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$\mathcal{H}_\infty$ Design of General Multirate Sampled-Data Control Systems[1]

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Abstract

Direct digital design of general multirate sampled-data systems is considered. To tackle causality constraints, a new and natural framework is proposed using nest operators and nest algebras. Based on this framework an explicit solution to the multirate $\mathcal{H}_\infty$ control problem is developed in the frequency domain.

I. Introduction

There are several reasons to use an MR (multirate) sampling scheme in digital control systems: (1) In complex, multivariable control systems, often it is unrealistic to sample all physical signals uniformly at one single rate. (2) For signals with different bandwidths, better trade-offs between performance and implementation cost can be obtained using A/D and D/A converters at different rates. (3) MR control systems can achieve what single-rate systems cannot; for example, gain margin improvement and simultaneous stabilization [16]. (4) Like single-rate controllers, many MR controllers do not violate the finite memory constraint in microprocessors.

The study of MR systems began in late 1950's [17]; recent interests are reflected in the LQG/LQR designs [1, 5, 21], the parametrization of all stabilizing controllers [19, 24], and the work in [2, 13]. Based on [19, 24], optimal MR control is potentially possible; but the causality constraint in controllers must be respected in design. This is similar to the case of discrete-time periodic control [9, 11, 31].

Our work has been influenced by the recent trend in SD (sampled-data) research, namely, direct digital design based on continuous-time performance specs. Related work on single-rate $\mathcal{H}_\infty$ design has been completed in [14, 29, 4, 26, 28, 15, 27]. In [23], we performed direct designs for an MR system with a uniform sampling rate and a uniform hold rate and proposed effective ways to tackle the causality constraint. Our goal in this paper is to treat *general* MR systems and give explicit solution to the $\mathcal{H}_\infty$ problem.

Two basic elements in SD systems are S_τ, the periodic sampler, and H_τ, the (zero-order) hold, both with period τ and synchronized at $t = 0$. The general MR system is shown in Figure 1. Here, G is the continuous-time generalized plant with two inputs, the exogenous input w and the control input u, and two outputs, the signal z to be controlled and the measured signal y. $\mathcal{S}$ and $\mathcal{H}$ are MR sampling and hold operators and are defined as follows:

$$\mathcal{S} = \begin{bmatrix} S_{m_1 h} & & \\ & \ddots & \\ & & S_{m_p h} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H_{n_1 h} & & \\ & \ddots & \\ & & H_{n_q h} \end{bmatrix}.$$

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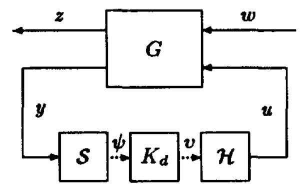

Figure 1: The general MR setup

If we partition the signals conformably

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \ \psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_p \end{bmatrix}, \ v = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix}, \ u = \begin{bmatrix} u_1 \\ \vdots \\ u_q \end{bmatrix},$$

then

$$\begin{aligned} \psi_i(k) &= y_i(km_i h), \ i = 1, \cdots, p \\ u_j(t) &= v_j(k), \ kn_j h \le t < (k+1)n_j h, \ j = 1, \cdots, q. \end{aligned}$$

K_d is the discrete-time MR controller, implemented via a microprocessor; it is synchronized with $\mathcal{S}$ and $\mathcal{H}$ in the sense that it inputs a value from the i-th channel at times $k(m_i h)$ and outputs a value to the j-th channel at $k(n_j h)$.

Introduce a useful notation: Given an operator K and an operator matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

the associated linear fractional transformation is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}.$$

Here we assume that the domains and co-domains of the operators are compatible and the inverse exists. With this notation, the closed-loop map $w \mapsto z$ in Figure 1 is $\mathcal{F}(G, \mathcal{H} K_d \mathcal{S})$.

Throughout the paper, G is LTI and finite-dimensional and K_d is linear; additional properties of K_d will be discussed in Section 3. Our purpose is to solve the following MR $\mathcal{H}_\infty$ control problem: Design a K_d to give closed-loop stability and achieve $\|\mathcal{F}(G, \mathcal{H} K_d \mathcal{S})\| < \gamma$ for a give $\gamma > 0$; here the norm is $\mathcal{L}_2$-induced and by proper scaling we can take $\gamma = 1$.

This paper is organized as follows. In Section 2, we give some concepts and facts on nest operators and nest algebras. Section 3 discusses desirable properties for MR controllers; in particular, causality is characterized using nest operators. Section 4 deals with internal stability of the setup in Figure 1. Section 5 contains the main contribution of this paper, namely, an explicit solution to the MR $\mathcal{H}_\infty$ control problem.

Throughout the paper, we choose to use λ-transforms instead of z-transforms, where $\lambda = z^{-1}$; in this case, discrete-time spaces such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ are defined on the open unit disk. Finally, $\hat{G}$ denotes the transfer matrix of G.

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II. Preliminaries

In this section we address some topics and computation on nests and nest algebras which are useful in the sequel. We shall restrict our attention to finite-dimensional spaces; more general treatment can be found in [3, 7].

Let $\mathcal{X}$ be a finite-dimensional space. A *nest* in $\mathcal{X}$, denoted $\{\mathcal{X}_i\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the nonincreasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{n-1} \supseteq \mathcal{X}_n = \{0\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be both finite-dimensional inner-product spaces with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ respectively. Assume the two nests have the same number of subspaces, say, $n+1$ as above. A linear map $T : \mathcal{X} \to \mathcal{Y}$ is a *nest operator* if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_i, \quad i = 0, 1, \cdots, n. \qquad (1)$$

Let $\Pi_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i$ and $\Pi_{\mathcal{Y}_i} : \mathcal{Y} \to \mathcal{Y}_i$ be orthogonal projections. Then the condition in (1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i} = 0, \quad i = 0, 1, \cdots, n.$$

The set of all such operators is denoted $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and abbreviated $\mathcal{N}(\{\mathcal{X}_i\})$ if $\{\mathcal{X}_i\} = \{\mathcal{Y}_i\}$. The following properties are straightforward to verify.

Lemma 1:

(a) If $T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, then $T_2T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$.

(b) $\mathcal{N}(\{\mathcal{X}_i\})$ forms an algebra, called *nest algebra*.

(c) If $T \in \mathcal{N}(\{\mathcal{X}_i\})$ and T is invertible, then $T^{-1} \in \mathcal{N}(\{\mathcal{X}_i\})$.

It is a useful fact that every operator on $\mathcal{X}$ can be factored as the product of a unitary operator and a nest operator.

Lemma 2: Let T be an operator on $\mathcal{X}$.

(a) There exists a unitary operator U_1 on $\mathcal{X}$ and an operator R_1 in $\mathcal{N}(\{\mathcal{X}_i\})$ such that $T = U_1R_1$.

(b) There exists an operator R_2 in $\mathcal{N}(\{\mathcal{X}_i\})$ and a unitary operator U_2 on $\mathcal{X}$ such that $T = R_2U_2$.

Since $\mathcal{X}_i \supseteq \mathcal{X}_{i+1}$, we write $(\mathcal{X}_{i+1})^{\perp}_{\mathcal{X}_i}$ as the orthogonal complement of $\mathcal{X}_{i+1}$ in $\mathcal{X}_i$. Decompose $\mathcal{X}$ into

$$\mathcal{X} = (\mathcal{X}_1)^{\perp}_{\mathcal{X}_0} \oplus (\mathcal{X}_2)^{\perp}_{\mathcal{X}_1} \oplus \cdots \oplus (\mathcal{X}_n)^{\perp}_{\mathcal{X}_{n-1}}.$$

It follows that under this decomposition any operator R belongs to $\mathcal{N}(\{\mathcal{X}_i\})$ iff its matrix is block lower-triangular, all the diagonal blocks being square. Thus the results in Lemma 2 follow from the well-known QR factorization.

Finally, we look at a distance problem. Let T be a linear operator $\mathcal{X} \to \mathcal{Y}$. We want to find the distance (via induced norms) of T to $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$, abbreviated $\mathcal{N}$:

$$\text{dist}\,(T, \mathcal{N}) := \inf_{Q \in \mathcal{N}} \|T - Q\|. \qquad (2)$$

Theorem 1:

$$\text{dist}\,(T, \mathcal{N}) = \max_i \|(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i}\|.$$

This is Corollary 9.2 in [7] specialized to operators on finite-dimensional spaces; it is an extension of a result in [22] on norm-preserving dilation of operators, which is specialized to matrices below. We denote the Moore-Penrose generalized inverse of a matrix M by $M^{\dagger}$.

Lemma 3: Assume that A, B, C are fixed matrices of appropriate dimensions. Then

$$\inf_X \left\| \begin{bmatrix} C & A \\ X & B \end{bmatrix} \right\| = \max\left\{ \left\| \begin{bmatrix} C & A \end{bmatrix} \right\|, \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\| \right\} := \alpha.$$

Moreover, a minimizing X is given by

$$X = -BA^*(\alpha I - AA^*)^{\dagger}C.$$

It will be of interest to us how to compute a Q to achieve the infimum in (2); this can be done by sequentially applying Lemma 3:

Step 1 Decompose the spaces $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{aligned} \mathcal{X} &= (\mathcal{X}_1)^{\perp}_{\mathcal{X}_0} \oplus (\mathcal{X}_2)^{\perp}_{\mathcal{X}_1} \oplus \cdots \oplus (\mathcal{X}_n)^{\perp}_{\mathcal{X}_{n-1}} \\ \mathcal{Y} &= (\mathcal{Y}_1)^{\perp}_{\mathcal{Y}_0} \oplus (\mathcal{Y}_2)^{\perp}_{\mathcal{Y}_1} \oplus \cdots \oplus (\mathcal{Y}_n)^{\perp}_{\mathcal{Y}_{n-1}}. \end{aligned}$$

We get matrix representations for T and Q, partitioned in the obvious way as $n \times n$ block matrices, with $Q_{ij} = 0, j > i$.

Step 2 Define $X_{ij} = T_{ij} - Q_{ij}$, $i \geq j$, and

$$P = \begin{bmatrix} X_{11} & T_{12} & \cdots & T_{1n} \\ X_{21} & X_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix}.$$

The problem reduces to

$$\min_{X_{ij}} \|P\|,$$

where T_{ij} are fixed. Minimizing X_{ij} can be selected as follows. First, set $X_{11}, \cdots, X_{n1}$ and $X_{n2}, \cdots, X_{nn}$ to zero. Second, choose X_{22} by Lemma 3 such that $\|(I - \Pi_{\mathcal{Y}_2})P\Pi_{\mathcal{X}_1}\|$ is minimized. Fix this X_{22}. Third, choose $\begin{bmatrix} X_{32} & X_{33} \end{bmatrix}$ again by Lemma 3 to minimize $\|(I - \Pi_{\mathcal{Y}_3})P\Pi_{\mathcal{X}_2}\|$. In this way, we can find all X_{ij} such that

$$\min_{X_{ij}} \|P\| = \max_i \|(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i}\|.$$

This procedure also gives a constructive proof of the theorem.

III. Multirate Systems

In this section we shall examine the MR controller K_d in Figure 1 as a discrete-time linear operator. To be studied are three basic properties: periodicity, causality, and finite dimensionality.

First, we look at periodicity. Let l be the least common multiple of the set of integers $\{m_1, \cdots, m_p, n_1, \cdots, n_q\}$. Thus $\sigma := lh$ is the least common period for all sampling and hold channels. K_d can be chosen so that $\mathcal{H}K_d\mathcal{S}$ is σ-periodic in continuous time. For this, we need a few definitions.

Let ℓ be the space of sequences, perhaps vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. Let U be the unit time delay on ℓ and U^* the unit time advance. Define

$$\bar{m}_i = l/m_i, \quad i = 1, \cdots, p, \quad \bar{n}_j = l/n_j, \quad j = 1, \cdots, q.$$

We say K_d is (m_i, n_j)*-periodic* if

$$\begin{bmatrix} (U^*)^{n_1} & & \\ & \ddots & \\ & & (U^*)^{n_q} \end{bmatrix} K_d \begin{bmatrix} U^{m_1} & & \\ & \ddots & \\ & & U^{m_p} \end{bmatrix} = K_d.$$

It follows easily that $\mathcal{H}K_d\mathcal{S}$ is σ-periodic in continuous time iff K_d is (m_i, n_j)-periodic. Since G is LTI, the SD system in Figure 1 is σ-periodic if K_d is (m_i, n_j)-periodic. We shall refer to σ as the *system period.*

Now we lift K_d to get an LTI system. For an integer $m > 0$, define the discrete lifting operator L_m via $\underline{v} = L_m v$,

$$\{v(0), v(1), \cdots\} \mapsto \left\{ \begin{bmatrix} v(0) \\ \vdots \\ v(m-1) \end{bmatrix}, \begin{bmatrix} v(m) \\ \vdots \\ v(2m-1) \end{bmatrix}, \cdots \right\}$$

and the operator matrices

$$\mathcal{L}_n := \begin{bmatrix} L_{n_1} & & \\ & \ddots & \\ & & L_{n_q} \end{bmatrix}, \quad \mathcal{L}_m := \begin{bmatrix} L_{m_1} & & \\ & \ddots & \\ & & L_{m_p} \end{bmatrix}.$$

The lifted controller is $\underline{K_d} = \mathcal{L}_n K_d \mathcal{L}_m^{-1}$. It is an easy matter to check, see, e.g., [20], that $\underline{K_d}$ is LTI iff K_d is (m_i, n_j)-periodic.

Next is causality. For $\underline{K_d}$ to be implementable in real time, $\mathcal{H}K_d\mathcal{S}$ must be causal in continuous time. This implies that $\underline{K_d}$, as a single-rate system, must be causal; and moreover, the feedthrough term $\underline{D}$ in $\underline{K_d}$ must satisfy a certain constraint, that is, some blocks in $\underline{D}$ must be zero [19, 24]. Now let us characterize this constraint on $\underline{D}$ using nest operators.

Write $\underline{v} = \underline{K_d}\,\underline{\psi}$; then $\underline{v}(0) = \underline{D}\underline{\psi}(0)$. Let Σ be the set of sampling or hold instants in the interval $[0, \sigma)$ (modulo the base period h). This is a finite set of, say, $n+1$ integers; order Σ increasingly ($\sigma_r < \sigma_{r+1}$):

$$\Sigma = \{\sigma_r : \; r = 0, 1, \cdots, n\}.$$

Let $\underline{\psi}(0)$ and $\underline{v}(0)$ live in the finite-dimensional spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively. For $r = 0, 1, \cdots, n$, define

$$\begin{aligned} \mathcal{X}_r &= \text{span}\,\{\underline{\psi}(0) : \; \psi_i(k) = 0 \text{ if } km_i < \sigma_r\} \\ \mathcal{Y}_r &= \text{span}\,\{\underline{v}(0) : \; v_j(k) = 0 \text{ if } kn_j < \sigma_r\}. \end{aligned}$$

$\mathcal{X}_r$ and $\mathcal{Y}_r$ correspond to, respectively, the inputs and outputs occurring from time $\sigma_r h$ on. It is easily checked that $\{\mathcal{X}_r\}$ and $\{\mathcal{Y}_r\}$ are nests and that the causality condition on $\underline{D}$ (the output at time $\sigma_r h$ depends only on inputs up to $\sigma_r h$) is exactly

$$\underline{D}\mathcal{X}_r \subseteq \mathcal{Y}_r, \quad r = 0, 1, \cdots, n.$$

Thus we define $\underline{D}$ to be (m_i, n_j)*-causal* if $\underline{D} \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$. For completeness, we define $\underline{D}$ to be (m_i, n_j)*-strictly causal* if

$$\underline{D}\mathcal{X}_r \subseteq \mathcal{Y}_{r+1}, \quad r = 0, 1, \cdots, n-1.$$

This means that the output at time $\sigma_{r+1}h$ depends only on inputs up to time $\sigma_r h$.

The following lemma, which is easy to prove, justifies our use of terminology from a continuous-time viewpoint.

Lemma 4:

(a) $\mathcal{H}K_d\mathcal{S}$ is causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-causal.

(b) $\mathcal{H}K_d\mathcal{S}$ is strictly causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-strictly causal.

Some conclusions on causality issues [19] are transparent from Lemmas 1 and 4 under this new formulation.

Lemma 5:

(a) If $\underline{D}_1$ is (m_i, p_k)-causal and $\underline{D}_2$ is (p_k, n_j)-causal, then $\underline{D}_2\underline{D}_1$ is (m_i, n_j)-causal; furthermore, if $\underline{D}_1$ or $\underline{D}_2$ is strictly causal, then $\underline{D}_2\underline{D}_1$ is also strictly causal.

(b) If $\underline{D}$ is (m_i, m_i)-causal and invertible, then $\underline{D}^{-1}$ is (m_i, m_i)-causal.

(c) If $\underline{D}$ is (m_i, m_i)-strictly causal, then $(I - \underline{D})^{-1}$ exists and is (m_i, m_i)-causal.

We assume K_d is (m_i, n_j)-periodic and -causal. Then $\underline{K_d}$ is LTI and causal. To get finite-dimensional difference equations for K_d, we further assume $\underline{K_d}$ is finite-dimensional. Thus $\underline{K_d}$ has state space equations

$$\begin{aligned} \eta(k+1) &= A\eta(k) + \sum_{i=1}^{p} B_i \underline{\psi}_i(k), \\ \underline{v}_j(k) &= C_j\eta(k) + \sum_{i=1}^{p} D_{ji}\underline{\psi}_i(k), \quad j = 1, 2, \cdots, q. \end{aligned}$$

Note that $\underline{\psi}_i = L_{m_i}\psi_i$ and $\underline{v}_j = L_{n_j}v_j$. Partitioning the matrices accordingly

$$B_i = \begin{bmatrix} (B_i)_0 & \cdots & (B_i)_{m_i-1} \end{bmatrix}, \quad C_j = \begin{bmatrix} (C_j)_0 \\ \vdots \\ (C_j)_{n_j-1} \end{bmatrix},$$

$$D_{ji} = \begin{bmatrix} (D_{ji})_{00} & \cdots & (D_{ji})_{0,m_i-1} \\ \vdots & & \vdots \\ (D_{ji})_{n_j-1,0} & \cdots & (D_{ji})_{n_j-1,m_i-1} \end{bmatrix}$$

(certain blocks in D_{ji} must be zero for the causality constraint), we get the difference equations for K_d ($v = K_d\psi$):

$$\begin{aligned} \eta(k+1) &= A\eta(k) + \sum_{i=1}^{p}\sum_{s=0}^{m_i-1} (B_i)_s \psi_i(k\bar{m}_i + s) \\ v_j(k\bar{n}_j + r) &= (C_j)_r\eta(k) + \sum_{i=1}^{p}\sum_{s=0}^{m_i-1} (D_{ji})_{rs}\psi_i(k\bar{m}_i + s), \end{aligned}$$

where the indices in (4) go as follows: $j = 1, 2, \cdots, q$ and $r = 0, 1, \cdots, \bar{n}_j - 1$. These are the equations for implementing K_d on computers and they require only finite memory. Note that the state vector η for K_d is updated every system period σ.

In summary, the *admissible* class of $\underline{K_d}$ is characterized by LTI, causal, and finite-dimensional $\underline{K_d}$ with $\underline{D}$ (m_i, n_j)-causal.

IV. Internal Stability

In this section we look at stability of Figure 1. We assume the continuous G has a state model:

$$\hat{G} = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]. \qquad (3)$$

Let the plant state be x and the controller state be η (K_d is admissible). Note that the system in Figure 1 is σ-periodic. Define the continuous-time vector

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ \eta(k) \end{bmatrix}, \quad k\sigma \le t < (k+1)\sigma.$$

The (autonomous) system in Figure 1 is *internally stable*, or K_d *internally stabilizes* G, if for any initial value $x_{sd}(t_0)$, $0 \leq t_0 < \sigma$, $x_{sd}(t) \to 0$ as $t \to \infty$.

Introduce $G_{22d} = \mathcal{S}G_{22}\mathcal{H}$, the MR discretization of G_{22}. Now lift K_d as before and G_{22d} by $\underline{G_{22d}} = \mathcal{L}_m G_{22d} \mathcal{L}_n^{-1}$. Because G_{22} is LTI and strictly causal, G_{22d} is (n_j, m_i)-periodic and -strictly causal. Thus $\underline{G_{22d}}$ is LTI and causal with $\underline{D}_{22d}$ (n_j, m_i)-strictly causal. In fact, a state model for $\underline{G_{22d}}$ can be obtained (Lemma 6 below).

Theorem 2: K_d internally stabilizes G iff $\underline{K_d}$ internally stabilizes $\underline{G_{22d}}$.

The proof is contained in [6]. Sufficient conditions for the internal stability to be achievable are that (A, B_2) and (C_2, A) are stabilizable and detectable respectively and that the system period σ is non-pathological in a certain sense, see, e.g., [18, 23].

V. $\mathcal{H}_\infty$-Optimal Control

With reference to Figure 1, we now study the $\mathcal{H}_\infty$ synthesis problem: Design an admissible K_d that internally stabilizes G and achieves $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\| < 1$.

The general idea in the solution is to reduce the MR problem to a discrete $\mathcal{H}_\infty$ model-matching problem with the causality constraint and then solve the constrained problem explicitly using techniques presented in Section 2 on nest operators and nest algebras. A special case of the reduction process was reported in [23].

We start with a state model for G in (5) with $D_{11} = 0$ and $D_{21} = 0$. We shall assume that (A, B_2) is stabilizable and (C_2, A) is detectable.

$\mathcal{H}_\infty$ *Discretization*

The original problem is posed in continuous time; so the first step is to recast it as a discrete-time problem with time-varying control. The reduction is based on recent advances in $\mathcal{H}_\infty$ SD control in the single-rate setting.

Introduce the *discrete sampling operator* $S_m : \ell \to \ell$ defined via

$$\psi = S_m \phi \iff \psi(k) = \phi(km)$$

and the *discrete hold operator* $H_n : \ell \to \ell$ via

$$v = H_n \phi \iff v(kn + r) = \phi(k), \quad r = 0, 1, \cdots, n-1.$$

It is easily checked that $S_{m_i h} = S_{m_i} S_h$ and $H_{n_j h} = H_h H_{n_j}$. So the MR sampling and hold operators $\mathcal{S}$ and $\mathcal{H}$ can be factored as $\mathcal{S} = \mathcal{S}_m S_h$ and $\mathcal{H} = H_h \mathcal{H}_n$, where

$$\mathcal{S}_m = \begin{bmatrix} S_{m_1} & & \\ & \ddots & \\ & & S_{m_p} \end{bmatrix}, \quad \mathcal{H}_n = \begin{bmatrix} H_{n_1} & & \\ & \ddots & \\ & & H_{n_q} \end{bmatrix}.$$

Defining $K_{d1} = \mathcal{H}_n K_d \mathcal{S}_m$, we can view the MR system $\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})$ as a fictitious single-rate system $\mathcal{F}(G, S_h K_{d1} H_h)$. Now the results in, e.g., [4] are applicable.

Let $\underline{D}_{11h} : \mathcal{L}_2[0, h) \to \mathcal{L}_2[0, h)$ be defined by

$$(\underline{D}_{11h} w)(t) = C_1 \int_0^t e^{(t-\tau)A} B_1 w(\tau)\, d\tau$$

and assume $\|\underline{D}_{11h}\| < 1$. Since $\underline{D}_{11h}$ is the restriction of $\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})$ on $\mathcal{L}_2[0, h)$, this condition is necessary for $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\| < 1$; how to verify this condition was studied in [4]. For the MR $\mathcal{H}_\infty$ problem, invoke the single-rate results to get the equivalent discrete-time problem: Design K_{d1} to give internal stability and achieve $\|\mathcal{F}(G_d, K_{d1})\| < 1$, where the norm now is ℓ_2-induced and the $\mathcal{H}_\infty$ discretization G_d (for $\gamma = 1$) has a state model

$$\hat{G}_d = \begin{bmatrix} \hat{G}_{11d} & \hat{G}_{12d} \\ \hat{G}_{21d} & \hat{G}_{22d} \end{bmatrix} = \left[\begin{array}{c|cc} A_d & B_{1d} & B_{2d} \\ \hline C_{1d} & D_{11d} & D_{12d} \\ C_{2d} & 0 & 0 \end{array}\right].$$

The computation of the matrices in $\hat{G}_d$ is given in, e.g., [4]. In this way, we arrive at an equivalent discrete $\mathcal{H}_\infty$ problem; however, K_{d1} is constrained to be of the form $K_{d1} = \mathcal{H}_n K_d \mathcal{S}_m$ with K_d admissible.

Discrete Lifting

The system $\mathcal{F}(G_d, K_{d1})$ is single-rate with period h. The next step is to lift to get an LTI system with period σ. Define $\underline{K_d}$ as before and

$$\underline{G_d} = \begin{bmatrix} L_l & 0 \\ 0 & \mathcal{L}_m \mathcal{S}_m \end{bmatrix} G_d \begin{bmatrix} L_l^{-1} & 0 \\ 0 & \mathcal{H}_n \mathcal{L}_n^{-1} \end{bmatrix}$$

to get the lifted system $\mathcal{F}(\underline{G_d}, \underline{K_d})$. Since G_d is LTI, causal, and finite-dimensional with G_{22d} strictly causal, we can show that $\underline{G_d}$ is LTI, causal, and finite-dimensional. Moreover, the feedthrough term $\underline{D}_{22d}$ of $\underline{G_{22d}}$ is (n_j, m_i)-strictly causal. In fact, a state model for $\underline{G_d}$ can be obtained using the lemma below.

Let P be a discrete-time system with state ξ and the corresponding realization (A, B, C, D). Let $m, n, \bar{m}, \bar{n}, l$ be positive integers such that $m\bar{m} = n\bar{n} = l$. Define

$$\underline{P} := L_m S_m P H_n L_n^{-1}$$

and the characteristic function on integers

$$\chi_{[p,q)}(r) = \begin{cases} 1, & p \leq r < q \\ 0, & \text{else.} \end{cases}$$

Lemma 6: A state model for $\underline{P}$ is

$$\hat{\underline{P}} = \left[\begin{array}{c|ccc} A^l & \sum_{r=0}^{n-1} A^{l-1-r}B & \cdots & \sum_{r=l-n}^{l-1} A^{l-1-r}B \\ \hline C & D_{00} & \cdots & D_{0,\bar{n}-1} \\ CA^m & D_{10} & \cdots & D_{1,\bar{n}-1} \\ \vdots & \vdots & & \vdots \\ CA^{l-m} & D_{\bar{m}-1,0} & \cdots & D_{\bar{m}-1,\bar{n}-1} \end{array}\right],$$

where

$$D_{ij} = D\chi_{[jn,(j+1)n)}(im) + \sum_{r=jn}^{(j+1)n-1} CA^{im-1-r}B\chi_{[0,im)}(r).$$

The corresponding state vector is $\underline{\xi} = S_l \xi$.

The lemma can be proven by manipulating the input-output equations for P. Note that the transfer matrices for all blocks in $\underline{G_d}$ can be obtained from this lemma.

From the definitions of $\underline{K_d}$ and $\underline{G_d}$, we get after some algebra that $\mathcal{F}(\underline{G_d}, \underline{K_d}) = L_l \mathcal{F}(G_d, K_{d1}) L_l^{-1}$. So $\|\mathcal{F}(\underline{G_d}, \underline{K_d})\| = \|\mathcal{F}(G_d, K_{d1})\|$ since L_l is norm-preserving. Thus the equivalent LTI problem is now: Design an admissible $\underline{K_d}$ that internally stabilizes $\underline{G_d}$ and achieves $\|\mathcal{F}(\hat{\underline{G_d}}, \hat{\underline{K_d}})\|_\infty < 1$. Notice that the feedthrough term $\hat{\underline{K_d}}(0)$ must be (m_i, n_j)-causal; so this is a constrained $\mathcal{H}_\infty$ control problem in discrete time.

Constrained Model-Matching Problem

Parametrizing the stabilizing controllers for $\underline{G_d}$ as in [10], we get

$$\mathcal{F}(\underline{\hat{G}_d}, \underline{\hat{K}_d}) = \hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3.$$

The causality constraint on $\underline{\hat{K}_d}(0)$ translates exactly to $\hat{Q}(0)$ [19, 24]. In this way we arrive at the constrained $\mathcal{H}_\infty$ model-matching problem: Find $\hat{Q} \in \mathcal{RH}_\infty$ with $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ (the nests $\{\mathcal{X}_r\}$ and $\{\mathcal{Y}_r\}$ were defined in Section 3) such that

$$\|\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3\|_\infty < 1.$$

If such a $\hat{Q}$ exists, we say the MR $\mathcal{H}_\infty$ problem is *solvable*.

We note here that a different procedure was reported which converts an MR $\mathcal{H}_\infty$ problem into a discrete model-matching problem [30].

An Explicit Solution

We write $\hat{T}^\sim(\lambda)$ for $\hat{T}(\lambda^{-1})'$. For regularity, we need the following assumption:

For each $|\lambda| = 1$, $\hat{T}_2(\lambda)$ and $\hat{T}_3^\sim(\lambda)$ are both injective.

Under this assumption, there exists an inner-outer factorization $\hat{T}_2 = \hat{T}_{2i}\hat{T}_{2o}$ and an co-inner-outer factorization $\hat{T}_3 = \hat{T}_{3co}\hat{T}_{3ci}$, where $\hat{T}_{2o}$ and $\hat{T}_{3co}$ are both invertible over $\mathcal{RH}_\infty$. Furthermore, these factorizations can be performed in such a way that $\hat{T}_{2o}(0) \in \mathcal{N}(\{\mathcal{Y}_r\})$ and $\hat{T}_{3co}(0) \in \mathcal{N}(\{\mathcal{X}_r\})$. To see this, let us assume that an inner-outer factorization $\hat{T}_2 = \hat{T}_{2i}\hat{T}_{2o}$ is obtained with $\hat{T}_{2o}(0) \notin \mathcal{N}(\{\mathcal{Y}_r\})$. By Lemma 2, we have factorization $\hat{T}_{2o}(0) = U_1R_1$ where U_1 is orthogonal and $R_1 \in \mathcal{N}(\{\mathcal{Y}_r\})$. Then a new inner-outer factorization of $\hat{T}_{2o}$ is given by $\hat{T}_2 = (\hat{T}_{2i}U_1)(U_1'\hat{T}_{2o})$ with $(U_1'\hat{T}_{2o})(0) = R_1 \in \mathcal{N}(\{\mathcal{Y}_r\})$. A similar argument applies to the co-inner-outer factorization of $\hat{T}_{3co}$.

Now bring in an inner-outer factorization $\hat{T}_2 = \hat{T}_{2i}\hat{T}_{2o}$ and a co-inner-outer factorization $\hat{T}_3 = \hat{T}_{3co}\hat{T}_{3ci}$ with $\hat{T}_{2o}(0) \in \mathcal{N}(\{\mathcal{Y}_r\})$ and $\hat{T}_{3co}(0) \in \mathcal{N}(\{\mathcal{X}_r\})$. Apply unitary transformations to $\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3$ and define $\hat{Q}_1 = \hat{T}_{2o}\hat{Q}\hat{T}_{3co}$ and $\hat{R}$ via

$$\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} = \begin{bmatrix} \hat{T}_{2i}^\sim \\ I - \hat{T}_{2i}\hat{T}_{2i}^\sim \end{bmatrix} \hat{T}_1 \begin{bmatrix} \hat{T}_{3ci}^\sim & I - \hat{T}_{3ci}^\sim\hat{T}_{3ci} \end{bmatrix}.$$

The constrained model-matching problem is equivalent to the following four-block problem of finding a $\hat{Q}_1 \in \mathcal{RH}_\infty$ with $\hat{Q}_1(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ such that

$$\left\| \begin{bmatrix} \hat{R}_{11} - \hat{Q}_1 & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} \right\|_\infty < 1. \tag{4}$$

Dropping the causality constraint on $\hat{Q}_1(0)$ temporarily allows us to parametrize all $\hat{Q}_1$ in $\mathcal{RH}_\infty$ achieving (6). We know from [8] that there exists a $\hat{Q}_1 \in \mathcal{RH}_\infty$ such that (6) holds iff

$$\left\| \begin{bmatrix} \Pi_{\mathcal{H}_2^\perp} & 0 \\ 0 & I \end{bmatrix} \hat{R}|_{\mathcal{H}_2 \oplus \mathcal{L}_2} \right\| < 1. \tag{5}$$

If (7) is satisfied, then a procedure in [12] allows us to find an $\mathcal{RH}_\infty$ matrix

$$\hat{K} = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix}$$

with $\hat{K}_{12}^{-1}, \hat{K}_{21}^{-1} \in \mathcal{RH}_\infty$ and $\|\hat{K}_{22}\|_\infty < 1$ such that all $\hat{Q}_1 \in \mathcal{RH}_\infty$ satisfying (6) are characterized by

$$\hat{Q}_1 = \mathcal{F}(\hat{K}, \hat{Q}_2), \quad \hat{Q}_2 \in \mathcal{RH}_\infty, \quad \|\hat{Q}_2\|_\infty < 1. \tag{6}$$

We refer to [12] for the details of checking inequality (7) and the expression of $\hat{K}$. Hereafter, we shall assume that (7) is true. This is also necessary for the solvability of the MR $\mathcal{H}_\infty$ problem.

In general $\hat{K}_{22}(0) \neq 0$, so $\hat{Q}_1(0)$ depends on $\hat{Q}_2(0)$ in a linear fractional manner. However, it is possible to simplify this relation by introducing another linear fractional transformation [23]:

$$\hat{Q}_2 = \mathcal{F}(V, \hat{Q}_3).$$

Here V, partitioned as usual, is a constant unitary matrix. It follows that the mapping $\hat{Q}_3 \mapsto \hat{Q}_2$ is bijective from the open unit ball of $\mathcal{RH}_\infty$ onto itself [25]. Thus all $\hat{Q}_1$ satisfying (6) can be re-parametrized by

$$\begin{aligned} \hat{Q}_1 &= \mathcal{F}[\hat{K}, \mathcal{F}(V, \hat{Q}_3)] \\ &= \mathcal{F}(\hat{L}, \hat{Q}_3), \quad \hat{Q}_3 \in \mathcal{RH}_\infty, \quad \|\hat{Q}_3\|_\infty < 1. \end{aligned}$$

For $\hat{L}_{22}(0) = 0$, we choose the unitary matrix V to be

$$V = \begin{bmatrix} \hat{K}_{22}'(0) & \left[I - \hat{K}_{22}'(0)\hat{K}_{22}(0)\right]^{1/2} \\ \left[I - \hat{K}_{22}(0)\hat{K}_{22}'(0)\right]^{1/2} & -\hat{K}_{22}(0) \end{bmatrix}.$$

$\hat{L}$ can be obtained from $\hat{K}$ and V. It can be checked that $\hat{L}_{12}(0)$ and $\hat{L}_{21}(0)$ are still nonsingular.

To recap, the set of all $Q_1 \in \mathcal{RH}_\infty$ achieving (6) is parametrized by

$$\hat{Q}_1 = \mathcal{F}(\hat{L}, \hat{Q}_3), \quad \hat{Q}_3 \in \mathcal{RH}_\infty, \quad \|\hat{Q}_3\|_\infty < 1.$$

Here $\hat{L}$ has the desirable properties that $\hat{L}_{22}(0) = 0$, $\hat{L}_{12}(0)$ and $\hat{L}_{21}(0)$ are nonsingular. Thus

$$\hat{Q}_1(0) = \hat{L}_{11}(0) + \hat{L}_{12}(0)\hat{Q}_3(0)\hat{L}_{21}(0). \tag{7}$$

This is an affine function $\hat{Q}_3(0) \mapsto \hat{Q}_1(0)$.

Now we bring in the causality constraint on $\hat{Q}_1(0)$. Our goal is to find a $\hat{Q}_3 \in \mathcal{RH}_\infty$ with $\|\hat{Q}_3\|_\infty < 1$ such that $\hat{Q}_1(0)$ in (9) lies in $\mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$. Since $\hat{Q}_1(0)$ depends only on $\hat{Q}_3(0)$ and in general $\|\hat{Q}_3\|_\infty \geq \|\hat{Q}_3(0)\|$, the equivalent problem is to find a constant matrix $\hat{Q}_3(0)$ with $\|\hat{Q}_3(0)\| < 1$ such that $\hat{Q}_1(0) \in \mathcal{N}$.

Now we use Lemma 2 to reduce the problem to a distance problem. Introduce matrix factorizations (Lemma 2)

$$L_{12}(0) = R_1U_1, \quad L_{21}(0) = -U_2R_2,$$

where R_1, R_2, U_1, U_2 are all invertible, U_1, U_2 are orthogonal, and R_1, R_2 belongs to the nest algebras $\mathcal{N}(\{\mathcal{Y}_r\}), \mathcal{N}(\{\mathcal{X}_r\})$ respectively.

Substitute the factorizations into (9) and pre- and post-multiply by R_1^{-1} and R_2^{-1} respectively to get

$$R_1^{-1}\hat{Q}_1(0)R_2^{-1} = R_1^{-1}\hat{L}_{11}(0)R_2^{-1} - U_1\hat{Q}_3(0)U_2.$$

Define

$$W = R_1^{-1}\hat{Q}_1(0)R_2^{-1}, \quad T = R_1^{-1}\hat{L}_{11}(0)R_2^{-1}, \quad P = U_1\hat{Q}_3(0)U_2.$$

It follows that $\hat{Q}_1(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ iff $W \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ (Lemma 1) and $\|\hat{Q}_3(0)\| < 1$ iff $\|P\| < 1$. Therefore, we arrive at the following equivalent matrix problem: Given T, find P with $\|P\| < 1$ such that $W = T - P \in \mathcal{N}$; or equivalently, find

$W \in \mathcal{N}$ such that $\|T - W\| < 1$. This can be solved via the distance problem studied in Theorem 1:

$$\text{dist}\,(T,\mathcal{N}) = \max_r\{\|(I - \Pi_{\mathcal{Y}_r})T\Pi_{\mathcal{X}_r}\|\} =: \mu.$$

Let $W_{opt} \in \mathcal{N}$ achieve the distance, i.e., $\|T - W_{opt}\| = \mu$. The following result summarizes what we have derived.

Theorem 3: The matrix problem is solvable, i.e., there exists a matrix P with $\|P\| < 1$ such that $T - P \in \mathcal{N}$, iff $\mu < 1$. Moreover, if $\mu < 1$, $P := T - W_{opt}$ solves the problem with $\|P\| = \mu$.

How to compute μ and W_{opt} were discussed in the procedure given at the end of section 2.

To summarize, let us list the solvability conditions for the MR $\mathcal{H}_\infty$ control problem $\|\mathcal{F}(G, \mathcal{H}K_d\mathcal{S})\| < 1$:

(a) $\|\underline{D}_{11h}\| < 1$;

(b) $\left\| \begin{bmatrix} P_{\mathcal{H}_2^\perp} & 0 \\ 0 & I \end{bmatrix} \hat{R}|_{\mathcal{H}_2 \oplus \mathcal{L}_2} \right\| < 1$;

(c) $\mu < 1$.

Condition (a) was studied in detail in [4] and would usually be satisfied for a reasonable design. Condition (b) is the solvability condition for a standard four-block $\mathcal{H}_\infty$ problem, see, e.g., [12] for checking this condition. When conditions (a-b) hold, condition (c) amounts to computing the norms of several constant matrices.

Finally, we remark that an MR $\mathcal{H}_2$-optimal control problem is also solved explicitly in the full paper [6].

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