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$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Designs of Multirate Sampled-Data Systems[1]

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Abstract

Treating causality constraints, this paper studies the optimal syntheses of multirate sampled-data systems with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criteria. Explicit solutions to both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems are obtained by input-output space extensions (lifting) and frequency-domain techniques.

1 Introduction

Designing digital controllers directly using continuous-time performance measures is receiving considerable attention recently; this is evidenced by work in the $\mathcal{H}_2$ framework [6, 17, 3] and $\mathcal{H}_\infty$ framework [14, 28, 2, 25, 27, 15]. A general mathematical tool, the lifting technique, has been developed [28, 31, 2, 4] for attacking problems in single-rate sampled-data systems.

All work mentioned above is in the single-rate setting. However, multirate sampled-data systems arise in a more natural way. In general, faster A/D and D/A conversions lead to better performance in digital control systems but also mean higher cost in implementation. Allowing different speeds for A/D and D/A conversions results in better trade-offs between performance and implementation cost.

The concept of multirate sampling was pioneered by Kranc [18]. Recent interests in multirate systems are reflected in the LQG/LQR designs [5, 1, 19, 7], the parametrization of stabilizing controllers [20, 23], and among others. While the research on single-rate direct digital design has been active, little work has been done on multirate systems using the direct design approach. The main obstacle is perhaps the so-called causality constraint [20, 23], which presents a unique difficulty for synthesizing the feedthrough term in lifted controllers. A similar constraint also arises in discrete-time periodic control; interesting solutions were obtained for the $\mathcal{H}_\infty$ problem [10, 11, 30] and the $\mathcal{H}_2$ problem [30]. In this paper we treat multirate designs directly from a sampled-data point of view and use matrix factorization theory to tackle causality constraints.

The organization of this paper is as follows. Section 2 presents the multirate setup for our study and discusses desirable properties of multirate controllers. Section 3 extends the lifting idea in [4] to the multirate case. Section 4 formulates and solves explicitly the multirate $\mathcal{H}_2$-optimal control problem using the lifting presented in Section 3. Section 5 is devoted to the multirate $\mathcal{H}_\infty$ control problem. We show how to reduce the multirate sampled-data problem to a discrete-time $\mathcal{H}_\infty$ problem with causality constraint. The latter problem is then solved explicitly using frequency-domain methods. We refer to [22] for proofs and details.

The notation is quite standard. We use ℓ to denote the space of sequences, perhaps vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. The external direct sum of n copies of ℓ is denoted ℓ^n. The space ℓ_2 is a subspace of ℓ of square-summable sequences. Similarly for the external direct sum ℓ_2^n. If G is a linear time-invariant (LTI) system, we shall not distinguish G from its transfer function. Finally, for an operator K and an operator matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

the associated linear fractional transformation is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

Of course, the domains and co-domains of the operators must be compatible and the inverse must exist.

2 Setup

The setup of the paper is shown in Figure 1, where G is an analog plant, S_{mh} an ideal sampler with period mh, H_{nh} a zero-order hold

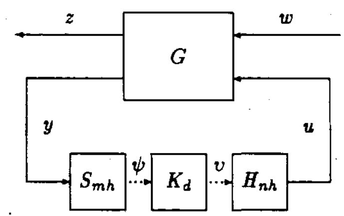

Figure 1: A multirate control system

with period nh, and K_d a multirate digital controller which is synchronized with S_{mh} and H_{nh} by a clock in the sense that K_d takes in a value of the sampled measurement ψ at times $t = k(mh)$, $k \geq 0$, and outputs a value of the control sequence v to the hold device at $t = k(nh)$, $k \geq 0$. We shall assume throughout the paper that m and n are coprime integers.

This setup is not the most general one as in [20, 23]; in fact, it has a uniform sampling rate and a uniform hold rate. But since the ratio of the two rates can be any positive rational number, this setup captures all the essential features in multirate systems while maintains some clarity in the exposition. Extensions to more general setup are possible [8].

We shall consider only the analog G which are LTI, causal, and finite-dimensional. What are the corresponding concepts for the multirate controller K_d? Throughout K_d is regarded as a linear map from ℓ to ℓ. Since the input and output time scales are not compatible, the single-rate definitions must be modified.

The sampled-data controller $H_{nh}K_dS_{mh}$ as a continuous-time operator is in general time-varying. However, note that both S_{mh} and H_{nh} are periodic elements, their least common period being $T = mnh$; so, by proper choice of K_d it is possible that $H_{nh}K_dS_{mh}$ is T-periodic in continuous time. Now let U be the unit time delay on ℓ and U^* the unit time advance. We define K_d to be (m, n)-*periodic* if

$$(U^*)^m K_d U^n = K_d.$$

Then it is not hard to see that $H_{nh}K_dS_{mh}$ is T-periodic iff K_d is (m, n)-periodic.

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This periodicity implies a deeper fact if we lift K_d properly. Define the discrete lifting operator $L_m : \ell \to \ell^m$ via $\underline{v} = L_m v$:

$$\{v(0), v(1), \cdots\} \mapsto \left\{ \begin{bmatrix} v(0) \\ \vdots \\ v(m-1) \end{bmatrix}, \begin{bmatrix} v(m) \\ \vdots \\ v(2m-1) \end{bmatrix}, \cdots \right\}.$$

Similarly for L_n. Now define the lifted controller

$$\underline{K_d} := L_m K_d L_n^{-1}. \quad (1)$$

This is now single-rate with the underlying period being T. Then $\underline{K_d}$ is time-invariant iff K_d is (m,n)-periodic.

Next is causality. Again we require that $H_{nh} K_d S_{mh}$ be causal in continuous time. This condition translates to an interesting constraint on K_d. To see this more clearly, we look at the lifted controller $\underline{K_d}$. The feedthrough term $\underline{D}$ in $\underline{K_d}$ is an $m \times n$ block matrix, namely,

$$\underline{D} = \begin{bmatrix} D_{00} & \cdots & D_{0,n-1} \\ \vdots & & \vdots \\ D_{m-1,0} & \cdots & D_{m-1,n-1} \end{bmatrix}.$$

Now the causality of $H_{nh} K_d S_{mh}$ translates exactly to the causality of $\underline{K_d}$ and a constraint on $\underline{D}$, namely,

$$D_{ij} = 0, \quad \text{whenever } jm > in.$$

This condition on $\underline{D}$ will be called the (m,n)-*causality constraint*. For ease of reference, the set of all $\underline{D}$ satisfying the (m,n)-causality constraint is denoted by $\Omega(m,n)$.

We say K_d is (m,n)-*causal* if the single-rate $\underline{K_d}$ is causal and $\underline{D}$ satisfies the (m,n)-causality constraint. It follows then that the sampled-data controller $H_{nh} K_d S_{mh}$ is causal in continuous time iff K_d is (m,n)-causal. More general treatment of these concepts can be found in, e.g., [20, 23].

A similar notion is that of strict causality. We say $\underline{D}$ satisfies the *strict* (m,n)-*causality constraint* if

$$D_{ij} = 0, \quad \text{whenever } jm \geq in.$$

The set of all such $\underline{D}$ is $\Omega_s(m,n)$. It follows that $H_{nh} K_d S_{mh}$ is strictly causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D} \in \Omega_s(m,n)$.

Finally, we turn to finite dimensionality of the controller K_d. This is again best explained in terms of $\underline{K_d}$. Assume K_d is (m,n)-periodic and (m,n)-causal. Then from the previous discussion $\underline{K_d}$ is LTI and causal. We furthermore assume $\underline{K_d}$ is finite-dimensional. Thus $\underline{K_d}$ has a state model

$$\underline{K_d} = \left[\begin{array}{c|ccc} A & B_0 & \cdots & B_{n-1} \\ \hline C_0 & D_{00} & \cdots & D_{0,n-1} \\ \vdots & \vdots & & \vdots \\ C_{m-1} & D_{m-1,0} & \cdots & D_{m-1,n-1} \end{array} \right].$$

The corresponding difference equations for K_d ($v = K_d \psi$) are

$$\eta(k+1) = A\eta(k) + \sum_{j=0}^{n-1} B_j \psi(nk+j), \quad (2)$$

$$v(mk+i) = C_i \eta(k) + \sum_{j=0}^{n-1} D_{ij} \psi(nk+j), \; i = 0, 1, \cdots m-1. \quad (3)$$

Here η, the state for $\underline{K_d}$, is updated every $T = mnh$ seconds and v every nh seconds. Such difference equations can be implemented on microprocessors with only finite memory because the vector η is finite-dimensional.

In summary, in this paper we are interested in the class of multirate controllers K_d which are (m,n)-periodic, (m,n)-causal, and finite-dimensional; this class is called the *admissible* class of K_d and can be modeled by the difference equations (2) and (3) with $D_{ij} = 0$ when $jm > in$. The corresponding admissible class of $\underline{K_d}$ is characterized by LTI, causal, and finite-dimensional $\underline{K_d}$ with the same constraint on $\underline{D}$.

3 Multirate Lifting

The single-rate lifting technique [28, 31, 2, 4] is very powerful in sampled-data control because it converts a periodic sampled-data system into an LTI discrete system with infinite-dimensional input and output spaces. In this section we shall extend this technique to the multirate case.

In Figure 1, partition G according to its inputs and outputs and bring in a state model:

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & D_{22} \end{array} \right]. \quad (4)$$

Now move S_{mh} and H_{nh} into the plant to get $\mathcal{F}(G, H_{nh} K_d S_{mh}) = \mathcal{F}(G_{sd}, K_d)$, where

$$G_{sd} = \begin{bmatrix} G_{11} & G_{12} H_{nh} \\ S_{mh} G_{21} & S_{mh} G_{22} H_{nh} \end{bmatrix}.$$

With our assumptions on K_d, $\mathcal{F}(G_{sd}, K_d)$ is T-periodic in continuous time. So the idea of lifting can be used.

Following [4], let $\mathcal{E}$ be any finite-dimensional Euclidean space, $\mathcal{E}^n$ be the external direct sum of n copies of $\mathcal{E}$, $\mathcal{K}$ be $\mathcal{L}_2[0,T)$, and $\ell_2(\mathcal{K})$ be the function-valued sequence space [4]. To handle unbounded signals, we bring in the two extended spaces $\mathcal{L}_{2e}[0,\infty)$ and $\ell_{2e}(\mathcal{K})$ defined in the obvious way. The lifting operator L_T, mapping $\mathcal{L}_{2e}[0,\infty)$ to $\ell_{2e}(\mathcal{K})$ is defined by

$$\psi = L_T y \Leftrightarrow \psi_k(t) = y(t+kT), \quad 0 \leq t < T.$$

Now we lift the system $\mathcal{F}(G_{sd}, K_d)$ with respect to the period T. Define

$$\underline{G_{sd}} = \begin{bmatrix} L_T G_{11} L_T^{-1} & L_T G_{12} H_{nh} L_m^{-1} \\ L_n S_{mh} G_{21} L_T^{-1} & L_n S_{mh} G_{22} H_{nh} L_m^{-1} \end{bmatrix}. \quad (5)$$

and $\underline{K_d}$ again as in (1) to get the lifted system $\mathcal{F}(\underline{G_{sd}}, \underline{K_d})$. We saw in Section 2 that $\underline{K_d}$ is LTI; it is not hard to see that $\underline{G_{sd}}$ too is LTI. So $\mathcal{F}(\underline{G_{sd}}, \underline{K_d})$ represents a discrete LTI system. It is easily verified that $\mathcal{F}(\underline{G_{sd}}, \underline{K_d}) = L_T \mathcal{F}(G, H_{nh} K_d S_{mh}) L_T^{-1}$. The usefulness of this relationship is due to the fact that the operators L_T and L_T^{-1} preserve norms.

Now with a state model for G in (4), we can derive a state-space representation for $\underline{G_{sd}}$ which maps $\ell_{2e}(\mathcal{K}) \oplus \ell^m$ to $\ell_{2e}(\mathcal{K}) \oplus \ell^n$:

$$\underline{G_{sd}} = \left[\begin{array}{c|cc} A_d & \underline{B_1} & B_{2d} \\ \hline \underline{C_1} & \underline{D_{11}} & \underline{D_{12}} \\ C_{2d} & \underline{D_{21}} & D_{22d} \end{array} \right].$$

Here A_d, B_{2d}, C_{2d}, D_{22d} are matrices and the rest are operators as follows:

$$\begin{array}{llll} \underline{B_1}: & \mathcal{K} \to \mathcal{E}, & \underline{C_1}: & \mathcal{E} \to \mathcal{K}, \\ \underline{D_{11}}: & \mathcal{K} \to \mathcal{K}, & \underline{D_{12}}: & \mathcal{E}^m \to \mathcal{K}, \\ \underline{D_{21}}: & \mathcal{K} \to \mathcal{E}^n. & & \end{array}$$

The explicit expressions in terms of the realization of G are given in [22] and are omitted here for space consideration. Note that $D_{22d} \in \Omega(n,m)$. Furthermore, $D_{22d} \in \Omega_s(n,m)$ if G_{22} is strictly causal ($D_{22} = 0$).

4 $\mathcal{H}_2$-Optimal Control

This section treats the first synthesis problem: Design an admissible K_d to achieve internal stability and minimize some generalized $\mathcal{H}_2$ performance measure.

First of all, let us look at the performance measure. Recall that for an admissible K_d, the closed-loop system $\mathcal{F}(G, H_{nh}K_dS_{mh})$ in Figure 1 is T-periodic. Thus we adopt the generalized $\mathcal{H}_2$ measure proposed for periodic systems in [17, 3].

Let F be a continuous-time, T-periodic, causal system described by the following integral operator

$$(Fu)(t) = \int_0^t f(t,\tau)u(\tau)\,d\tau.$$

We assume that f, the matrix-valued impulse response of F, is locally square-integrable. The periodicity of F implies $f(t+T,\tau+T) = f(t,\tau)$, and the causality implies that $f(t,\tau) = 0$ if $\tau > t$. If f is square-integrable on $[0,\infty)\times[0,T)$, we can define a norm for F as follows [17, 3]:

$$\|F\|_{per} = \left\{ \frac{1}{T}\int_0^T \int_0^\infty \text{trace}\,[f'(t,\tau)f(t,\tau)]\,dt\,d\tau \right\}^{1/2}$$

Now we lift F to get $\underline{F} := L_T F L_T^{-1}$. The lifted system $\underline{F} : \ell_{2e}(\mathcal{K}) \mapsto \ell_{2e}(\mathcal{K})$ can be described by ($\underline{y} = \underline{F}\,\underline{u}$)

$$\underline{y}_k = \sum_{j=0}^{k} \underline{f}_{k-j}\underline{u}_j, \quad k \geq 0,$$

where $\underline{f}_i$, $i \geq 0$, map $\mathcal{K}$ to $\mathcal{K}$ via

$$(\underline{f}_i u)(t) = \int_0^T f(t+iT,\tau)u(\tau)\,d\tau, \quad 0 \leq t < T.$$

$\underline{F}$ is LTI in discrete time; its transfer function is defined as

$$\underline{F}(\lambda) = \sum_{i=0}^{\infty} \underline{f}_i \lambda^i.$$

The local square-integrability of $f(t,\tau)$ implies that the operators $\underline{f}_i$, $i \geq 0$, are Hilbert-Schmidt operators [32]. Moreover, the set of Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, $\|\cdot\|_{HS}$, is a Hilbert space [13]. Thus the transfer function $\underline{F}$ is a Hilbert-space vector-valued function on some subset of $\mathcal{C}$. We say the function $\underline{F}$ belongs to $\mathcal{H}_2$ if

$$\left(\sum_{i=0}^{\infty} \|\underline{f}_i\|_{HS}^2 \right)^{1/2} < \infty,$$

and the left-hand side is defined to be its $\mathcal{H}_2$ norm, denoted $\|\underline{F}\|_2$ [26]. It follows from [3] that $\underline{F}$ is in $\mathcal{H}_2$ iff every element of f is square-integrable on $[0,\infty)\times[0,T)$; in this case, $\frac{1}{\sqrt{T}}\|\underline{F}\|_2 = \|F\|_{per}$.

For internal stability of Figure 1, let the plant state be x and the controller state be η. Define the continuous-time vector

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ \eta(k) \end{bmatrix}, \quad kT \leq t < (k+1)T.$$

The (autonomous) multirate sampled-data system is *internally stable*, or K_d *internally stabilizes* G, if for any initial value $x_{sd}(t_0)$, $0 \leq t_0 < T$, $x_{sd} \to 0$ as $t \to \infty$.

We need a few standing assumptions in this section about the plant G in (4):

1. (A, B_2) is stabilizable and (C_2, A) is detectable;
2. the period T is non-pathological with respect to G [16, 6];
3. $D_{22} = 0$.

Assumptions 1 and 2 are mild and standard. Assumption 3 is for the well-posedness of the closed-loop system. It follows that K_d internally stabilizes G iff $\underline{K_d}$ internally stabilizes $\underline{G}_{22}$ in discrete time, where $\underline{G}_{22}$ is a standard discrete system.

We can now state the $\mathcal{H}_2$-optimal control problem precisely: Given G, m, n, and h, design an admissible K_d to provide internal stability and minimize $\|\mathcal{F}(G, H_{nh}K_dS_{mh})\|_{per}$. By the above discussion, we can recast the problem exactly in the lifted spaces: Design an admissible $\underline{K_d}$ to internally stabilize $\underline{G}_{22}$ and minimize the $\mathcal{H}_2$ norm of $\mathcal{F}(\underline{G_{sd}}, \underline{K_d})$. This $\mathcal{H}_2$ problem will be solved using a frequency-domain approach. The problem is harder than the single-rate one [17, 3] due to the facts that $\underline{D}_{21}$ is nonzero and that $\underline{K_d}$ must satisfy the causality constraint.

Now bring in a doubly-coprime factorization for the real rational transfer matrix $\underline{G}_{22}$:

$$\underline{G}_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N},$$

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I,$$

with $M(0) = I$ and $\tilde{M}(0) = I$. It follows from [20, 23] that the set of admissible K_d which internally stabilize G is parametrized by

$$\underline{K_d} = (Y - MQ)(X - NQ)^{-1}, \quad Q \in \mathcal{RH}_\infty, \quad Q(0) \in \Omega(m,n).$$

With this controller applied, the closed-loop map is

$$\mathcal{F}(\underline{G_{sd}}, \underline{K_d}) = T_1 - T_2 Q T_3,$$

where T_1, T_2, T_3 are given by

$$\begin{aligned} T_1 &= \underline{G}_{11} + \underline{G}_{12}M\tilde{Y}\underline{G}_{21}, \\ T_2 &= \underline{G}_{12}M, \\ T_3 &= \tilde{M}\underline{G}_{21}. \end{aligned}$$

Therefore, the multirate $\mathcal{H}_2$ problem is equivalent to the following constrained $\mathcal{H}_2$ model-matching problem

$$\inf_{Q \in \mathcal{RH}_\infty, Q(0)\in\Omega} \|T_1 - T_2 Q T_3\|_2. \qquad (6)$$

Here we used Ω for $\Omega(m,n)$ to simplify notation. Note that T_1, T_2, T_3 are all operator-valued. For an operator-valued transfer function $T(\lambda)$, denote the transfer function of the adjoint system by $T^\sim(\lambda) := T^*(1/\lambda)$. To proceed further, we need one additional assumption:

4. For every λ on the unit circle, $T_2(\lambda)$ and $T_3^\sim(\lambda)$ are both injective.

Note that $T_2^\sim T_2$ and $T_3 T_3^\sim$ are both matrix-valued. It follows that $T_2^\sim T_2$ and $T_3 T_3^\sim$ are both para-symmetric real-rational matrices and have full ranks on the unit circle (Assumption 4). So we can perform spectral factorizations $T_2^\sim T_2 = T_{2o}^\sim T_{2o}$ and $T_3 T_3^\sim = T_{3co}T_{3co}^\sim$ with T_{2o}, T_{2o}^{-1}, T_{3co}, $T_{3co}^{-1} \in \mathcal{RH}_\infty$. An inner-outer factorization $T_2 = T_{2i}T_{2o}$ and a co-inner-outer factorization $T_3 = T_{3co}T_{3ci}$ can be obtained by defining

$$T_{2i} = T_2 T_{2o}^{-1}, \quad T_{3ci} = T_{3co}^{-1}T_3.$$

Define the real-rational matrix in $\mathcal{L}_2$

$$R_{11} = T_{2i}^\sim T_1 T_{3ci}^\sim$$

and denote the constant term of R_{11} by R_{110}. Let $\Pi_{\mathcal{H}_2} : \mathcal{L}_2 \to \mathcal{H}_2$ and $\Pi_{\mathcal{H}_2^\perp} : \mathcal{L}_2 \to \mathcal{H}_2^\perp$ be the orthogonal projections. We are now set up to state the main result of this section.

Theorem 1 *The optimal Q in (6) is given by*

$$Q_{opt} = Q_0 + \lambda T_{2o}^{-1} \left\{ \Pi_{\mathcal{H}_2} \left[\lambda^{-1}(R_{11} - T_{2o}Q_0T_{3co}) \right] \right\} T_{3co}^{-1},$$

where the constant matrix Q_0 is the optimal $Q(0)$ solving

$$\min_{Q(0)\in\Omega} \|R_{110} - T_{2o}(0)Q(0)T_{3co}(0)\|_2. \tag{7}$$

Now we look at how to use matrix factorization theory to find Q_0 solving (7). For square and nonsingular matrices $T_{2o}(0)$ and $T_{3co}(0)$, bring in factorizations

$$T_{2o}(0) = U_2R_2, \quad T_{3co}(0) = R_3U_3,$$

where U_2, R_2, U_3, R_3 are all square, U_2, U_3 are orthogonal ($U_2'U_2 = I$, $U_3'U_3 = I$), and R_2, R_3 are lower-triangular. The existence and computation of such factorizations follow analogously from those of the well-known QR factorization. Recall that the 2-norm for matrices is induced by the inner product:

$$\langle A, B\rangle := \text{trace } (A'B).$$

Thus the subspace Ω has its orthogonal complement $\Omega^\perp$ in the space of matrices of appropriate dimensions. Let Π_Ω and $\Pi_{\Omega^\perp}$ be the orthogonal projections to Ω and $\Omega^\perp$ respectively. It follows then that Π_Ω amounts to simply retaining the blocks corresponding to the unconstrained blocks in Ω and zeroing the other blocks.

Lemma 1 *The optimal $Q(0)$ solving (7) is*

$$Q_0 = R_2^{-1}\Pi_\Omega[U_2'R_{110}U_3']R_3^{-1}.$$

Finally, we refer to [22] for the proofs of the results and for an explicit and detailed procedure for computation.

5 $\mathcal{H}_\infty$-Optimal Control

In this section we shall study the multirate $\mathcal{H}_\infty$ control problem: Design an admissible K_d to provide internal stability and achieve a pre-specified level of $\mathcal{H}_\infty$ performance, i.e., $\|\mathcal{F}(G, H_{nh}K_dS_{nh})\| < \gamma$, where γ is positive and the norm is $\mathcal{L}_2$-induced. By proper scaling, we can always take $\gamma = 1$.

In principle, the multirate lifting procedure in Section 3 could be employed to reduce the problem to a discrete-time $\mathcal{H}_\infty$ problem with causality constraint. However, in this section we shall present a simpler reduction process which is based on recent single-rate results [2, 15] and the discrete lifting. Then the constrained discrete $\mathcal{H}_\infty$ problem is solved explicitly.

With the state model of G in (4), Assumptions 1-3 made in Section 4 are in force in this section. Let $\underline{D}_{11h} : \mathcal{L}_2[0,h) \to \mathcal{L}_2[0,h)$ be defined by

$$(\underline{D}_{11h}w)(t) = C_1 \int_0^t e^{(t-\tau)A}B_1w(\tau)\,d\tau.$$

An additional assumption is needed:

4′. $\|\underline{D}_{11h}\| < 1$.

This is a necessary condition for $\|\mathcal{F}(G, H_{nh}K_dS_{mh})\| < 1$; its computation was studied in [2].

Corresponding to the two integers m and n, introduce the discrete sampling operator $S_m : \ell \to \ell$ defined via

$$\psi = S_m\phi \Leftrightarrow \psi(k) = \phi(mk)$$

and the discrete hold operator $H_n : \ell \to \ell$ via

$$\psi = H_n\phi \Leftrightarrow \psi(kn+j) = \phi(k), \quad j = 0, 1, \cdots, n-1.$$

Now we bring in a discrete LTI system

$$G_d := \left[\begin{array}{c|cc} A_d & B_{1d} & B_{2d} \\ \hline C_{1d} & D_{11d} & D_{12d} \\ C_{2d} & 0 & 0 \end{array}\right]. \tag{8}$$

Here G_d is an equivalent system for the single-rate $\mathcal{H}_\infty$ sampled-data problem with sampling period h; several sets of realization matrices were given in several recent papers, e.g., [2, 15]. Define the lifted discrete system $\underline{K_d}$ as in Section 2 and

$$\underline{G_d} = \begin{bmatrix} L_{mn} & 0 \\ 0 & L_nS_m \end{bmatrix} G_d \begin{bmatrix} L_{mn}^{-1} & 0 \\ 0 & H_nL_m^{-1} \end{bmatrix}.$$

It is not hard to check that $\underline{G_d}$ is LTI, causal, and finite-dimensional. The following result establishes the connection between the multirate $\mathcal{H}_\infty$ problem and a discrete $\mathcal{H}_\infty$ problem.

Theorem 2 *Under Assumptions 1-3 and 4′, we have*

(i) K_d internally stabilizes G iff $\underline{K_d}$ internally stabilizes $\underline{G_d}$;

(ii) $\|\mathcal{F}(G, H_{nh}K_dS_{mh})\| < 1$ iff $\|\mathcal{F}(\underline{G_d}, \underline{K_d})\|_\infty < 1$.

A different reduction process was recently reported in [29]. This theorem also implies that the multirate $\mathcal{H}_\infty$ problem can be recast as a constrained $\mathcal{H}_\infty$ model-matching problem. To see this, we note that the (2,2) block in $\underline{G_d}$, $\underline{G_{22d}}$, is (n,m)-strictly causal. Parametrize all the stabilizing and admissible controllers $\underline{K_d}$ for $\underline{G_{22d}}$ as in Section 4 to get

$$\mathcal{F}(\underline{G_d}, \underline{K_d}) = T_1 - T_2QT_3,$$

where T_1, T_2, T_3 are real-rational matrices in $\mathcal{H}_\infty$ and can be found from $\underline{G_d}$. Then the multirate $\mathcal{H}_\infty$ problem is equivalent to the discrete $\mathcal{H}_\infty$ model-matching problem of finding a $Q \in \mathcal{RH}_\infty$ with the constraint $Q(0) \in \Omega(m,n)$ such that

$$\|T_1 - T_2QT_3\|_\infty < 1. \tag{9}$$

If such a Q exists, we say the multirate $\mathcal{H}_\infty$ problem is *solvable*.

From now on we shall focus on this constrained $\mathcal{H}_\infty$ problem. For regularity, we need an assumption similar to Assumption 4 in Section 4:

5′. For every λ on the unit circle, $T_2(\lambda)$ and $T_3^\sim(\lambda)$ are both injective.

Under this assumption, perform an inner-outer factorization $T_2 = T_{2i}T_{2o}$ and an co-inner-outer factorization $T_3 = T_{3co}T_{3ci}$, where T_{2o} and T_{3co} are both invertible over $\mathcal{RH}_\infty$. Apply unitary transformations to $T_1 - T_2QT_3$ and define

$$R = \begin{bmatrix} T_{2i}^\sim \\ I - T_{2i}T_{2i}^\sim \end{bmatrix} T_1 \begin{bmatrix} T_{3ci}^\sim & I - T_{3ci}^\sim T_{3ci} \end{bmatrix}.$$

We shall consider the causality constraint at a later stage; let us now drop this constraint on $Q(0)$ and look at the unconstrained problem. This allows us to use the powerful result in [12] to parametrize all Q in $\mathcal{RH}_\infty$ achieving (9). The unconstrained problem in (9) is solvable iff

$$\left\| \begin{bmatrix} \Pi_{\mathcal{H}_2^\perp} & 0 \\ 0 & I \end{bmatrix} R|_{\mathcal{H}_2\oplus\mathcal{L}_2} \right\| < 1. \tag{10}$$

Moreover, if (10) is satisfied, then there exists an $\mathcal{RH}_\infty$ matrix

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

with $K_{12}^{-1}, K_{21}^{-1} \in \mathcal{RH}_\infty$ and $\|K_{22}\|_\infty < 1$ such that all $Q \in \mathcal{RH}_\infty$ satisfying (9) are characterized by

$$Q = \mathcal{F}(K, Q_1), \quad Q_1 \in \mathcal{RH}_\infty, \quad \|Q_1\|_\infty < 1. \tag{11}$$

We refer to [12] for the details of checking inequality (10) and the expression of K. Hereafter, we shall assume that (10) is true.

By (11), $Q(0)$ depends on $Q_1(0)$ in a linear fractional manner. To simplify this, introduce another linear fractional transformation $Q_1 = \mathcal{F}(V, Q_2)$, where V, partitioned as usual, is a constant unitary matrix:

$$V = \begin{bmatrix} K_{22}'(0) & [I - K_{22}'(0)K_{22}(0)]^{1/2} \\ [I - K_{22}(0)K_{22}'(0)]^{1/2} & -K_{22}(0) \end{bmatrix}.$$

It follows that the mapping $Q_2 \mapsto Q_1$ is bijective from the open unit ball of $\mathcal{RH}_\infty$ onto itself [24]. Thus all Q satisfying (9) can be re-parametrized by

$$\begin{aligned} Q &= \mathcal{F}[K, \mathcal{F}(V, Q_2)] \\ &= \mathcal{F}(L, Q_2), \quad Q_2 \in \mathcal{RH}_\infty, \quad \|Q_2\|_\infty < 1. \end{aligned}$$

It can be checked that $L_{22}(0) = 0$ and $L_{12}(0)$, $L_{21}(0)$ are still nonsingular. Thus

$$Q(0) = L_{11}(0) + L_{12}(0)Q_2(0)L_{21}(0). \tag{12}$$

Now we bring in the causality constraint on $Q(0)$. Our goal is to find the necessary and sufficient condition for the existence of a $Q_2 \in \mathcal{RH}_\infty$ with $\|Q_2\|_\infty < 1$ such that $Q(0)$ in (12) lies in $\Omega(m,n)$. Since $Q(0)$ depends only on $Q_2(0)$ and in general $\|Q_2\|_\infty \geq \|Q_2(0)\|$, the problem is the same as searching a constant matrix $Q_2(0)$ with $\|Q_2(0)\| < 1$ such that $Q(0) \in \Omega(m,n)$, the norm being the largest singular value of $Q_2(0)$.

As in Section 4, introduce matrix factorizations

$$L_{12}(0) = R_1U_1, \quad L_{21}(0) = -U_2R_2,$$

where R_1, R_2, U_1, U_2 are all square, R_1, R_2 are lower-triangular, and U_1, U_2 are orthogonal. Substitute the factorizations into (12) and pre- and post-multiply by R_1^{-1} and R_2^{-1} respectively to get

$$R_1^{-1}Q(0)R_2^{-1} = R_1^{-1}L_{11}(0)R_2^{-1} - U_1Q_2(0)U_2.$$

Define

$$W := R_1^{-1}L_{11}(0)R_2^{-1}, \quad P := U_1Q_2(0)U_2.$$

It follows that $\|Q_2(0)\| < 1$ iff $\|P\| < 1$ and $Q(0) \in \Omega(m,n)$ iff $R_1^{-1}Q(0)R_2^{-1} \in \Omega(m,n)$ [20]. Therefore, we arrive at the following equivalent matrix problem: Given W, find P with $\|P\| < 1$ such that $W - P \in \Omega(m,n)$.

Partition W and P as required in $\Omega(m,n)$. Apparently, P must cancel the $\Omega^\perp$-part of W. The solution is somewhat complicated. First, let us distinguish two cases: The fixed blocks in P, or the zero blocks in $\Omega(m,n)$, take the (block) row-echelon form if $m < n$ and the (block) column-echelon form if $n < m$. Next, we need to locate all the *maximum fixed submatrices* of P, namely, the submatrices which consist of only the fixed blocks and have maximum sizes. To do this, denote the integer part of a positive real number x by $\lfloor x \rfloor$. If $m < n$, let $l = m$ and for $k = 0, 1, \ldots, l-1$, define

$$M_k = \underbrace{\begin{bmatrix} I & & \\ & \ddots & 0 \\ & I & \end{bmatrix}}_{m \text{ blocks}} \Bigg\} k+1 \text{ blocks}$$

$$N_k = \underbrace{\begin{bmatrix} & 0 & \\ I & & \\ & \ddots & \\ & & I \end{bmatrix}}_{n-1-\lfloor \frac{km}{n} \rfloor \text{ blocks}} \Bigg\} n \text{ blocks}$$

If $n < m$, define $l = n - 1$ and for $k = 0, 1, \ldots, l-1$, define

$$M_k = \underbrace{\begin{bmatrix} I & & \\ & \ddots & 0 \\ & I & \end{bmatrix}}_{m \text{ blocks}} \Bigg\} 1 + \lfloor \tfrac{(k+1)m}{n} \rfloor \text{ blocks}$$

$$N_k = \underbrace{\begin{bmatrix} & 0 & \\ I & & \\ & \ddots & \\ & & I \end{bmatrix}}_{n-1-k \text{ blocks}} \Bigg\} n \text{ blocks}$$

Then it can be checked that $M_kWN_k, k = 0, 1, \ldots, l-1$, are exactly those maximum fixed submatrices of P. Define

$$\mu := \max\{\|M_kWN_k\| : \ k = 0, 1, \ldots, l-1\}.$$

Theorem 3 *Under Assumptions 1-3 and 4'-5', the multirate $\mathcal{H}_\infty$ problem is solvable, i.e., there exists a matrix P with $\|P\| < 1$ such that $W - P \in \Omega(m,n)$, iff $\mu < 1$.*

The proof is based on a result on norm preserving dilations from operator theory [21, 9], which also provides a constructive procedure to determine the free blocks in P to get $\|P\| = \mu$; for details, see [22].

To summarize, let us list the solvability conditions for the multirate $\mathcal{H}_\infty$ control problem $\|\mathcal{F}(G, H_{nh}K_dS_{mh})\| < 1$:

(a) $\|\underline{D}_{11h}\| < 1$;

(b) $\left\| \begin{bmatrix} P_{\mathcal{H}_2^\perp} & 0 \\ 0 & I \end{bmatrix} R|_{\mathcal{H}_2 \oplus \mathcal{L}_2} \right\| < 1$;

(c) $\mu < 1$.

Condition (a) was studied in detail in [2]. Condition (b) is the solvability condition for a standard $\mathcal{H}_\infty$ problem. When conditions (a-b) hold, a necessary and sufficient test for condition (c) is given in Theorem 3; it amounts to computing the norms of several constant matrices.

6 Concluding Remarks

In this paper we have addressed causality constraints in direct designs of multirate sampled-data control systems using $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures. Explicit solutions are given for the $\mathcal{H}_2$-optimal controller and the $\mathcal{H}_\infty$-suboptimal controllers which achieve the performance requirement $\|\mathcal{F}(G, H_{nh}K_dS_{mh})\| < 1$. $\mathcal{H}_\infty$ controllers which are arbitrarily close to optimality can be computed based on the solvability conditions (a-c) (with proper scaling) and a standard bisection search. Finally, we mention that extensions to the more general setup have been made using operators between appropriate nests [8].

References

[1] H. Al-Rahmani and G. F. Franklin, "A new optimal multirate control of linear periodic and time-varying systems," *IEEE Trans. Automat. Control*, vol. 35, pp. 406-415, 1990.

[2] B. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with application to $\mathcal{H}_\infty$ sampled-data control," *IEEE Trans. Automat. Control*, vol. 37, pp. 418-435, 1992.

[3] B. Bamieh and J. B. Pearson, "The $\mathcal{H}_2$ problem for sampled-data systems," *Systems and Control Letters*, vol. 19, pp. 1-12, 1992.

[4] B. Bamieh, J. B. Pearson, B. A. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control," *Systems and Control Letters*, vol. 17, pp. 79-88, 1991.

[5] M. C. Berg, N. Amit, and J. Powell, "Multirate digital control system design," *IEEE Trans. Automat. Control*, vol. 33, pp. 1139-1150, 1988.

[6] T. Chen and B. A. Francis, "$\mathcal{H}_2$-optimal sampled-data control," *IEEE Trans. Automat. Control*, vol. 36, pp. 387-397, 1991.

[7] T. Chen and B. A. Francis, "Linear time-varying $\mathcal{H}_2$-optimal control of sampled-data systems," *Automatica*, vol. 27, No. 6, pp. 963-974, 1991.

[8] T. Chen and L. Qiu, "$\mathcal{H}_\infty$ design of general multirate sampled-data control systems," IMA Preprint Series # 1090, University of Minnesota, December 1992.

[9] C. Davis, M. M. Kahan, and H. F. Weinsberger, "Norm-preserving dilations and their applications to optimal error bounds," *SIAM J. Numer. Anal.*, vol. 19, pp. 445-469, 1982.

[10] A. Feintuch, P. P. Khargonekar, and A. Tannenbaum, "On the sensitivity minimization problem for linear time-varying periodic systems," *SIAM J. Control and Optimization*, vol. 24, pp. 1076-1085, 1986.

[11] T. T. Georgiou and P. P. Khargonekar, "A constructive algorithm for sensitivity optimization of periodic systems," *SIAM J. Control and Optimization*, vol. 25, pp. 334-340, 1987.

[12] K. Glover, D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally, and M. G. Safonov, "A characterization of all solution to the four block general distance problem," *SIAM J. Control and Optimization*, vol. 29, pp. 283-324, 1991.

[13] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, American Mathematical Society, Providence, Rhode Island, 1969.

[14] S. Hara and P. T. Kabamba, "Worst case analysis and design of sampled-data control systems," *Proc. CDC*, 1990.

[15] Y. Hayakawa, Y. Yamamoto, and S. Hara, "$\mathcal{H}_\infty$ type problem for sampled-data control system — a solution via minimum energy characterization," *Proc. CDC*, 1992.

[16] R. Kalman, Y. C. Ho, and K. Narendra, "Controllability of linear dynamical systems," in *Contributions to Differential Equations*, vol. 1, Interscience, New York, 1963.

[17] P. P. Khargonekar and N. Sivashankar, "$\mathcal{H}_2$ optimal control for sampled-data systems," *Systems and Control Letters*, vol. 18, No. 3, pp. 627-631, 1992.

[18] G. M. Kranc, "Input-output analysis of multirate feedback systems," *IRE Trans. Automat. Control*, vol. 3, pp. 21-28, 1957.

[19] D. G. Meyer, "A theorem on translating the general multi-rate LQG problem to a standard LQG problem via lifts," *Proc. ACC*, 1991.

[20] D. G. Meyer, "A parametrization of stabilizing controllers for multirate sampled-data systems," *IEEE Trans. Automat. Control*, vol. 35, pp. 233-236, 1990.

[21] S. Parrott, "On a quotient norm and the Sz.-Nagy-Foias lifting theorem", *J. Funct. Anal.*, vol. 30, pp. 311-328, 1978.

[22] L. Qiu and T. Chen, "$\mathcal{H}_2$ and $\mathcal{H}_\infty$ designs of multirate sampled-data systems," IMA Preprint Series # 1062, University of Minnesota, November 1992.

[23] R. Ravi, P. P. Khargonekar, K. D. Minto, and C. N. Nett, "Controller parametrization for time-varying multirate plants," *IEEE Trans. Automat. Control*, vol. 35, pp. 1259-1262, 1990.

[24] R. M. Redheffer, "On a certain linear fractional transformation," *J. Math. Phys.*, vol. 39, pp. 269-286, 1960.

[25] W. Sun, P. P. Khargonekar, K. M. Nagpal, and K. Poolla, "Digital control systems: $\mathcal{H}_\infty$ controller design with a zero-order hold function," *Proc. CDC*, 1992.

[26] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland Publishing Company, Amsterdam, 1970.

[27] G. Tadmor, "Optimal $\mathcal{H}_\infty$ sampled-data control in continuous time systems," *Proc. ACC*, 1991.

[28] H. T. Toivonen, "Sampled-data control of continuous-time systems with an $\mathcal{H}_\infty$ optimality criterion," *Automatica*, vol. 28, No. 1, pp. 45-54, 1992.

[29] P. G. Voulgaris and B. Bamieh, "Optimal $\mathcal{H}_\infty$ control of hybrid multirate systems," *Proc. CDC*, 1992.

[30] P. G. Voulgaris, M. A. Dahleh, and L. S. Valavani, "$\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers for periodic and multi-rate systems," *Proc. CDC*, 1991.

[31] Y. Yamamoto, "A new approach to sampled-data control systems: a function space method," *Proc. CDC*, 1990.

[32] N. Young, *An Introduction to Hilbert Spaces*, Cambridge University Press, Cambridge, UK, 1988.