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# Limitations on Maximal Tracking Accuracy, Part 2: Tracking Sinusoidal and Ramp Signals *

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## Abstract

This paper continues an earlier work on optimal tracking problems with respect to multivariable feedback systems. We examine the error between a system's output response and its command input, which is either a sinusoidal or a ramp signal. This error is quantified under an integral square criterion, and is adopted as a measure of tracking accuracy. For both sinusoidal and ramp signals, we derive explicit expressions of the minimal tracking error attainable by all possible stabilizing controllers. These results show that the tracking accuracy may be fundamentally limited by plant nonminimum phase zeros and unstable poles together with their spatial properties, in ways dependent upon the command input signals.

## 1 Introduction

In an earlier companion paper [2], the authors studied a classical tracking problem [5, 6, 7] pertaining to finite dimensional, linear, time invariant, multivariable feedback systems. An integral square criterion was adopted in our study to quantify the error between a system's output response and its input, which, as a measure of tracking accuracy, determines how well the output may track the input signal. In the case when the input is a certain generalized unit step signal, we obtained an explicit expression of the minimal tracking error attainable by all possible stabilizing controllers. It became clear from this work that a system's tracking performance can be severely limited by plant characteristics such as nonminimum phase zeros, unstable poles, and time delays. One particularly noteworthy outcome is that in a multivariable system the tracking error depends not only upon the locations of the zeros and poles, but also on their directions. This phenomenon is most interesting, and it points to a distinguishing feature only seen in multivariable systems. As an immediate implication, the result indicates that while for a single-input single-output nonminimum phase plant it is impossible to achieve perfect tracking, it can be accomplished for a multivariable nonminimum phase plant whenever the input signal is properly aligned with the zero directions.

The present paper continues the investigation in [2]. We revisit the same tracking problem but consider different classes of input signals. The motive here is a simple one. While our previous study has yielded important insight toward limitations upon a system's tracking ability in general, we hope a deeper investigation of these issues by making use of other different signals will lead to a better understanding. This motivation appears to be well-founded, and is rooted in the observation that tracking performance generally varies with command signals, and that in tracking different types of signals a system may behave in a drastically different way; indeed, classical analysis of steady-state tracking error serves as a good example. We shall consider specifically sinusoidal and ramp signals. Our consideration of these signals stems from the fact that they, like the unit step signal, are among the most frequently used signals for assessing transient and steady-state performance. We shall derive an explicit formula for the minimal tracking error in each case. Similar to their counterparts in [2], these results provide explicit characterizations on the intrinsic tracking error irreducible via compensator design, and they shed new lights on how plant nonminimum phase zeros and unstable poles may fundamentally limit the tracking performance. Specifically, while the results obtained herein bear much resemblence to [2], in that the minimal tracking error is shown to depend on plant nonminimum phase zeros and unstable poles together with their spatial properties, it will be seen that in each case the zeros and poles lead to performance degradation in rather different a manner. This difference is clarified through our study of the problem by use of different input signals.

We collect below some relevant notations from [2]. For any complex number $z$, we denote its complex conjugate by $\bar{z}$. For any vector $u$, we denote its conjugate transpose by $u^H$. For any signal $u(t)$, we denote its Laplace transform by $\hat{u}(s)$. The conjugate transpose of a matrix $A$ is denoted by $A^H$. All the vectors and matrices involved in the sequel are assumed to have compatible dimensions, and for simplicity, their dimensions are omitted. We denote the open right half plane by $\mathbb{C}_+$ and the imaginary axis by $\mathbb{C}_0$. Moreover, let $\|\cdot\|$ denote the Euclidean vector norm and define,

$$\mathcal{L}_2 := \left\{ f : \quad f(s) \text{ measurable in } \mathbb{C}_0, \ \|f\|_2 < \infty \right\},$$

where

$$\|f\|_2 := \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(j\omega)\|^2 d\omega \right)^{1/2}.$$

Finally, let $\mathcal{H}_\infty$ denote the set of all bounded analytic functions defined in the open right half plane and $\mathbb{R}\mathcal{H}_\infty$ denote the set of all rational functions in $\mathcal{H}_\infty$.

## 2 Preliminaries

Consider the finite dimensional linear time-invariant feedback system depicted in Figure 1, where the signals $u$, $y$, and $e$ represent the command input, the system output, and the error signal between $y$ and $u$. The transfer function matrices $P$ and $K$ are those of the plant model and the compensator, respectively. Let the system be at rest initially. Then, For a given input signal $u$, we define the tracking error between $u$ and its output response $y$ as

$$J := \int_0^\infty \|e(t)\|^2 dt,$$

and we use $J$ as a measure of the system's tracking ability, specifically in reference to the signal $u$. We will be particularly interested in the minimal error attainable by all possible stabilizing

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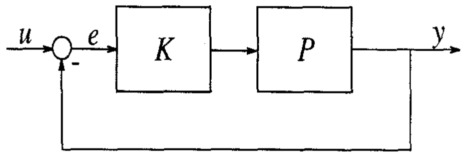


Figure 1: The feedback system

compensators. This quantity provides an intrinsic measure of the difficulty in tracking an input signal and is defined below.

Let the right and left coprime factorizations of $P$ be given by

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{2.1}$$

where $N,\ M,\ \tilde{N},\ \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$ and satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I, \tag{2.2}$$

for some $X,\ Y,\ \tilde{X},\ \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$. It is well-known that the set of all stabilizing compensators $K$ can be described via the so-called Youla parameterization [3]

$$\mathcal{K} := \left\{K:\ K = (Y - MQ)(NQ - X)^{-1},\ Q \in \mathbb{R}\mathcal{H}_\infty\right\}. \tag{2.3}$$

Based upon this characterization, we define the minimal tracking error by

$$J^* := \inf_{K \in \mathcal{K}} J.$$

Let the system sensitivity function be denoted by

$$S(s) := (I + P(s)K(s))^{-1}.$$

It follows that

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|S(j\omega)\hat{u}(j\omega)\|^2 d\omega = \|S(s)\hat{u}(s)\|_2^2, \tag{2.4}$$

and hence

$$J^* = \inf_{K \in \mathcal{K}} \|S(s)\hat{u}(s)\|_2^2. \tag{2.5}$$

The latter expression indicates the possibility of a frequency domain approach for finding $J^*$. Such an approach has been previously adopted in [6, 2] and will be used in the present paper as well.

Throughout this paper we shall need the following assumption in order for the tracking problem to be well-posed.

**Assumption 2.1** *The transfer function matrix $P(s)$ has full row rank for at least one $s$.*

Stated in words,, this condition implies that the plant transfer function is right invertible, which is required to insure that the tracking error be finite.

We now describe an allpass factorization of nonminimum phase transfer functions matrices. Let $(A,\ B,\ C,\ D)$ be a minimal realization of $P(s)$. Then, a point $z \in \mathbb{C}_+$ is called a nonminimum phase zero of $P$ if there exist vectors $\eta$ and $\zeta$ such that the relation

$$\begin{bmatrix} \zeta^H & \eta^H \end{bmatrix} \begin{bmatrix} zI - A & -B \\ -C & -D \end{bmatrix} = 0,$$

holds, where $\|\eta\| = 1$, and $\eta$ is called the output zero direction vector associated with $z$. Moreover, a complex number $p \in \mathbb{C}_+$ is said to be an unstable pole of $P$ if it is an eigenvalue of $A$. It can be readily shown that $z \in \mathbb{C}_+$ is a zero of $P$ with an output direction vector $\eta$ if and only if $\eta^H N(z) = 0$. Similarly, it can be shown that $p \in \mathbb{C}_+$ is a pole of $P$ if there exists a vector $w$ such that $\tilde{M}(p)w = 0$, where $\|w\| = 1$. We shall call $w$ the input direction vector of the pole $p$. In the sequel, we shall assume that the plant transfer function matrix $P$ has only simple nonminimum phase zeros and unstable poles and that the set of poles and zeros are disjoint.

Suppose that $P(s)$ has nonminimum phase zeros at $z_i \in \mathbb{C}_+$, $i = 1,\ \cdots,\ k$ and unstable poles at $p_i \in \mathbb{C}_+$, $i = 1,\ \cdots,\ l$. Then, it is well-known [8, 9, 2] that $P(s)$ can be factorized in the form of

$$P(s) = L(s)P_m(s) = \left(\prod_{i=1}^{k} L_i(s)\right) P_m(s), \tag{2.6}$$

where $P_m(s)$ represents the minimum phase part of $P(s)$, and the allpass factor $L(s)$ can be constructed as

$$L(s) := \prod_{i=1}^{k} L_i(s), \qquad L_i(s) := I - \frac{2\mathrm{Re}z_i}{z_i}\frac{s}{s + \bar{z}_i}\eta_i\eta_i^H. \tag{2.7}$$

Here the unitary vectors $\eta_i$ can be constructed via an iterative procedure [9]. Note that when $P$ is stable, then $P_m$ will be an *outer factor*, and (2.6) amounts to an inner-outer factorization of $P$ [3]. Under this circumstance, there exists a right inverse of $P_m$ analytic in $\mathbb{C}_+$. Note also that the right and left coprime factors $N(s)$ and $\tilde{M}(s)$ can be factorized as

$$N(s) = L(s)N_m(s), \qquad \tilde{M}(s) = \tilde{M}_m(s)F(s), \tag{2.8}$$

where $L$ is given in (2.6-7), and $F$ is defined accordingly:

$$F(s) := \prod_{i=1}^{l} F_i(s), \qquad F_i(s) := I - \frac{2\mathrm{Re}p_i}{p_i}\frac{s}{s + \bar{p}_i}w_i w_i^H. \tag{2.9}$$

Finally, we introduce the following angular measure between two subspaces. Given a unitary vector $u$, we call the one dimensional subspace spanned by $u$ the *direction* of $u$. The angle between the directions of two unitary vectors $u$, $v$ is defined to be the *principal angle* [1] between the two corresponding subspaces spanned by the two vectors. We denote this angle by $\angle(u,\ v)$:

$$\cos\angle(u,\ v) := |u^H v|.$$

We shall say that the two directions are parallel if $\cos\angle(u,\ v) = 1$ and that they are orthogonal if $\cos\angle(u,\ v) = 0$.

## 3 Tracking Sinusoidal Signals

### 3.1 Complex Sinusoids

The complex sinusoidal signals to be considered in this section are described by

$$u(t) := \begin{cases} ve^{j\omega_0 t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{3.1}$$

where $\omega_0$ is a known frequency, and $v$ is a given unitary vector. The Laplace transform of $u(t)$ is

$$\hat{u}(s) = \frac{v}{s - j\omega_0}.$$

It is clear that in order for $J$ to be finite the sensitivity function $S(s)$ must have a zero at $j\omega_0$, which in turn requires that the open loop transfer function have a pole at $j\omega_0$. To prevent unstable pole-zero cancelation, the plant transfer function should then have no zero at $j\omega_0$. Thus, the following assumption will be imposed throughout this subsection.

**Assumption 3.1** *The matrix* $\begin{bmatrix} j\omega_0 I - A & -B \\ -C & -D \end{bmatrix}$ *has full rank.*

The minimal tracking error is characterized explicitly as follows.

**Theorem 3.1** *Let $u$ be given in (3.1), and suppose that Assumption 2.1 and Assumption 3.1 hold. Furthermore, let $N(s)$ be factorized as in (2.8). Then,*

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i - j\omega_0|^2} \cos^2 \angle(\eta_i, v) \quad + \quad v^H H v, \tag{3.2}$$

*where*

$$H = \sum_{i,j \in \mathbb{I}} \left( \frac{4\mathrm{Re}(p_i)\mathrm{Re}(p_j)}{(\overline{p}_i + p_j)(p_i - j\omega_0)(\overline{p}_j + j\omega_0)\overline{c}_i c_j} \times \left(I - L^{-1}(p_i)\right)^H \left(I - L^{-1}(p_j)\right), \right)$$

$$c_i := \prod_{\substack{j \in \mathbb{I} \\ j \neq i}} \frac{\overline{p}_j + j\omega_0}{p_j - j\omega_0} \frac{p_j - p_i}{\overline{p}_j + p_i},$$

*and* $\mathbb{I} := \{i : \tilde{M}(p_i)v = 0\}$.

Much like its counterpart in the step input case [2], Theorem 3.1 shows that the tracking error depends upon both the locations and directions of the plant nonminimum phase zeros and unstable poles, and that this dependence is fully captured by the principal angles between the input direction and those of the zeros and poles. While any nonminimum phase zero will affect the tracking error whenever its direction is not orthogonal to that of the input signal, an unstable pole will do so only when its direction is perfectly aligned with that of input, and in particular, when the plant is nonminimum phase. It is clear that nonminimum phase zeros far away from the imaginary axis have a negligible effect in general. On the other hand, zeros close to $j\omega_0$ can lead to rather poor tracking quality. This can be observed by weakening (3.2) to the following lower bound

$$J^* \geq \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i - j\omega_0|^2} \cos^2 \angle(\eta_i, v).$$

Moreover, it is worth noting that whenever an unstable pole does affect the tracking error, i.e., when its pole direction is parallel to that of the input, then it may couple with any nonminimum phase zero to lead to a particularly serious consequence. Indeed, suppose that the plant has a single nonminimum phase zero $z$ with direction vector $\eta$, and a single unstable pole $p$ with direction vector $w$ such that $\cos \angle(w, v) = 1$. Then, it can be readily shown that

$$J^* = \left( \frac{2\mathrm{Re}(z)}{|z - j\omega_0|^2} + \frac{8|p|^3}{|p - j\omega_0|^2 |p - z|^2} \right) \cos^2 \angle(\eta, v).$$

As such, $J^*$ may become excessively large when the zero and pole are close to each other, and when their directions are closely aligned.

## 3.2 Real Sinusoids

We now examine real sinusoidal signals. This class of signals are described by

$$u(t) := \begin{cases} v \sin \omega_0 t & t \geq 0 \\ 0 & t < 0 \end{cases}, \tag{3.3}$$

with $\omega_0$ being a fixed frequency and $v$ a given unitary vector. Noting that

$$\hat{u}(s) = \frac{v\omega_0}{s^2 + \omega_0^2},$$

one immediately realizes that $P$ should not have zeros at $j \pm \omega_0$. Since $P$ is real rational, this possibility is ruled out under Assumption 3.1.

It is a household knowledge that the real sinusoids can be decomposed as a linear combination of complex ones:

$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}.$$

Motivated by this fact, one is tempted to conjecture that the minimal tracking error in response to the former class of signals may be the combination of those to the latter signals. A little thought, however, indicates that this is not true in general. The observation here is a simple one: while the error signals with respect to the complex sinusoidal inputs do satisfy the superposition principle, they do not constitute an orthogonal pair. Our following result shows that this is indeed not the case.

**Theorem 3.2** *Let $u$ be given in (3.3), and suppose that Assumption 2.1 and Assumption 3.1 hold. Furthermore, let $N(s)$ be factorized as in (2.8). Then,*

$$J^* = \omega_0^2 \sum_{i,j=1}^{k} \frac{4\mathrm{Re}(z_i)\mathrm{Re}(z_j)\overline{z}_i z_j \overline{\alpha}_i \alpha_j}{z_i \overline{z}_j (\overline{z}_i + z_j)(\overline{z}_i^2 + \omega_0^2)(z_j^2 + \omega_0^2)} \beta_i^H \beta_j + \omega_0^2 v^H H_1 v, \tag{3.4}$$

*where*

$$\alpha_i : = \eta_i^H \left( \prod_{j=1}^{i-1} L_j(z_i) \right)^{-1} v,$$

$$\beta_i : = \left( \prod_{j=i+1}^{k} L_j(z_i) \right)^{-1} \eta_i,$$

$$\Phi(s) : = \frac{s + j\omega_0}{2j\omega_0} L^{-1}(j\omega_0) - \frac{s - j\omega_0}{2j\omega_0} L^{-1}(-j\omega_0),$$

$$H_1 := \sum_{i,j \in \mathbb{I}} \left( \frac{4\mathrm{Re}(p_i)\mathrm{Re}(p_j)\overline{p}_i p_j}{p_i \overline{p}_j \overline{b}_i b_j (\overline{p}_i + p_j)(\overline{p}_i^2 + \omega_0^2)(p_j^2 + \omega_0^2)} \times \left(\Phi(p_i) - L^{-1}(p_i)\right)^H \left(\Phi(p_i) - L^{-1}(p_j)\right) \right),$$

$$b_i := \prod_{\substack{j \in \mathbb{I} \\ j \neq i}} \frac{\overline{p}_j}{p_j} \frac{p_j - p_i}{\overline{p}_j + p_i},$$

*and* $\mathbb{I} := \left\{i : \tilde{M}(p_i)v = 0\right\}$.

Theorem 3.1 and Theorem 3.2 serve to demonstrate the complex behavior of how plant nonminimum phase zeros and unstable poles may affect the tracking performance in response to sinusoidal signals. A simple observation from this result is that sine waves of a lower frequency are easier to track. This, of course, is consistent with one's intuition. Note that while Theorem 3.1 and its derivation bears much similarity to the case of step signals, the statement and derivation of Theorem 3.2 is quite different and it exhibits several new aspects not found previously. One notable difference here is that the zero effects are entangled in a rather complicated manner, unlike in the complex sinusoids case where each of the nonminimum phase zeros seemingly exerts its effect independently and they together contribute in an additive fashion. This makes the analysis of the zero effects rather difficult, and it deters a further interpretation of the nature in how the zeros and poles may limit the achievable performance. In spite of this difficulty, however, it remains possible to gain useful insight by examining certain limiting cases. Consider, for example, that the plant has only one nonminimum phase zero $z$ with direction $\eta$, together with only one unstable pole $p$ whose direction is completely aligned with the input vector $v$. In this case, it is not difficult to deduce from (3.4) that

$$J^* = \omega_0^2 \frac{2\mathrm{Re}(z)}{|z^2+\omega_0^2|^2} \left|\frac{\bar{z}+p}{z-p}\right|^2 \cos^2 \angle(\eta,\ v).$$

This again shows that zeros and poles lying in proximity will in general make tracking performance goal particularly difficult to attain. Note also that $J^*$ achieves the maximum value at $\omega_0 = \pm\sqrt{z}$.

## 4 Tracking Ramp Signals

Similar to step inputs, the unit ramp signal is typically used for analyzing transient and steady state behavior in classical system analysis. This signal is described by

$$u(t) := \begin{cases} vt & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{4.1}$$

and its Laplace transform is

$$\hat{u}(s) = \frac{v}{s^2}.$$

Again, we assume that $\|v\| = 1$. Note that to warrant a finite tracking error $J$, similar assumption must be imposed on the system's behavior at the origin. In the present case, the open loop transfer function must be at least of type two. This implies that neither $P(s)$ nor its derivative $P'(s)$ can have a zero at the origin. An equivalent statement for this assumption can be easily shown to be the following

**Assumption 4.1** *The matrices* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *and* $\begin{bmatrix} A^2 & B \\ C & 0 \end{bmatrix}$ *have full rank.*

The following result gives an explicit expression of $J^*$ with respect to the ramp signal.

**Theorem 4.1** *Let $u$ be given in (4.1), and suppose that Assumption 2.1 and Assumption 4.1 hold. Furthermore, let $N(s)$ be factorized as in (2.8). Then,*

$$J^* = \sum_{i,j=1}^{k} \frac{4\mathrm{Re}(z_i)\mathrm{Re}(z_j)\bar{\alpha}_i\alpha_j}{|z_i|^2|z_j|^2(\bar{z}_i+z_j)}\beta_i^H\beta_j + v^H H_1 v, \tag{4.2}$$

*where*

$$H_1 := \sum_{i,j\in\mathbb{I}} \left( \frac{4\mathrm{Re}(p_i)\mathrm{Re}(p_j)}{|p_i|^2|p_j|^2\bar{b}_i b_j(\bar{p}_i+p_j)} \times \left(\Psi(p_i) - L^{-1}(p_i)\right)^H \left(\Psi(p_i) - L^{-1}(p_j)\right) \right),$$

$$\Psi(s) := I + s\sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2}\eta_i\eta_i^H.$$

*Furthermore, $\alpha_i$, $\beta_i$, $b_i$, and $\mathbb{I}$ are defined as in Theorem 3.2.*

At the outset, one notes that

$$u(t) = \lim_{\omega_0\to 0} \frac{v\sin\omega_0 t}{\omega_0},$$

or equivalently,

$$\hat{u}(s) = \lim_{\omega_0\to 0} \frac{1}{\omega_0}\frac{v\omega_0}{s^2+\omega_0^2}.$$

This suggests that the ramp signal can be considered as a limiting case of the real sinusoid, averaged over frequency. As a consequence, the error signal in response to the former input can be considered as the limit of that to the latter. The implication then is that the same relationship may exist between the minimal tracking errors in the two cases. While a direct justification of this observation resorting to a limiting procedure requires certain subtle technical details and can be provided, a comparison of Theorem 3.2 and Theorem 4.1 indicates that the minimal error expression in (4.2) is indeed the limit of that in (3.4). More specifically, let $J_s^*$ and $J_r^*$ denote the minimal tracking errors in response to the real sinusoidal and the ramp signals, respectively. Then,

$$J_r^* = \lim_{\omega_0\to 0} \frac{J_s^*}{\omega_0^2}.$$

This is clear by observing that

$$\begin{aligned} \lim_{\omega_0\to 0} \Phi(s) &= I + s\left.\frac{dL^{-1}(s)}{ds}\right|_{s=0} \\ &= \Psi(s). \end{aligned}$$

Here the equality follows from the well-known L'Hopital rule.

## 5 Average Tracking

As a final contribution, we now formulate and solve a related problem, which amounts to an averaged version of the tracking problems considered thus far. We shall first need to generalize several concepts concerning norms and spaces. We collect these generalizations below.

- Frobenius norm of matrices:

$$\|F\|_F^2 := \mathrm{Tr}\left(F^H F\right).$$

- The Hilbert space $\mathcal{L}_2$:

$$\mathcal{L}_2 := \Bigg\{ F : F(s) \text{ measurable in } \mathbb{C}_0, \quad \|F\|_2 := \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \|F(j\omega)\|_F^2\, d\omega\right]^{1/2} < \infty \Bigg\}.$$

These notions are now standard and are known to possess the same properties as those defined for vector functions.

While in our preceding development the tracking error is measured in terms of input signals lying in certain specific directions, it is of interest to know *a priori* to what extent tracking performance may be achieved independently of input directions. This motivates our study of the minimization problem

$$\hat{J}^* := \inf_{K \in \mathcal{K}} \left\| \frac{S(s)}{s} \right\|_2^2, \tag{5.1}$$

which quantifies the minimal error in a spatially uniform fashion, and hence may considered as a uniform measure of the tracking performance. Note also that the problem may be viewed as an extension to an $\mathcal{H}_2$ optimal control problem formulated in [6]. It is clear that $S(s)$ must have a blocking zero at the origin, for which Assumption 3.1 constitutes a necessary condition. We provide below the solution to this problem.

**Theorem 5.1** *Suppose that Assumption 2.1 and Assumption 3.1 hold. Let $N(s)$ and $\tilde{M}(s)$ be factorized as in (2.8), together with $F(s)$ and $F_i(s)$ given by (2.9). Then,*

$$\hat{J}^* = \sum_{i=1}^{k} \left( \frac{2\mathrm{Re}(z_i)}{|z_i|^2} + \right.$$
$$\left. + \sum_{i,j=1}^{l} \frac{4\mathrm{Re}(p_i)\mathrm{Re}(p_j)}{p_i \bar{p}_j (\bar{p}_i + p_j)} (w_i^H H_i^H H_j w_j)(w_j^H G_j^H G_i w_i) \right), \tag{5.2}$$

*where*

$$G_i : = F_1^{-H}(p_i) \cdots F_{i-1}^{-H}(p_i),$$
$$H_i : = F_l^{-1}(p_i) \cdots F_{i+1}^{-1}(p_i).$$

For stable plants, the error quantity alluded to above may be interpreted in the following sense as an average of the system's tracking errors in response to signals from different directions. Let $m$ be the number of the system's outputs. In addition, let $e_j$ be the $j$th Euclidean coordinate of the $m$-dimensional Euclidean space. Then,

$$\left\| \frac{I - PQ}{s} \right\|_F^2 = \sum_{j=1}^{m} e_j^H \left[ \left( \frac{I-PQ}{s} \right)^H \left( \frac{I-PQ}{s} \right) \right] e_j$$
$$= \sum_{j=1}^{m} \left\| \frac{I-PQ}{s} e_j \right\|^2 .$$

Since for a stable plant $P$ its right and left coprime factors can be selected as $N = \tilde{N} = P$, $M = \tilde{M} = I$, which leads to $X = \tilde{X} = I$, $Y = \tilde{Y} = 0$. It follows that

$$\hat{J}^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \sum_{j=1}^{m} \left\| \frac{I-PQ}{s} e_j \right\|_2^2 .$$

It is clear that

$$\hat{J}^* \geq \sum_{j=1}^{m} \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{I-PQ}{s} e_j \right\|_2^2 .$$

On the other hand,

$$\inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \sum_{j=1}^{m} \left\| \frac{I-PQ}{s} e_j \right\|_2^2 \leq$$
$$\sum_{j=1}^{m} \left\| \frac{I-L}{s} e_j \right\|_2^2 = \sum_{j=1}^{m} \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{I-PQ}{s} e_j \right\|_2^2 .$$

As a result, we have

$$\hat{J}^* = \sum_{j=1}^{m} \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{I-PQ}{s} e_j \right\|_2^2 . \tag{5.3}$$

It follows immediately from [2] that

$$\hat{J}^* = \sum_{j=1}^{m} \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i,\ e_j) = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2}.$$

Note that $e_j$ can be replaced by any orthonormal basis of the $m$-dimensional Euclidean space, and as such $\hat{J}^*$ may be interpreted as a quantity resulted from averaging the tracking errors in response to all orthogonal input signals. Note also that this property in general does not hold for unstable plants. Indeed, while for stable plants the optimal $Q$ is independent of input directions, it is not so when the plant is unstable.

## 6 Conclusion

In this paper we have derived explicit formulas for the minimal tracking error in response to sinusoidal and ramp signals. On the one hand these results share much in common with our earlier work [2], showing that tracking performance in general depends on both the locations and directions of plant nonminimum phase zeros and unstable poles. On the other hand, they help display new features useful for clarifying the rather intricate behavior in how nonminimum phase zeros and unstable poles may lead to poor performance in tracking different input signals. Overall, the following general statements can be made from this work and [2].

- Nonminimum phase zeros will limit the tracking performance to a significant extent only when they are close to the imaginary axis. The effect of nonminimum phase zeros is completely determined by zero locations and the principal angles between zero and input directions.
- Unstable poles will affect the tracking performance only when the plant is also nonminimum phase, and when the input and certain pole directions are completely aligned. When this is the case, approximate unstable pole-zero cancelation can lead to particularly poor tracking performance.

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