible based on difference equations related to their state-space models [3].

3. REALIZATION VIA BLOCK DECIMATION

A linear dual-rate system which is (m, n)-shift-invariant cannot be represented by the structure in Figure 1 for some LTI F, if m and n have non-trivial common factor [11]. In this case, as we will show, block decimators and expanders, and LPTV systems should be used instead. Block decimation has been studied in multirate filter banks [5, 9].

Two positive integers p and q characterize the *block decimator* shown in Figure 5, where q represents the block size and p the decimation ratio. For any integer k, the input-output equation is

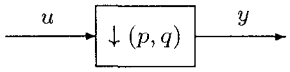

Figure 5. The block decimator.

$$y(kq + i) = u(kpq + i), \quad i = 0, 1, \cdots, q - 1.$$

If one groups the input into blocks of size q (starting from time $k = 0$), this block decimator retains only every p-th block. It can be verified that the block decimator is (q, pq)-shift-invariant and causal; moreover, if $q = 1$, it reduces to the usual decimator: $\downarrow (p, 1) = \downarrow p$.

The *block expander* is the dual system, shown in Figure 6, where q again is the block size and p the expansion ratio.

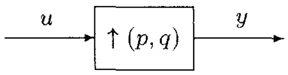

Figure 6. The block expander.

For any integer k, the block expander is defined via

$$y(kpq + i) = \begin{cases} u(kq + i), & i = 0, 1, \cdots, q - 1, \\ 0, & i = q, q + 1, \cdots, pq - 1. \end{cases}$$

This corresponds to inserting $p - 1$ blocks of zeros if the input and output are blocked with size q. The block expander is (pq, q)-shift-invariant but noncausal. If $q = 1$, it reduces to the usual expander $\uparrow p$. (For the frequency-domain input-output relations, see [5].)

Let G be any linear dual-rate system which is (m, n)-shift-invariant. If m and n are not coprime, we can find the largest common factor q and write $m = \bar{m}q$, $n = \bar{n}q$, so that $\bar{m}$ and $\bar{n}$ are coprime. Thus we can state the main result of this section:

Theorem 1 *The dual-rate system G is realizable by the cascade structure, shown in Figure 7, of the block expander $\uparrow (\bar{m}, q)$, a single-rate, LPTV system F with period q, and the block decimator $\downarrow (\bar{n}, q)$.*

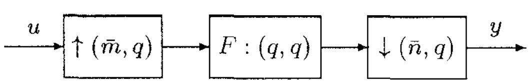

Figure 7. An equivalent structure.

Note that if m and n are coprime, then $q = 1$. The structure in Figure 7 reduces to that in Figure 1 with F LTI; and the theorem reduces to a known result. The proof of this theorem is omitted due to space limitation; a procedure can be given to compute the LPTV F in Figure 7 associated with a given dual-rate system G.

4. NONUNIFORM FILTER BANKS

In this section we study using the general structures in nonuniform filter banks to achieve what are otherwise impossible.

Consider the three-channel nonuniform filter bank in Figure 3, built via traditional blocks. It is shown that this system is *incompatible* [4] and hence alias cancellation is impossible using LTI and causal filters, let along perfect reconstruction.

Now we propose to use general dual-rate systems as in Figure 4 to replace the analysis and synthesis subsystems; note that H_0, H_1, F_0 and F_1 in Figure 4 are more general because they have common factors in their m and n. Denote the system $u \mapsto y$ in Figure 4 by T. It is easily verified that T is LPTV with period 6; the equivalent blocked system ($\underline{T} = L_6 T L_6^{-1}$) is

$$\begin{aligned} \underline{T} &= L_6 \begin{bmatrix} F_0 & F_1 & F_2 \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix} L_6^{-1} \\ &= \begin{bmatrix} L_6 F_0 L_3^{-1} & L_6 F_1 L_2^{-1} & L_6 F_2 \end{bmatrix} \begin{bmatrix} L_3 H_0 L_6^{-1} \\ L_2 H_1 L_6^{-1} \\ H_2 L_6^{-1} \end{bmatrix} \\ &=: \begin{bmatrix} \underline{F}_0 & \underline{F}_1 & \underline{F}_2 \end{bmatrix} \begin{bmatrix} \underline{H}_0 \\ \underline{H}_1 \\ \underline{H}_2 \end{bmatrix} \\ &=: \underline{F}\,\underline{H}, \end{aligned}$$

where $\underline{H}$ and $\underline{F}$ are both LTI and 6×6: $\hat{\underline{H}}(z)$ is the analysis matrix and $\hat{\underline{F}}(z)$ the synthesis matrix. Note that all entries in $\hat{\underline{H}}$ and $\hat{\underline{F}}$ are freely designable (no structural dependency). Hence T can perfectly match any causal, LPTV system with period 6; in particular, it can perfectly match any time-delay systems by proper choice of the dual-rate systems. Therefore, perfect reconstruction for the structure in Figure 4 is *possible*.

A more interesting scenario is perhaps given in Figure 8 in which the analysis structure in Figure 3 is combined with the synthesis structure in Figure 4. Is perfect reconstruction possible in this case?

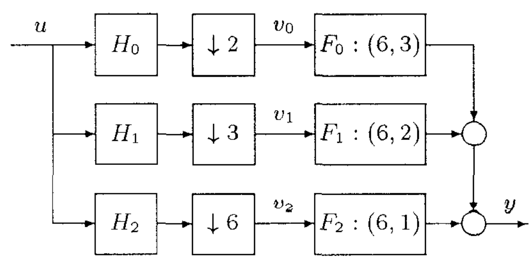

Figure 8. A filter bank with mixed structures.

The answer is positive and is illustrated by the simple example below. Let the LTI analysis filters in Figure 8 be

$$\hat{H}_0(z) = 1, \quad \hat{H}_1(z) = z^{-4} + z^{-5}, \quad \hat{H}_2(z) = z^{-3}.$$

The associated analysis matrix after blocking can be computed via procedures in [1]:

$$\underline{\hat{H}}(z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & z^{-1} & z^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^{-1} & z^{-1} \\ 0 & 0 & 0 & z^{-1} & 0 & 0 \end{bmatrix}.$$

Define the synthesis matrix after blocking to be

$$\underline{\hat{F}}(z) = \begin{bmatrix} z^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -z^{-1} & 0 & 1 & 0 & 0 \\ 0 & z^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & 0 & -z^{-1} & 0 & 1 & 0 \end{bmatrix}. \qquad (4)$$

It follows that $\underline{\hat{F}}(z)\underline{\hat{H}}(z) = z^{-1}I$ and hence the system achieves perfect reconstruction with time delay z^{-6}. The synthesis matrix in (4) corresponds to some general dual-rate structures in Figure 8.

The above observation can be generalized to nonuniform multirate filter banks with multiple channels and arbitrary decimation ratios [10, 6]:

> If the synthesis subsystems are replaced by appropriate dual-rate structures, incompatibility [4] and structural dependency [6, 1] disappear; and perfect reconstruction is always possible.

One advantage of eliminating structural dependency is to allow optimal design of synthesis systems; we conclude this paper by considering such a design example. We use the structure in Figure 8 and pre-select the linear-phase, FIR analysis filters: H_0 (order 40) is lowpass with cutoff frequency $\omega = \pi/2$; H_1 (order 30) is bandpass with passband $\pi/2 \leq \omega \leq 5\pi/6$; H_2 (order 14) is highpass with cutoff frequency $5\pi/6$. All three filters are designed using MATLAB function $fir1$ with their magnitude Bode plots given in Figure 9. The synthesis systems now can be de-

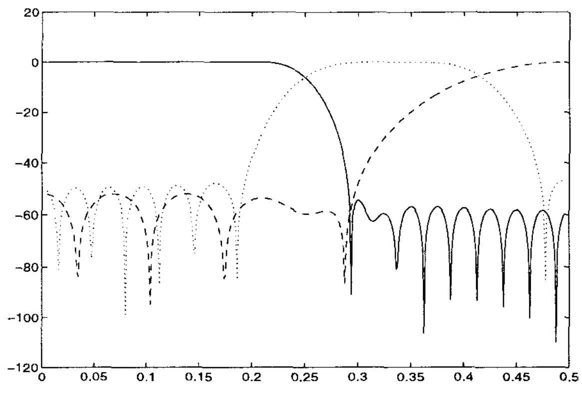

Figure 9. $|\hat{H}_0|$ **(solid),** $|\hat{H}_1|$ **(dot), and** $|\hat{H}_2|$ **(dash) in dB versus** $\omega/2\pi$**.**

signed by minimizing the $\mathcal{H}_\infty$ norm [2] of the error system between the ideal time delay (z^{-80}) and the filter-bank system, yielding a reconstruction error 0.31% in $\mathcal{H}_\infty$ norm. This means [8] that the alias and magnitude distortions of the designed system are both $\leq 0.31\%$ and the phase distortion is $\leq \sin^{-1} 0.31\% = 0.18°$. Of course, the synthesis systems designed use the general dual-rate structures.

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