$\mathcal{H}_2$-Optimal Design of Multirate Sampled-Data Systems

Li Qiu and Tongwen Chen

***Abstract*— Treating causality constraints, this paper studies optimal synthesis of multirate sampled-data systems with an $\mathcal{H}_2$ performance criteria. An explicit solution is obtained by input–output space extensions (lifting) and frequency-domain techniques.**

I. Introduction

In industry, most control systems are implemented digitally via microprocessors. Many digital designs are performed by rules of thumb. There are essentially two conventional methods: design an analog controller and then implement it digitally, sampling "fast enough" or discretize the plant and then design a discrete controller, ignoring intersample behavior.

The recent trend is to perform direct digital design, i.e., design digital controllers directly using continuous-time performance measures. This should be the preferred approach because most sampled-data systems operate in real time and the input and output signals are naturally in continuous time. Many pieces of work treating design issues have been completed in this direction; they include solutions to several $\mathcal{H}_2$ sampled-data control problems [7], [18], [4] and several solutions to the $\mathcal{H}_\infty$ sampled-data control problem [16], [28], [3], [25], [27], [24], [15]. A general mathematical tool, the lifting technique, has been developed [28], [31], [3], [5] for attacking problems in single-rate sampled-data systems.

All work mentioned above is in the single-rate setting. However, multirate sampled-data systems arise in a more natural way. In general, faster A/D and D/A conversions lead to better performance in digital control systems but also mean higher implementation cost. Allowing different speeds for A/D and D/A conversions results in better trade-offs between performance and implementation cost. Furthermore, multirate controllers can outperform single-rate linear-time invariant (LTI) controllers in certain design situations [19], [12], due to their time-varying nature.

The concept of multirate sampling was pioneered by Kranc [20]. Recent interests in multirate systems are reflected in the parametrization of stabilizing controllers [21], [23], the LQG/LQR designs [6], [1], [22], [9], the $\mathcal{H}_2$ design [30], [29], the $\mathcal{H}_\infty$ design [29], [8], [30], and the l_1 design [10]. While a lifting technique has been successfully used in the analysis of multirate system since the 1950's [20], the research on the optimal design of multirate controllers has lagged behind. The main obstacle is perhaps the so-called causality constraint [21], [23], which presents a unique difficulty for synthesizing the feedthrough terms in lifted controllers.

In this paper we shall study the direct digital design of multirate controller and show how to treat the causality constraint in the $\mathcal{H}_2$ design framework. Instead of treating the general multirate system, we consider the dual-rate case where all A/D converters operate at one rate and all D/A converters at another rate. This setup captures most of the essential features of multirate systems while maintains some clarity in exposition. The general multirate $\mathcal{H}_2$ design is studied in [29]. Our solution is different and more explicit, however, and gives closed-form formulas in many steps involved. The extension of some results in the paper to more general setup can be done via tools from nest algebra [8].

Manuscript received October, 1992; revised January 21, 1994. This work was supported in part by the Ministry of Colleges and Universities of Ontario, and the Natural Sciences and Engineering Research Council of Canada.

L. Qiu is with the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

T. Chen is with the Department of Electrical and Computer Engineering, University of Calgary, Calgary, Alberta, Canada T2N 1N4.

IEEE Log Number 9405672.

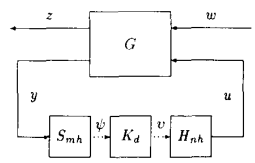

Fig. 1. A multirate control system.

Finally, we remark that causality constraints also arise in discrete-time periodic control [19], where the feedthrough terms in lifted controllers must be block lower-triangular. Tools developed in this paper are directly applicable to periodic control problems.

The organization of this paper is as follows. Section II presents the multirate sampled-data configuration for our subsequent study; in particular, desirable properties of multirate controllers are discussed. Section III extends the lifting macros in [5] to the multirate case; these formulas are useful in converting a sampled-data problem to an associated discrete-time one. Section IV formulates and solves explicitly the multirate $\mathcal{H}_2$-optimal control problem using frequency domain method; matrix projection is used to tackle the causality constraint.

The notation is quite standard. We use l to denote the space of sequences, perhaps vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. The external direct sum of n copies of l is denoted l^n. The space l_2 is the subspace of l consisting of all square-summable sequences. Similarly for the external direct sum l_2^n. Finally, if G is a LTI system, we shall not distinguish G from its transfer function.

II. Setup

The setup of the paper is shown in Fig. 1. Here we have used continuous lines for continuous signals and dotted lines for discrete signals. In Fig. 1, G is an analog plant, S_{mh} an ideal sampler with period mh, H_{nh} a zero-order hold with period nh, and K_d a multirate digital controller which is synchronized with S_{mh} and H_{nh} by a clock in the sense that K_d takes in a value of the sampled measurement ψ at times $t = k(mh)$, $k \geq 0$, and outputs a value of the control sequence v to the hold device at $t = k(nh)$, $k \geq 0$. We shall assume throughout the paper that m and n are coprime integers without loss of generality.

We shall consider only the analog G which are LTI, causal, and finite-dimensional. What are the corresponding concepts for the multirate controller K_d? Throughout K_d is regarded as a linear map from l to l. Since the input and output time scales are not compatible, the single-rate definitions must be modified.

The sampled-data controller $H_{nh}K_dS_{mh}$ as a continuous-time operator is in general time varying. Note, however, that both S_{mh} and H_{nh} are periodic elements, their least common period being $T = mnh$; so, by proper choice of K_d it is possible that $H_{nh}K_dS_{mh}$ is T-periodic in continuous time. Now let U be the unit time delay on l and U^* the unit time advance. We define K_d to be (m, n)-periodic if

$$(U^*)^m K_d U^n = K_d.$$

Then it is not hard to see that $H_{nh} K_d S_{mh}$ is T-periodic iff K_d is (m, n)-periodic.

This periodicity implies a deeper fact if we use the standard discrete-time lifting procedure [19] and extend the input and output spaces of K_d so as to be compatible with the period T. Define the discrete lifting operator L_m: $l \to l^m$ via $\underline{v} = L_m v$

$$\{v(0), v(1), \cdots\} \mapsto \left\{ \begin{bmatrix} v(0) \\ \vdots \\ v(m-1) \end{bmatrix}, \begin{bmatrix} v(m) \\ \vdots \\ v(2m-1) \end{bmatrix}, \cdots \right\}.$$

Similarly for L_n. Now define the lifted controller

$$\underline{K_d} := L_m K_d L_n^{-1}. \tag{1}$$

This is now single-rate with the underlying period being T. Then $\underline{K_d}$ is time-invariant iff K_d is (m, n)-periodic [21].

Next is causality. Again we require that $H_{nh} K_d S_{mh}$ be causal in continuous time. This condition translates to an interesting constraint on K_d. To see this more clearly, we look at the lifted controller $\underline{K_d}$. The feedthrough term $\underline{D}$ in $\underline{K_d}$ is an $m \times n$ block matrix, namely

$$\underline{D} = \begin{bmatrix} D_{00} & \cdots & D_{0,n-1} \\ \vdots & & \vdots \\ D_{m-1,0} & \cdots & D_{m-1,n-1} \end{bmatrix}$$

where each D_{ij} is a matrix with dimensions compatible to the dimensions of ψ and v. Now the causality of $H_{nh} K_d S_{mh}$ translates exactly to the causality of $\underline{K_d}$ and a constraint on $\underline{D}$, namely

$$D_{ij} = 0, \quad \text{whenever } jm > in.$$

This condition on $\underline{D}$ will be called the (m, n)-causality constraint. For ease of reference, the set of all $\underline{D}$ satisfying the (m, n)-causality constraint is denoted by $\mathcal{C}(m, n)$.

We say K_d is (m, n)-causal if the single-rate $\underline{K_d}$ is causal and $\underline{D}$ satisfies the (m, n)-causality constraint. It follows then that the sampled-data controller $H_{nh} K_d S_{mh}$ is causal in continuous time iff K_d is (m, n)-causal. More general treatment of these concepts can be found in [21], [23], [8].

A similar notion is that of strict causality. We say $\underline{D}$ satisfies the strict (m, n)-causality constraint if

$$D_{ij} = 0, \quad \text{whenever } jm \geq in.$$

The set of all such $\underline{D}$ is $\mathcal{C}_s(m, n)$. It follows that $H_{nh} K_d S_{mh}$ is strictly causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D} \in \mathcal{C}_s(m, n)$.

Note that the notation $\mathcal{C}(m, n)$ and $\mathcal{C}_s(m, n)$, representing sets of block triangular matrices and block strictly triangular matrices respectively, do not give information on the size of blocks and we assume this information can be inferred from the context. With compatibility assumption, the following lemma can be easily verified using matrix manipulation.

Lemma:

1) If $M_1 \in \mathcal{C}(p, q)$, $M_2 \in \mathcal{C}(q, r)$, then $M_2 M_1 \in \mathcal{C}(p, r)$.
2) If $M_1 \in \mathcal{C}_s(p, q)$, $M_2 \in \mathcal{C}(q, r)$ or $M_1 \in \mathcal{C}(p, q)$, $M_2 \in \mathcal{C}_s(q, r)$, then $M_2 M_1 \in \mathcal{C}_s(p, r)$.
3) If $M \in \mathcal{C}(p, p)$ and M is invertible, then $M^{-1} \in \mathcal{C}(p, p)$.
4) If $M \in \mathcal{C}_s(p, p)$, then $I - M$ is always invertible.

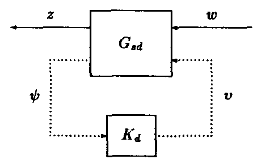

Fig. 2. An equivalent multirate system.

Finally, we turn to finite dimensionality of the controller K_d. This is again best explained in terms of $\underline{K_d}$. Assume K_d is (m, n)-periodic and (m, n)-causal. Then from the previous discussion $\underline{K_d}$ is LTI and causal. We furthermore assume $\underline{K_d}$ is finite-dimensional. Thus $\underline{K_d}$ has a state model

$$\underline{K_d} = \left[\begin{array}{c|ccc} A & B_0 & \cdots & B_{n-1} \\ \hline C_0 & D_{00} & \cdots & D_{0,n-1} \\ \vdots & \vdots & & \vdots \\ C_{m-1} & D_{m-1,0} & \cdots & D_{m-1,n-1} \end{array}\right].$$

The corresponding difference equations for K_d ($v = K_d \psi$) are

$$\eta(k+1) = A\eta(k) + \sum_{j=0}^{n-1} B_j \psi(nk+j), \tag{2}$$

$$v(mk+i) = C_i \eta(k) + \sum_{j=0}^{n-1} D_{ij} \psi(nk+j), \quad i = 0, 1, \cdots m-1. \tag{3}$$

Here η, the state for $\underline{K_d}$, is updated every $T = mnh$ seconds and v every nh seconds. Such difference equations can be implemented on microprocessors with only finite memory because the vector η is finite-dimensional.

In summary, in this paper we are interested in the class of multirate K_d which are (m, n)-periodic, (m, n)-causal, and finite-dimensional; this class is called the admissible class of K_d and can be modeled by the difference equations (2) and (3) with $D_{ij} = 0$ when $jm > in$. The corresponding admissible class of $\underline{K_d}$ is characterized by LTI, causal, and finite-dimensional $\underline{K_d}$ with the same constraint on $\underline{D}$.

III. Multirate Lifting

In Fig. 1, partition G according to its inputs and outputs and bring in a state model

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & D_{22} \end{array}\right]. \tag{4}$$

The zero block in D_{21} guarantees the proper functioning of the sampler when the exogeous input is impulsive. The zero block in D_{11} is necessary for the finiteness of the $\mathcal{H}_2$ measure in the next section. Now move S_{mh} and H_{nh} into the plant to get Fig. 2, where

$$G_{sd} = \begin{bmatrix} G_{11} & G_{12} H_{nh} \\ S_{mh} G_{21} & S_{mh} G_{22} H_{nh} \end{bmatrix}.$$

With our assumptions on K_d, this multirate system is T-periodic in continuous time. So the idea of lifting can be used to convert it into an LTI discrete system with infinite-dimensional input and output spaces.

Following [5], let $\mathcal{E}$ be any finite-dimensional Euclidean space, $\mathcal{E}^n$ be the external direct sum of n copies of $\mathcal{E}$, and $\mathcal{K}$ be $\mathcal{L}_2[0, T)$. The sequence space $l_2(\mathcal{K})$ is defined to be

$$l_2(\mathcal{K}) := \left\{ \psi: \ \psi_k \in \mathcal{K}, \sum_{k=0}^{\infty} \|\psi_k\|^2 < \infty \right\}.$$

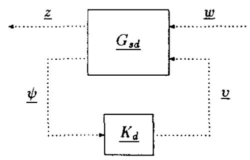

Fig. 3. The lifted system.

The norm for ψ_k is the one on $\mathcal{K}$ and the norm for $l_2(\mathcal{K})$ is given by

$$\|\psi\|_2 := \left(\sum_{k=0}^{\infty}\|\psi_k\|^2\right)^{1/2}.$$

To handle unbounded signals, we bring in the two extended spaces $\mathcal{L}_{2e}[0, \infty)$ and $l_{2e}(\mathcal{K})$ defined in the obvious way. The lifting operator L_T, mapping $\mathcal{L}_{2e}[0, \infty)$ to $l_{2e}(\mathcal{K})$ is defined by

$$\psi = L_T y \Rightarrow \psi_k(t) = y(t + kT), \qquad 0 \le t < T.$$

As before, we denote the lifted signal $L_T y$ by $\underline{y}$.

Now we lift the system in Fig. 2 with respect to the period T. Define

$$\begin{aligned}\underline{G_{sd}} &:= \begin{bmatrix} L_T & 0 \\ 0 & L_n \end{bmatrix} G_{sd} \begin{bmatrix} L_T^{-1} & 0 \\ 0 & L_m^{-1} \end{bmatrix} \\ &= \begin{bmatrix} L_T G_{11} L_T^{-1} & L_T G_{12} H_{nh} L_m^{-1} \\ L_n S_{mh} G_{21} L_T^{-1} & L_n S_{mh} G_{22} H_{nh} L_m^{-1} \end{bmatrix} \end{aligned} \tag{5}$$

and $\underline{K_d}$ again as in (1) to get the lifted system configuration in Fig. 3. Here the signals are all lifted; e.g., $\underline{w} = L_T w$ and $\underline{\psi} = L_n \psi$. We saw in Section II that $\underline{K_d}$ is LTI; it is not hard to see that $\underline{G_{sd}}$ too is LTI. So Fig. 3 represents a discrete LTI system. Let T_{zw} be the closed-loop map $w \mapsto z$ in Fig. 1. Then the closed-loop map $\underline{T_{zw}}$: $\underline{w} \mapsto \underline{z}$ in Fig. 3 is the lifted T_{zw}, namely, $L_T T_{zw} L_T^{-1}$. The usefulness of this relationship is due to the fact that the operators L_T and L_T^{-1} preserve norms.

Now with a state model for G in (4), we can derive a state-space representation for $\underline{G_{sd}}$. But first, let us find state models for the four blocks of $\underline{G_{sd}}$ in (5) as they may be of independent interest.

Lifting G_{11}

The lifted G_{11}, namely, $\underline{G_{11}} := L_T G_{11} L_T^{-1}$, maps $l_{2e}(\mathcal{K})$ to $l_{2e}(\mathcal{K})$ and can be represented by a state model with finite-dimensional state space [5]

$$\underline{G_{11}} = \left[\begin{array}{c|c} A_d & \underline{B}_1 \\ \hline \underline{C}_1 & \underline{D}_{11} \end{array}\right]$$

where

$$\begin{aligned} &A_d: \ \mathcal{E} \to \mathcal{E}, && A_d = e^{TA}, \\ &\underline{B}_1: \ \mathcal{K} \to \mathcal{E}, && \underline{B}_1 w = \int_0^T e^{(T-\tau)A} B_1 w(\tau)\, d\tau, \\ &\underline{C}_1: \ \mathcal{E} \to \mathcal{K}, && (\underline{C}_1 \xi)(t) = C_1 e^{tA} \xi, \\ &\underline{D}_{11}: \ \mathcal{K} \to \mathcal{K}, && (\underline{D}_{11} w)(t) = \int_0^t C_1 e^{(t-\tau)A} B_1 w(\tau) d\tau. \end{aligned}$$

Lifting $S_{mh} G_{21}$

Now we derive a state model for $\underline{S_{mh} G_{21}}$, namely, $L_n S_{mn} G_{21} L_T^{-1}$, which maps $l_{2e}(\mathcal{K})$ to l^n.

Write $\underline{\psi} = \underline{S_{mh} G_{21}}\, \underline{w}$ and let the state for the realization in (4) be x. Then the state equations for G_{21} ($y = G_{21} w$) are

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t), \\ y(t) &= C_2 x(t). \end{aligned} \tag{6}$$

Integrate (6) from kT to $(k+1)T$ and define the sequence ξ by $\xi(k) = x(kT)$ to get

$$\xi(k+1) = A_d \xi(k) + \underline{B}_1 \underline{w}_k.$$

By the definition of L_n

$$\underline{\psi}(k) = \begin{bmatrix} \psi(kn) \\ \vdots \\ \psi(kn + n - 1) \end{bmatrix}$$

where for $j = 0, 1, \cdots, n-1$

$$\begin{aligned} \psi(kn + j) &= C_2 x(kT + jmh) \\ &= C_2 e^{jmhA} \xi(k) + \int_0^{jmh} C_2 e^{(jmh-\tau)A} B_1 \underline{w}_k(\tau)\, d\tau. \end{aligned}$$

For notational convenience, define

$$\Psi_j(\tau) = C_2 e^{(jmh-\tau)A} B_1 \chi_{[0,\, jmh)}(\tau) \tag{7}$$

for $j = 0, 1, \cdots, n-1$, where $\chi_{[a,\, b)}$ is the characteristic function on the interval $[a, b)$. Then

$$\psi(kn + j) = C_2 e^{jmhA} \xi(k) + \int_0^T \Psi_j(\tau) \underline{w}_k(\tau)\, d\tau.$$

Putting things together, we get the state model

$$\begin{aligned} \xi(k+1) &= A_d \xi(k) + \underline{B}_1 \underline{w}_k, \\ \underline{\psi}(k) &= C_{2d} \xi(k) + \underline{D}_{21} \underline{w}_k \end{aligned}$$

or equivalently

$$\underline{S_{mh} G_{21}} = \left[\begin{array}{c|c} A_d & \underline{B}_1 \\ \hline C_{2d} & \underline{D}_{21} \end{array}\right]$$

where A_d and $\underline{B}_1$ were given before and

$$C_{2d}: \ \mathcal{E} \to \mathcal{E}^n, \qquad C_{2d} = \begin{bmatrix} C_2 \\ C_2 e^{mhA} \\ \vdots \\ C_2 e^{(n-1)mhA} \end{bmatrix},$$

$$\underline{D}_{21}: \ \mathcal{K} \to \mathcal{E}^n, \qquad \underline{D}_{21} w = \int_0^T \begin{bmatrix} \Psi_0(\tau) \\ \vdots \\ \Psi_{n-1}(\tau) \end{bmatrix} w(\tau)\, d\tau.$$

The lifting formulas for the other two blocks in $\underline{G_{sd}}$ can be derived similarly; they are summarized below.

Lifting $G_{12} H_{nh}$

For $i = 0, 1, \cdots, (m-1)$, define

$$\Phi_i(t) = D_{12} \chi_{[inh,(i+1)nh)}(t) + \int_0^t C_1 e^{(t-\tau)A} B_2 \chi_{[inh,(i+1)nh)}(\tau) d\tau. \tag{8}$$

Next define

$$B_{2d}: \ \mathcal{E}^m \to \mathcal{E}, \qquad B_{2d} = \int_0^T e^{(T-\tau)A} B_2 [\chi_{[0,\, nh)}(\tau) \ \cdots \ \chi_{[(m-1)nh,\, T)}(\tau)]\, d\tau,$$

$$\underline{D}_{12}: \ \mathcal{E}^m \to \mathcal{K}, \qquad (\underline{D}_{12} v)(t) = [\Phi_0(t) \ \cdots \ \Phi_{m-1}(t)] v.$$

Then the lifted operator $\underline{G_{12} H_{nh}}$, namely, $L_T G_{12} H_{nh} L_m^{-1}$: $\mathcal{E}^m \to l_{2e}(\mathcal{K})$, has a state model

$$\underline{G_{12} H_{nh}} = \left[\begin{array}{c|c} A_d & B_{2d} \\ \hline \underline{C}_1 & \underline{D}_{12} \end{array}\right].$$

Here A_d and $\underline{C}_1$ were given before.

Lifting $S_{mh}G_{22}H_{nh}$

The lifted operator $\underline{S_{mh}G_{22}H_{nh}} := L_n S_{mh} G_{22} H_{nh} L_m^{-1}$ maps l^m to l^n. A state model is

$$\underline{S_{mh}G_{22}H_{nh}} = \left[\begin{array}{c|c} A_d & B_{2d} \\ \hline C_{2d} & D_{22d} \end{array}\right]$$

where A_d, B_{2d}, C_{2d} were already given and

$$D_{22d}: \mathcal{E}^m \to \mathcal{E}^n,$$

$$[D_{22d}]_{ji} = D_{22}\chi_{[inh,\,(i+1)nh)}(jmh) + \int_{inh}^{(i+1)nh} C_2 e^{(jmh-\tau)A} B_2 \chi_{[0,\,jmh)}\, d\tau.$$

It can be verified that D_{22d} satisfies the (n, m)-causality constraint. Furthermore, D_{22d} satisfies the strict (n, m)-causality constraint if G_{22} is strictly causal ($D_{22} = 0$).

Lifting G_{sd}

We remark that all the four lifted blocks in $\underline{G_{sd}}$ share the same state vector $\xi(k) = x(kT)$. Moreover, their state models fit nicely together to form a state model for $\underline{G_{sd}}$ which maps $l_{2e}(\mathcal{K}) \oplus l^m$ to $l_{2e}(\mathcal{K}) \oplus l^n$

$$\underline{G_{sd}} = \begin{bmatrix} \underline{G}_{11} & \underline{G}_{12} \\ \underline{G}_{21} & \underline{G}_{22} \end{bmatrix} := \left[\begin{array}{c|cc} A_d & \underline{B}_1 & B_{2d} \\ \hline \underline{C}_1 & \underline{D}_{11} & \underline{D}_{12} \\ C_{2d} & \underline{D}_{21} & D_{22d} \end{array}\right]. \tag{9}$$

IV. $\mathcal{H}_2$-Optimal Control

Now we are ready to treat the synthesis problem: Design an admissible K_d to achieve internal stability and minimize some generalized $\mathcal{H}_2$ performance measure.

We adopt the generalized $\mathcal{H}_2$ measure proposed for periodic systems in [18], [4]. Let F be a T-periodic, causal system mapping $\mathcal{L}_{2e}$ to $\mathcal{L}_{2e}$. The lifted system $\underline{F} := L_T F L_T^{-1}$, mapping $l_{2e}(\mathcal{K})$ to $l_{2e}(\mathcal{K})$, is LTI in discrete time. Hence it has a transfer function

$$\underline{F}(\lambda) = \sum_{i=0}^{\infty} \underline{f}_i \lambda^i$$

where $\underline{f}_i$, $i \geq 0$, are Hilbert–Schmidt operators. Denote the Hilbert–Schmidt norm by $\|\cdot\|_{HS}$. We say the function $\underline{F}$ belongs to $\mathcal{H}_2$ if

$$\left(\sum_{i=0}^{\infty} \|\underline{f}_i\|_{HS}^2\right)^{1/2} < \infty$$

and the left-hand side is defined to be $\mathcal{H}_2$ norm of $\underline{F}$, denoted $\|\underline{F}\|_2$. By a slight abuse of notation, we define $\|F\|_2 = \|\underline{F}\|_2$. For more details on this $\mathcal{H}_2$ norm and its stochastic interpretation, see [4].

Now we turn to internal stability of Fig. 1. Define the continuous-time vector

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ \eta(k) \end{bmatrix}, \qquad kT \leq t < (k+1)T.$$

The (autonomous) multirate sampled-data system is internally stable, or K_d internally stabilizes G, if for any initial value $x_{sd}(t_0)$, $0 \leq t_0 < T$, $x_{sd} \to 0$ as $t \to \infty$.

We need a few standing assumptions in this section about the plant G in (4):

1) (A, B_2) is stabilizable and (C_2, A) is detectable;
2) the period T is nonpathological with respect to G [17], [7];
3) $D_{22} = 0$.

Assumptions 1) and 2) are mild and standard. Assumption 3) ensures the well-posedness of the closed-loop system following the lemma in Section II. It follows from similar arguments as in [12] that under these assumptions K_d internally stabilizes G iff $\underline{K_d}$ internally stabilizes $\underline{G_{sd}}$ in discrete time.

We can now state the $\mathcal{H}_2$-optimal control problem precisely: Given G, m, n, and h, design an admissible K_d to provide internal stability and minimize $\|T_{zw}\|_2$ in Fig. 2. This can be recast exactly in the lifted spaces: Design an admissible $\underline{K_d}$ to internally stabilize $\underline{G_{sd}}$ and minimize $\|T_{\underline{zw}}\|_2$ in Fig. 3. In what follows we shall solve explicitly this $\mathcal{H}_2$ problem using a frequency-domain approach.

In (9), A_d, B_{2d}, C_{2d}, D_{22d} are matrices and $\underline{B}_1$, $\underline{D}_{11}$, $\underline{D}_{12}$, $\underline{D}_{21}$ are operators. All the operators but $\underline{D}_{11}$ are of finite rank. This fact can be exploited: Define the real-rational matrices

$$\overline{G}_{11} = \left[\begin{array}{c|c} A_d & I \\ \hline I & 0 \end{array}\right], \qquad \overline{G}_{12} = \left[\begin{array}{c|c} A_d & B_{2d} \\ \hline \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{array}\right],$$

$$\overline{G}_{21} = \left[\begin{array}{c|c} A_d & [I \;\; 0] \\ \hline C_{2d} & [0 \;\; I] \end{array}\right]$$

to get

$$\underline{G}_{11} = \underline{D}_{11} + \underline{C}_1 \overline{G}_{11} \underline{B}_1, \qquad \underline{G}_{12} = [\underline{C}_1 \quad \underline{D}_{12}]\overline{G}_{12},$$

$$\underline{G}_{21} = \overline{G}_{21} \begin{bmatrix} \underline{B}_1 \\ \underline{D}_{21} \end{bmatrix}.$$

Bring in a special doubly-coprime factorization for the real rational transfer matrix $\underline{G}_{22}$

$$\underline{G}_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \qquad \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I$$

with

$$N(0), \tilde{N}(0) \in \mathcal{C}_s(m, n); \qquad M(0), \tilde{X}(0) \in \mathcal{C}(n, n);$$

$$\tilde{M}(0), X(0) \in \mathcal{C}(m, m); \qquad Y(0) = \tilde{Y}(0) = 0.$$

Since $D_{22d} \in \mathcal{C}_s(n, m)$, the standard procedure in [11] generates such a factorization. It then follows from [21], [23] that the set of admissible K_d which internally stabilize G is parameterized by

$$\underline{K_d} = (Y - MQ)(X - NQ)^{-1}, \quad Q \in \mathcal{RH}_\infty, \quad Q(0) \in \mathcal{C}(m, n).$$

With this controller applied, the closed-loop map in Fig. 3 is

$$T_{\underline{zw}} = T_1 - T_2 Q T_3 \tag{10}$$

where T_1, T_2, T_3 are given by

$$T_1 = \underline{D}_{11} + [\underline{C}_1 \quad \underline{D}_{12}]\left(\begin{bmatrix} \overline{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \overline{G}_{12} M \tilde{Y} \overline{G}_{21}\right)\begin{bmatrix} \underline{B}_1 \\ \underline{D}_{21} \end{bmatrix},$$

$$T_2 = [\underline{C}_1 \quad \underline{D}_{12}]\overline{G}_{12} M,$$

$$T_3 = \tilde{M}\overline{G}_{21}\begin{bmatrix} \underline{B}_1 \\ \underline{D}_{21} \end{bmatrix}.$$

Therefore, the multirate $\mathcal{H}_2$ problem is equivalent to the following constrained $\mathcal{H}_2$ model-matching problem

$$\inf_{Q \in \mathcal{RH}_\infty,\, Q(0) \in \mathcal{C}} \|T_1 - T_2 Q T_3\|_2. \tag{11}$$

Here we used $\mathcal{C}$ for $\mathcal{C}(m, n)$ to simplify notation. Note that T_1, T_2, T_3 are all operator-valued. For an operator-valued transfer function $T(\lambda)$, denote the transfer function of the adjoint system by $T^\sim(\lambda) := T^*(1/\lambda)$. To proceed further, we need one additional assumption:

4) For every λ on the unit circle, $T_2(\lambda)$ and $T_3^{\sim}(\lambda)$ are both injective.

Note that $T_2^{\sim}T_2$ and $T_3T_3^{\sim}$ are both matrix-valued. Bring in constant matrices E_{12} and E_{21} satisfying

$$E_{12}'E_{12} = \begin{bmatrix} \underline{C}_1^* \\ \underline{D}_{12}^* \end{bmatrix} [\underline{C}_1 \quad \underline{D}_{12}],$$

$$E_{21}E_{21}' = \begin{bmatrix} \underline{B}_1 \\ \underline{D}_{21} \end{bmatrix} [\underline{B}_1^* \quad \underline{D}_{21}^*]$$

to get

$$T_2^{\sim}T_2 = (E_{12}\overline{G}_{12}M)^{\sim}(E_{12}\overline{G}_{12}M),$$

$$T_3T_3^{\sim} = (\tilde{M}\overline{G}_{21}E_{21})(\tilde{M}\overline{G}_{21}E_{21})^{\sim}.$$

It follows that $T_2^{\sim}T_2$ and $T_3T_3^{\sim}$ are both parasymmetric real-rational matrices and have full ranks on the unit circle [Assumption 4]. So we can perform spectral factorizations $T_2^{\sim}T_2 = T_{2o}^{\sim}T_{2o}$ and $T_3T_3^{\sim} = T_{3co}T_{3co}^{\sim}$ with $T_{2o}, T_{2o}^{-1}, T_{3co}, T_{3co}^{-1} \in \mathcal{RH}_\infty$ and $T_{2o}(0) \in \mathcal{C}(m, m), T_{3co}(0) \in \mathcal{C}(n, n)$. Note that the extra condition on $T_{2o}(0)$ and $T_{3co}(0)$ can be always achieved by performing some QR factorizations. An inner–outer factorization $T_2 = T_{2i}T_{2o}$ and a co-inner–outer factorization $T_3 = T_{3co}T_{3ci}$ can be obtained by defining

$$T_{2i} = T_2T_{2o}^{-1} = [\underline{C}_1 \quad \underline{D}_{12}]\overline{G}_{12}MT_{2o}^{-1},$$

$$T_{3ci} = T_{3co}^{-1}T_3 = T_{3co}^{-1}\tilde{M}\overline{G}_{21}\begin{bmatrix} \underline{B}_1 \\ \underline{D}_{21} \end{bmatrix}.$$

Define the constant matrix

$$E_{11} := \begin{bmatrix} \underline{C}_1^* \\ \underline{D}_{12}^* \end{bmatrix} \underline{D}_{11} [\underline{B}_1^* \quad \underline{D}_{21}^*]$$

and the real-rational matrix in $\mathcal{L}_2$

$$R_{11} = (\overline{G}_{12}MT_{2o}^{-1})^{\sim}\left[E_{11} + E_{12}'E_{12}\left(\begin{bmatrix} \overline{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \overline{G}_{12}M\tilde{Y}\overline{G}_{21}\right)E_{21}E_{21}'\right](T_{3co}^{-1}\tilde{M}\overline{G}_{21})^{\sim}. \quad (12)$$

Denote the constant term of R_{11} by R_{110}. (Since R_{11} is in general noncausal, it follows that in general $R_{110} \neq R_{11}(0)$). Let $\Pi_{\mathcal{H}_2}: \mathcal{L}_2 \to \mathcal{H}_2$ and $\Pi_{\mathcal{H}_2^\perp}: \mathcal{L}_2 \to \mathcal{H}_2^\perp$ be the orthogonal projections. Also let $\Pi_{\mathcal{C}}$ and $\Pi_{\mathcal{C}^\perp}$ be the orthogonal projections from the space of matrices of appropriate dimensions to $\mathcal{C}(m, n)$ and $\mathcal{C}(m, n)^\perp$ respectively. We are now set up to state the main result of this paper.

Theorem: The optimal Q in (11) is given by

$$Q_{\text{opt}} = T_{2o}^{-1}[\Pi_{\mathcal{H}_2}R_{11} - \Pi_{\mathcal{C}^\perp}R_{110}]T_{3co}^{-1} \quad (13)$$

and the optimal model matching error by

$$\min_{Q\in\mathcal{RH}_\infty,\, Q(0)\in\mathcal{C}} \|T_1 - T_2QT_3\|_2^2 = \|T_1\|_2^2 - \|\Pi_{\mathcal{H}_2}R_{11}\|_2^2 + \|\Pi_{\mathcal{C}^\perp}R_{110}\|_2^2. \quad (14)$$

Proof: Apply unitary transformations to $T_1 - T_2QT_3$ and define

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} T_{2i}^{\sim} \\ I - T_{2i}T_{2i}^{\sim} \end{bmatrix} T_1 [T_{3ci}^{\sim} \quad I - T_{3ci}^{\sim}T_{3ci}]$$

(simple calculation shows that this R_{11} is exactly the one given in (12)) to get

$$\begin{aligned} &\|T_1 - T_2QT_3\|_2^2 \\ &= \left\|\begin{bmatrix} T_{2i}^{\sim} \\ I - T_{2i}T_{2i}^{\sim} \end{bmatrix}(T_1 - T_2QT_3)[T_{3ci}^{\sim} \quad I - T_{3ci}^{\sim}T_{3ci}]\right\|_2^2 \\ &= \|R_{11} - T_{2o}QT_{3co}\|_2^2 + \|R_{12}\|_2^2 + \|R_{21}\|_2^2 + \|R_{22}\|_2^2. \end{aligned}$$

The last three terms are independent of Q. Define $Q_1 = T_{2o}QT_{3co}$. From the lemma in Section II, we see that $Q_1 \in \mathcal{RH}_\infty$ iff $Q \in \mathcal{RH}_\infty$ and $Q_1(0) \in \mathcal{C}(m, n)$ iff $Q(0) \in \mathcal{C}(m, n)$. It follows then

$$\begin{aligned} &\inf_{Q\in\mathcal{RH}_\infty,\, Q(0)\in\mathcal{C}} \|R_{11} - T_{2o}QT_{3co}\|_2^2 \\ &= \inf_{Q_1\in\mathcal{RH}_\infty,\, Q_1(0)\in\mathcal{C}} \|R_{11} - Q_1\|_2^2 \\ &= \|\Pi_{\mathcal{H}_2^\perp}R_{11}\|_2^2 + \|\Pi_{\mathcal{C}^\perp}R_{110}\|_2^2. \end{aligned}$$

The optimal Q_1 is given by the sum of $\Pi_{\mathcal{C}}R_{110}$ and the strictly causal part of R_{11}, or equivalently, by the causal part of R_{11} minus $\Pi_{\mathcal{C}^\perp}R_{110}$

$$Q_{1,\text{opt}} = \Pi_{\mathcal{H}_2}R_{11} - \Pi_{\mathcal{C}^\perp}R_{110}.$$

This proves (13).

The optimal cost is given by

$$\|T_1 - T_2QT_3\|_2^2 = \|R_{11}\|_2^2 - \|Q_{1,\text{opt}}\|_2^2 + \|R_{12}\|_2^2 + \|R_{21}\|_2^2 + \|R_{22}\|_2^2$$

which is clearly equal to the right-hand side of (14). □

The computation in the above theorem involves only real-rational matrices and constant matrices except T_1, which is an operator-valued function. Its norm, required in (14), can be computed with relative ease

$$\|T_1\|_2^2 = \|\underline{D}_{11}\|_{HS}^2 + \left\|E_{12}\left(\begin{bmatrix} \overline{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \overline{G}_{12}M\tilde{Y}\overline{G}_{21}\right)E_{21}\right\|_2^2.$$

Here we used the fact that $\overline{G}_{11}$ and $\tilde{Y}$ are strictly proper. The Hilbert–Schmidt norm of $\underline{D}_{11}$ follows from that of a general integral operator [13]

$$\|\underline{D}_{11}\|_{HS}^2 = \text{trace} \int_0^T \int_0^t B_1'e^{(t-\tau)A'}C_1'C_1e^{(t-\tau)A}B_1\, d\tau\, dt.$$

Finally, we conclude this section by presenting the explicit formulas for E_{11}, E_{12}, and E_{21}. To this end, we need to find several matrix blocks formed by compositions of operators such as $\underline{C}_1^*\underline{D}_{11}\underline{B}_1^*$. With the functions Ψ_j and Φ_i defined in (7) and (8), these blocks can be found to be

$$\underline{C}_1^*\underline{D}_{11}\underline{B}_1^* = \int_0^T \int_0^t e^{tA'}C_1'C_1e^{(t-\tau)A}B_1B_1'e^{(T-\tau)A'}\, d\tau\, dt,$$

$$\underline{C}_1^*\underline{D}_{11}\underline{D}_{21}^* = \int_0^T \int_0^t e^{tA'}C_1'C_1e^{(t-\tau)A} \cdot B_1[\Psi_0'(t) \quad \cdots \quad \Psi_{n-1}'(t)]\, d\tau\, dt,$$

$$\underline{D}_{12}^*\underline{D}_{11}\underline{B}_1^* = \int_0^T \int_0^t \begin{bmatrix} \Phi_0'(t) \\ \vdots \\ \Phi_{m-1}'(t) \end{bmatrix} C_1e^{(t-\tau)A}B_1B_1'e^{(T-\tau)A'}\, d\tau\, dt,$$

$$\underline{D}_{12}^*\underline{D}_{11}\underline{D}_{21}^* = \int_0^T \int_0^t \begin{bmatrix} \Phi_0'(t) \\ \vdots \\ \Phi_{m-1}'(t) \end{bmatrix} C_1e^{(t-\tau)A}B_1[\Psi_0'(t) \quad \cdots \quad \Psi_{n-1}'(t)]\, d\tau\, dt,$$

$$\underline{C}_1^*\underline{C}_1 = \int_0^T e^{tA'}C_1'C_1e^{tA}\, dt,$$

$$\underline{C}_1^*\underline{D}_{12} = \int_0^T e^{tA'}C_1'[\Phi_0(t) \quad \cdots \quad \Phi_{m-1}(t)]\, dt,$$

$$\underline{D}_{12}^*\underline{D}_{12} = \int_0^T \begin{bmatrix} \Phi_0'(t) \\ \vdots \\ \Phi_{m-1}'(t) \end{bmatrix} [\Phi_0(t) \quad \cdots \quad \Phi_{m-1}(t)]\, dt,$$

$$\underline{B}_1\underline{B}_1^* = \int_0^T e^{(T-\tau)A}B_1B_1'e^{(T-\tau)A'}\, d\tau,$$

$$\underline{B}_1\underline{D}_{21}^* = \int_0^T e^{(T-\tau)A}B_1[\Psi_0'(\tau) \quad \cdots \quad \Psi_{n-1}'(\tau)]\,d\tau,$$

$$\underline{D}_{21}\underline{D}_{21}^* = \int_0^T \begin{bmatrix} \Psi_0(\tau) \\ \vdots \\ \Psi_{n-1}(\tau) \end{bmatrix} [\Psi_0'(\tau) \quad \cdots \quad \Psi_{n-1}'(\tau)]\,d\tau.$$

With the two symmetric matrices $E_{12}'E_{12}$ and $E_{21}E_{21}'$ computed, there are many choices for E_{12} and E_{21}; for example, we can take them as the square roots or Cholesky factors of the two symmetric matrices respectively.

References

[1] H. Al-Rahmani and G. F. Franklin, "A new optimal multirate control of linear periodic and time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 406–415, 1990.

[2] M. Araki and K. Yamamoto, "Multivariable multirate sampled-data systems: state-space description, transfer characteristics, and Nyquist criterion," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 145–154, 1986.

[3] B. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with application to $\mathcal{H}_\infty$ sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 418–435, 1992.

[4] B. Bamieh and J. B. Pearson, "The $\mathcal{H}_2$ problem for sampled-data systems," *Syst. Contr. Lett.*, vol. 19, pp. 1–12, 1992.

[5] B. Bamieh, J. B. Pearson, B. A. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control," *Syst. Contr. Lett.*, vol. 17, pp. 79–88, 1991.

[6] M. C. Berg, N. Amit, and J. Powell, "Multirate digital control system design," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 1139–1150, 1988.

[7] T. Chen and B. A. Francis, "$\mathcal{H}_2$-optimal sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 387–397, 1991.

[8] T. Chen and L. Qiu, "$\mathcal{H}_\infty$ design of general multirate sampled-data control systems," *Automatica*, vol. 30, pp. 1139–1152, 1994.

[9] T. Chen and B. A. Francis, "Linear time-varying $\mathcal{H}_2$-optimal control of sampled-data systems," *Automatica*, vol. 27, no. 6, pp. 963–974, 1991.

[10] M. A. Dahleh, P. G. Voulgaris, and L. S. Valavani, "Optimal and robust controllers for periodic and multirate systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 90–99, 1992.

[11] B. A. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory.* New York: Springer-Verlag, 1987.

[12] B. A. Francis and T. T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 820–832, 1988.

[13] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators.* Providence, RI: American Mathematical Society, 1969.

[14] T. Hagiwara and M. Araki, "Design of a stable feedback controller based on the multirate sampling of the plant output," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 812–819, 1988.

[15] Y. Hayakawa, Y. Yamamoto, and S. Hara, "$\mathcal{H}_\infty$ type problem for sampled-data control system—A solution via minimum energy characterization," in *Proc. IEEE Conf. Decis. Contr.*, pp. 463–468, 1992.

[16] P. T. Kabamba and S. Hara, "Worst case analysis and design of sampled-data control systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1337–1357, 1993.

[17] R. Kalman, Y. C. Ho, and K. Narendra, "Controllability of linear dynamical systems," in *Contributions to Differential Equations*, vol. 1. New York: Interscience, 1963.

[18] P. P. Khargonekar and N. Sivashankar, "$\mathcal{H}_2$ optimal control for sampled-data systems," *Syst. Contr. Lett.*, vol. 18, no. 3, pp. 627–631, 1992.

[19] P. P. Khargonekar, K. Poolla, and A. Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 1088–1096, 1985.

[20] G. M. Kranc, "Input–output analysis of multirate feedback systems," *IRE Trans. Automat. Control*, vol. 3, pp. 21–28, 1957.

[21] D. G. Meyer, "A parameterization of stabilizing controllers for multirate sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 233–236, 1990.

[22] ——, "Cost translation and a lifting approach to the multirate LQG problem," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1411–1415, 1992.

[23] R. Ravi, P. P. Khargonekar, K. D. Minto, and C. N. Nett, "Controller parameterization for time-varying multirate plants," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1259–1262, 1990.

[24] N. Sivashankar and P. P. Khargonekar, "Characterization of the $\mathcal{L}_2$-induced norm for linear systems with jumps with applications to sampled-data systems," preprint, 1991.

[25] W. Sun, K. M. Nagpal, and P. P. Khargonekar, "$\mathcal{H}_\infty$ control and filtering with sampled measurements," in *Proc. Amer. Contr. Conf.*, pp. 1652–1657, 1991.

[26] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space.* Amsterdam: North-Holland, 1970.

[27] G. Tadmor, "$\mathcal{H}_\infty$ optimal sampled-data control in continuous time systems," *Int. J. Contr.*, vol. 56, no. 1, pp. 99–141, 1992.

[28] H. T. Toivonen, "Sampled-data control of continuous-time systems with an $\mathcal{H}_\infty$ optimality criterion," *Automatica*, vol. 28, no. 1, pp. 45–54, 1992.

[29] P. G. Voulgaris and B. Bamieh, "Optimal $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control of hybrid multirate systems," *Syst. Contr. Lett.*, vol. 20, pp. 249–261, 1993.

[30] P. G. Voulgaris, M. A. Dahleh, and L. S. Valavani, "$\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers for periodic and multi-rate systems," *Automatica*, vol. 32, pp. 251–264, 1994.

[31] Y. Yamamoto, "A new approach to sampled-data control systems: A function space method," in *Proc. IEEE Conf. Decis. Contr.*, 1990, pp. 1882–1887.

An Exact Solution to General SISO Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problems via Convex Optimization

Mario Sznaier

***Abstract*—The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be motivated as a nominal LQG optimal control problem, subject to robust stability constraints, expressed in the form of an $\mathcal{H}_\infty$ norm bound. A related modified problem consisting on minimizing an upper bound of the $\mathcal{H}_2$ cost subject to $\mathcal{H}_\infty$ constraints was introduced in [1]. Although there presently exist efficient methods to solve this modified problem, the original problem remains, to a large extent, still open. In this paper we propose a method for solving general discrete-time SISO $\mathcal{H}_2/\mathcal{H}_\infty$ problems. This method involves solving a sequence of problems, each one consisting of a finite-dimensional convex optimization and an unconstrained Nehari approximation problem.**

I. Introduction

During the last decade, a large research effort has been devoted to the problem of designing robust controllers capable of guaranteeing stability in the face of plant uncertainty. As a result, a powerful $\mathcal{H}_\infty$ framework has been developed, addressing the issue of robust stability in the presence of norm-bounded plant perturbations. Since its introduction, the original formulation of Zames [2] has been substantially simplified, resulting in efficient computational schemes for finding solutions. Of particular importance is [3] where a state-space approach is developed and an efficient procedure is given to compute suboptimal $\mathcal{H}_\infty$ controllers. Since these controllers are not unique, the extra degrees of freedom available can then be used to

Manuscript received January 8, 1993; revised June 19, 1993. This work was supported in part by NSF Grant ECS-9211169 and by a grant from the Florida Space Grant Consortium.

The author is with the Department of Electrical Engineering, Pennsylvania State University, University Park, PA 16802 USA.

IEEE Log Number 9405673.

0018-9286/94$04.00 © 1994 IEEE