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Limitations on Maximal Tracking Accuracy, Part I: Tracking Step Signals*

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Abstract

This paper studies optimal tracking performance issues pertaining to finite dimensional, linear, time invariant feedback control systems. The problem under consideration amounts to determining the minimal tracking error between the output and input signals of a system, attainable by all possible stabilizing compensators. An integral square error criterion is used as a measure for the tracking error, and explicit expressions are derived for this measure with respect to step signals. It is shown that plant nonminimum phase zeros have a negative effect upon a system's ability in reducing the tracking error, and that in a multivariable system this effect results in a way depending upon not only the zero locations, but also the zero directions. It is also shown that plant nonminimum phase zeros and unstable poles can together play a particularly detrimental role to tracking performance, especially when the zeros and poles are nearby and their directions are closely aligned. These results lead to new insights into the optimal tracking problem, and more generally, insights into certain fundamental issues concerning limitations on performance achievable via feedback control.

1 Introduction

In this paper we study optimal tracking performance issues pertaining to finite dimensional, linear, time invariant feedback control systems. The problem under consideration amounts to determining the maximal tracking accuracy, or the minimal tracking error between the output and input signals of a system, attainable by all possible stabilizing compensators. Here the tracking error is defined in the $\mathcal{H}_2$ sense using an integral square error criterion, and the tracking performance is measured by the minimal error in tracking certain specific classes of input signals; typical signals to be examined are unit step signals. It is worth noting that while in classical system analysis one is mainly concerned with steady state error, adoption of an integral square error criterion makes it possible to capture transient behavior as well.

The ability of tracking command input signals is a primary criterion used to assess the performance of feedback control systems and indeed constitutes a primary objective in control system design. As such, optimal tracking problems have over the years received a considerable amount of research interest. While in many such problems a main objective is to design an optimal compensator to minimize tracking error, in this work we are more interested in the intrinsic system properties that may limit the tracking performance achievable via feedback. In this vein, the problem being pursued here is closely related to a number of well-studied issues regarding cheap LQR control [8], servomechanism problems [11], and an $\mathcal{H}_2$ optimal control problem studied in [10]. Earlier investigation into these problems has led to several important discoveries on tracking performance limitations. It is now generally known that perfect tracking, or zero tracking error, can be achieved for minimum phase systems, and that this desirable property will vanish, however, when the system is nonminimum phase [8]. More recently, explicit expressions relating tracking error to the plant nonminimum phase zeros were made available in [10, 11], displaying how the error may be negatively affected by the zero locations.

We adopt a frequency domain approach similar to that in [10]. Using this approach, we derive explicit formulae for the minimal tracking error with respect to step signals; our consideration of step signals is motivated by the observation that in classical control design it serves as a benchmark testing signal for assessing transient and steady state performance. Note that while the same problem has been previously examined in [10] with respect to single-input single-output stable plants, we consider multivariable unstable systems. This leads to several new discoveries unique to the latter. A particularly interesting observation resulting from these formulae indicates that in a multivariable system the tracking error depend not only upon the location of the plant nonminimum phase zeros, but also upon how the input signal may interact with the zeros. This interaction is characterized by the principal angles between the input and zero directions. It is clear from this result that while for single-input single-output systems it is impossible to achieve perfect tracking in the presence of plant nonminimum phase zeros, it can be accomplished in the multivariable setting for signals that are properly aligned with zero directions, specifically when the input and zero directions are orthogonal. Additionally, as yet another interesting feature, our result also shows that for plants that are both nonminimum phase and unstable, a close coupling between the nonminimum phase zeros and unstable poles can be particularly detrimental. Here the coupling is determined not only by the closeness of pole and zero locations, but also in how the pole and zero directions are aligned. This phenomenon was unknown previously with respect to tracking performance.

For any complex number z, we denote its complex conjugate by $\bar{z}$. For any vector u, we denote its conjugate transpose by u^H. For any signal $u(t)$, we denote its Laplace transform by $\hat{u}(s)$. The conjugate transpose of a matrix A is denoted by A^H. All the vectors and matrices involved in the sequel are assumed to have compatible dimensions, and for simplicity, their dimensions are omitted. We denote the open right

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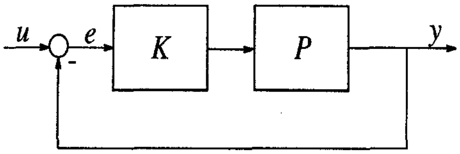

Figure 1: The feedback system

half plane by $\mathbb{C}_+$ and the imaginary axis by $\mathbb{C}_0$. Moreover, let $\|\cdot\|$ denote the Euclidean vector norm and define,

$$\mathcal{L}_2 := \left\{ f : \ \ f(s) \text{ measurable in } \mathbb{C}_0, \ \|f\|_2 < \infty \right\},$$

where

$$\|f\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(j\omega)\|^2 d\omega \right)^{1/2}.$$

Finally, let $\mathcal{H}_\infty$ denote the set of all bounded analytic functions defined in the open right half plane and $\mathbb{R}\mathcal{H}_\infty$ denote the set of all rational functions in $\mathcal{H}_\infty$.

2 Preliminaries

We consider the finite dimensional linear time-invariant feedback system depicted in Figure 1. In this setup, P denotes the plant model and K the compensator, whose transfer function matrices are $P(s)$ and $K(s)$, respectively[1]. The signals u, y, and e represent respectively the command input, the system output, and the error signal between y and u. We shall assume throughout the paper that the system is initially at rest. For a given input signal u, the tracking error of the system is defined as

$$J := \int_0^\infty \|e(t)\|^2 dt.$$

This quantity constitutes an important criterion in $\mathcal{H}_2$ optimal control, and more generally, serves as a useful measure for assessing a system's performance. An important objective in feedback control design is to achieve internal stability of the system and to minimize the tracking error.

It is well-known that, any stabilizing compensator K can be described via the so-called Youla parameterization [6]. Specifically, let the right and left coprime factorizations of P be given by

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{2.1}$$

where N, M, $\tilde{N}$, $\tilde{M} \in \mathbb{R}\mathcal{H}_\infty$ and satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I, \tag{2.2}$$

for some X, Y, $\tilde{X}$, $\tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$. Then, the set of all stabilizing compensators K is characterized by

$$\mathcal{K} := \left\{ K : \ K = (Y - MQ)(NQ - X)^{-1}, \ Q \in \mathbb{R}\mathcal{H}_\infty \right\}. \tag{2.3}$$

[1] In the sequel, we shall use a same symbol to denote a system and its transfer function, and whenever convenient, to omit the dependence upon the frequency variable s.

Note in particular that when P is stable, then one can select $N = \tilde{N} = P$, $\tilde{X} = M = I$, $X = \tilde{M} = I$, $Y = 0$, and $\tilde{Y} = 0$. As a result, the parameterization (2.3) reduces to

$$\mathcal{K} = \left\{ K : \ K = Q(I - PQ)^{-1}, \ Q \in \mathbb{R}\mathcal{H}_\infty \right\}. \tag{2.4}$$

In this paper we are interested in finding the minimal tracking error attainable with respect to certain specific classes of signals. This problem amounts to determining

$$J^* := \inf_{K \in \mathcal{K}} J.$$

Define the system sensitivity function by

$$S(s) := (I + P(s)K(s))^{-1}.$$

Then, $\hat{e}(s) = S(s)\hat{u}(s)$. It follows from the well-known Parseval identity that

$$J = \|S\hat{u}\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|S(j\omega)\hat{u}(j\omega)\|^2 d\omega. \tag{2.5}$$

A standing assumption throughout this paper requires that the plant transfer function matrix be right invertible, by which we mean

Assumption 2.1 *The transfer function matrix $P(s)$ has full row rank for at least one s.*

This assumption is standard, and has proven to be crucial in optimal tracking problems, as evidenced by the previous work (see, e.g., [8, 11]). Additional assumptions to be imposed will vary with the command signals to be considered, and will be made subsequently.

In our later development we shall be particularly interested in the behavior of nonminimum phase plants, by which we mean that $P(s)$ has zeros in the open right half plane. Here the notion of zeros of a multivariable system is that of the transmission zeros, which are often defined via the *Smith-McMillan form* (see e.g., [9]) and can be characterized using state space representations [12]. Let (A, B, C, D) be a minimal realization of $P(s)$. Then, a point $z \in \mathbb{C}_+$ is called a nonminimum phase zero of P if there exist vectors x and w such that the relation

$$\begin{bmatrix} zI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$$

holds, where $\|w\| = 1$, and w is called the input zero direction vector associated with z. Analogously, a zero z of P satisfies the relation

$$\begin{bmatrix} \zeta^H & \eta^H \end{bmatrix} \begin{bmatrix} zI - A & -B \\ -C & -D \end{bmatrix} = 0,$$

where ζ and η are some vectors with η being unitary, $\|\eta\| = 1$. The vector η is called the output zero direction vector associated with z. It can be readily shown that $z \in \mathbb{C}_+$ is a nonminimum phase zero of P with an input direction vector w if and only if $\tilde{N}(z)w = 0$, and that it is a nonminimum phase zero of P with an output direction vector η if and only if $\eta^H N(z) = 0$. A complex number $p \in \mathbb{C}_+$ is an unstable pole of P with a right pole direction vector w if $\tilde{M}(p)w = 0$, where $\|w\| = 1$. Similarly, p is an unstable pole of P with left direction vector η, $\|\eta\| = 1$, if $\eta^H M(p) = 0$. Throughout this paper, we shall assume that the plant transfer function P has only simple nonminimum phase zeros and unstable poles and that the set of poles and zeros are disjoint.

It is well-known that a nonminimum phase, right invertible transfer function admits a factorization consisting of a minimum phase part and an allpass factor. While such a factorization is not unique, a specific factorization can be explicitly constructed as follows, by a repeated use of a formula given in [12]. Let $z_i \in \mathbb{C}_+$, $i = 1, \cdots, k$, be the nonminimum phase zeros of $P(s)$. Define

$$\begin{aligned} B^{(0)} : &= B, \\ B^{(i)} : &= B^{(i-1)} - 2(\text{Re} z_i)\zeta_i \eta_i^H, \end{aligned}$$

where η_i is a unitary vector ($\|\eta_i\| = 1$) and it, together with ζ_i, satisfies the relation

$$\begin{bmatrix} \zeta_i^H & \eta_i^H \end{bmatrix} \begin{bmatrix} zI - A & -B^{(i-1)} \\ -C & -D \end{bmatrix} = 0.$$

Additionally, define

$$L_i(s) := I - \frac{2\text{Re} z_i}{z_i} \frac{s}{s + \overline{z}_i} \eta_i \eta_i^H \tag{2.6}$$

and

$$L(s) := \prod_{i=1}^{k} L_i(s). \tag{2.7}$$

Then, as shown in [14], the transfer function $P(s)$ can be factorized in the form of

$$P(s) = L(s) P_m(s) = \left(\prod_{i=1}^{k} L_i(s) \right) P_m(s), \tag{2.8}$$

where $P_m(s)$ represents the minimum phase part of $P(s)$, and $L(s)$ the all pass factor. Note that the unitary vector η_i in (2.6) need not be the zero direction vector of $P(s)$ associated with z_i, it is a zero direction vector of $L_i(s) ... L_k(s) P_m(s)$ associated with z_i. For ease of reference, however, we shall refer to η_i as a zero direction vector from this point onward. Note also that $L_i(s)$ can be alternatively written as

$$L_i(s) = \begin{bmatrix} \eta_i & U_i \end{bmatrix} \begin{bmatrix} \frac{\overline{z}_i}{z_i} \frac{z_i - s}{\overline{z}_i + s} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta_i^H \\ U_i^H \end{bmatrix}. \tag{2.9}$$

where U_i is a matrix whose columns, together with η_i, form an orthonormal basis of the corresponding Euclidean space, i.e., $\eta_i \eta_i^H + U_i U_i^H = I$. It is immediately clear from (2.9) that $L_i(s)$ is indeed all pass, and that $L_i(0) = I$. It is useful to point out that when P is stable, then P_m will be an *outer factor*, and (2.8) amounts to an inner-outer factorization of P [6]. Under this circumstance, there exists a right inverse of P_m analytic in $\mathbb{C}_+$.

Suppose now that in addition to the nonminimum phase zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, k$, P also has unstable poles at $p_i \in \mathbb{C}_+$, $i = 1, \cdots, l$. Let the coprime factorizations of P be given by (2.1). Since N is right invertible and the nonminimum phase zeros of N coincide with those of P it follows that N can be factorized as

$$N(s) = L(s) N_m(s) \tag{2.10}$$

where L is given in (2.6-7).

In closing, we introduce the following angular measure between two subspaces. Given a unitary vector u, we call the one dimensional subspace spanned by u the *direction* of u. The angle between the directions of two unitary vectors u, v is defined to be the *principal angle* [1] between the two corresponding subspaces spanned by the two vectors. We denote this angle by $\angle(u, v)$:

$$\cos \angle(u, v) := |u^H v|.$$

We shall say that the two directions are parallel if $\cos \angle(u, v) = 1$, and that they are orthogonal if $\cos \angle(u, v) = 0$. As discussed in [1], the principal angle between two subspaces serves as a distance measure and it quantifies how well the two subspaces are aligned.

3 Main results

In this paper we consider the problem of tracking a constant signal of the form

$$u(t) = \begin{cases} v & t \geq 0 \\ 0 & t < 0 \end{cases}, \tag{3.1}$$

where v is a constant unitary vector: $\|v\| = 1$. This signal may be viewed as a generalized unit step signal, whose Laplace transform is given by

$$\hat{u}(s) = \frac{v}{s}.$$

In light of (2.5), it is clear that in order for J to be finite, the sensitivity function $S(s)$ must have a zero at the origin. This implies that integral control action is needed, and that the open loop transfer function must be of at least type one. Since there should be no unstable pole-zero cancellation in the system, the following condition is necessary for the problem to be well-posed.

Assumption 3.1 $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *has full rank.*

Stated in words, P should have no zero, and hence no pole-zero cancellation will occur, at the origin.

3.1 Stable Plants

We shall examine first stable plants, for which the parameterization (2.4) results in $S = I - PQ$, and hence the minimal tracking error can be expressed as

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - PQ) \frac{v}{s} \right\|_2^2. \tag{3.2}$$

Our following result gives an explicit expression of J^*.

Theorem 3.1 *Let u be given in (3.1), and suppose that $P(s)$ is stable. Also, suppose that Assumption 2.1 and Assumption 3.1 hold, and that $P(s)$ is factorized as in (2.8). Then,*

$$J^* = \sum_{i=1}^{k} \frac{2\text{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i, v) \tag{3.3}$$

Theorem 3.1 is similar to its counterpart pertaining to single-input single-output systems, which shows that one generally cannot achieve perfect tracking with respect to nonminmum phase plants. A distinguishing feature about this result, however, shows that the tracking accuracy depends upon not only the zero locations, but also the zero and input directions, and that this dependence is fully captured by the principal angles between the zero and input directions. To further illustrate, consider the case that P has only one right half plane zero z with a zero direction vector η, to which (3.3) reduces to

$$J^* = \frac{2\text{Re}(z)}{|z|^2}\cos^2\angle(\eta,\ v).$$

It is clear from this expression that perfect tracking is still possible, provided that the input direction is orthogonal to the zero direction. Spatial properties of this kind have no analog in single-input single-output systems. It is worth noting that while zeros far away from the imaginary axis may have a negligible effect in general, those close to the imaginary axis, when they are complex, do not necessarily play a dominant role; their effect on J^* depends upon the imaginary parts also. Indeed, one can clearly observe that any complex nonminimum phase zero will have only a limited effect when it is far away from the origin, despite that it may be very close to the imaginary axis. This appears to be a phenomenon unique to the optimal tracking problem, contrary to certain other circumstances where nonminimum phase zeros close to the imaginary axis do pose a formidable difficulty regardless of their imaginary parts [5, 4].

As a related issue, it is useful to examine the largest and smallest value of J^* achievable among all possible v. This amounts to determining

$$J^*_{\text{max}} := \max_{\|v\|=1} J^*, \qquad J^*_{\text{min}} := \min_{\|v\|=1} J^*,$$

which can be readily characterized as follows. Note first that J^* can be alternatively written as

$$J^* = v^H\left(\sum_{i=1}^{k}\frac{2\text{Re}z_i}{|z_i|^2}\eta_i\eta_i^H\right)v.$$

This suggests that

$$J^*_{\text{max}} = \lambda_{\text{max}}\left(\sum_{i=1}^{k}\frac{2\text{Re}z_i}{|z_i|^2}\eta_i\eta_i^H\right),$$

$$J^*_{\text{min}} = \lambda_{\text{min}}\left(\sum_{i=1}^{k}\frac{2\text{Re}z_i}{|z_i|^2}\eta_i\eta_i^H\right),$$

and that the least and the most desirable input signals coincide with the respective eigenvectors. It is not difficult to see that J^*_{max} depends on the principal angles between the zero directions. On the other hand, J^*_{min} will depend on the angles only when $k \geq \dim(v)$, otherwise $J^*_{\text{min}} = 0$, which attests to the fact that perfect tracking is possible. To further illustrate, consider the case $k = \dim(v) = 2$. A simple calculation reveals

$$J^*_{\text{max}} = \frac{\text{Re}z_1}{|z_1|^2} + \frac{\text{Re}z_2}{|z_2|^2}$$
$$+\sqrt{\left(\frac{\text{Re}z_1}{|z_1|^2} + \frac{\text{Re}z_2}{|z_2|^2}\right)^2 - 4\frac{\text{Re}z_1}{|z_1|^2}\cdot\frac{\text{Re}z_2}{|z_2|^2}\sin^2\angle(\eta_1,\eta_2)},$$

$$J^*_{\text{min}} = \frac{\text{Re}z_1}{|z_1|^2} + \frac{\text{Re}z_2}{|z_2|^2}$$
$$-\sqrt{\left(\frac{\text{Re}z_1}{|z_1|^2} + \frac{\text{Re}z_2}{|z_2|^2}\right)^2 - 4\frac{\text{Re}z_1}{|z_1|^2}\cdot\frac{\text{Re}z_2}{|z_2|^2}\sin^2\angle(\eta_1,\eta_2)},$$

which exhibit an explicit dependence of J^*_{max} and J^*_{min} on the principal angle between the two zero directions.

3.2 Unstable Plants

We now consider unstable plants. Our main result in this subsection shows that together with nonminimum phase zeros plant unstable poles may have a severe negative effect on the achievable optimal tracking performance. Unlike Theorem 3.1, both the statement and derivation of this result differ considerably from the previous work [10, 11].

Theorem 3.2 *Let u be given in (3.1), and suppose that Assumption 2.1 and Assumption 3.1 hold. Suppose that $P(s)$ has unstable poles at $p_1, ..., p_\ell$. Then,*

$$J^* = \sum_{i=1}^{k}\frac{2\text{Re}(z_i)}{|z_i|^2}\cos^2\angle(\eta_i, v) \quad + \quad v^H H v, \quad (3.4)$$

where

$$H = \sum_{i,j\in\mathbb{I}}\frac{4\text{Re}(p_i)\text{Re}(p_j)}{(\overline{p_i}+p_j)p_i\overline{p}_j\overline{b}_i b_j}\left(I - L^{-1}(p_i)\right)^H\left(I - L^{-1}(p_j)\right),$$

$$b_i := \prod_{\substack{j\,\in\,\mathbb{I}\\ j\,\neq\,i}}\frac{\overline{p}_j}{p_j}\frac{p_j - p_i}{\overline{p}_j + p_i},$$

and $\mathbb{I} := \left\{i:\ \tilde{M}(p_i)v = 0\right\}$ is the index set of poles of $P(s)$ whose pole direction vectors are parallel to v.

It is clear from Theorem 3.2 that for an unstable, nonminimum phase plant, its unstable poles will in general worsen the optimal tracking performance. The significance of this result lies in that it leads to an important discovery previously unknown: in the presence of plant unstable poles, the tracking error depends not only on the nonminimum phase zeros, but also on the unstable poles. From a conceptual standpoint, the present result appears to bear a close similarity to [5, 3, 4]; the latter works studied performance limitation issues based upon the classical Bode and Poisson type integrals and lead to the conclusion that unstable plant poles will indeed impose limitations upon the achievable performance, especially when nonminimum phase zeros are also present in the plant transfer function.

While plant unstable poles may indeed limit the achievable tracking performance, it is interesting to note that they affect the tracking error in a rather intricate way. First, such poles will have an effect on J^* only when the plant is also nonminimum phase. Indeed, if the plant is minimun phase, then $H = 0$, for in this case $L = I$. Secondly, even for a nonminimum phase plant, they will have an effect only when the input vector v lies in certain pole directions. The latter property appears rather intriguing, and it points to yet another major conceptual difference between multivariable and single-input single-output systems. While in a multivariable system plant unstable poles may or may not affect the tracking performance and it depends on the alignment between input and pole directions, in a single-input single-output system they always do. It is clear that in the latter case $\mathbb{I} = \{1, 2, \cdots, l\}$, and (3.4) reduces to

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} +$$

$$\sum_{i,j=1}^{l} \frac{4\mathrm{Re}(p_i)\mathrm{Re}(p_j)}{(\overline{p_i}+p_j)p_i\overline{p}_j\overline{b}_i b_j} \left(1 - L^{-1}(p_i)\right)^H \left(1 - L^{-1}(p_j)\right).$$

There is also a strong indication from Theorem 3.2 that nonminimum phase zeros and unstable poles can particularly limit the tracking performance when they are close to each other. To better observe this property, it is instructive to examine (3.4) with respect to certain special cases. Consider the case that there is only one unstable pole p whose direction contains v. Under this circumstance,

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i, v) + \frac{2\mathrm{Re}(p)}{|p|^2} \|(I - L^{-1}(p))v\|^2. \tag{3.5}$$

Additionally, suppose that the plant has only one nonminimum phase zero z together with η. Then, it follows from (3.5) that

$$J^* = \frac{2\mathrm{Re}(z)}{|z|^2} \left(1 + \frac{4\mathrm{Re}(z)\mathrm{Re}(p)}{|z-p|^2}\right) \cos^2 \angle(\eta, v). \tag{3.6}$$

Since

$$\frac{4\mathrm{Re}(z)\mathrm{Re}(p)}{|z-p|^2} = \left|\frac{\overline{p}+z}{p-z}\right|^2 - 1, \tag{3.7}$$

we may rewrite (3.6) as

$$J^* = \frac{2\mathrm{Re}(z)}{|z|^2} \left|\frac{\overline{z}+p}{z-p}\right|^2 \cos^2 \angle(\eta, v).$$

This expression is particularly reminiscent of the performance bounds obtained in [5, 4, 3, 7, 13]. It is clear from this result that distribution of plant nonminimum phase zeros and unstable poles at nearby locations can be detrimental. Nevertheless, one can also see that the extent to which the zero and pole may affect J^* depends upon how the zero and pole directions are aligned, noticing that in this case v lies in the pole direction.

As a final note, we point out that when the plant has more than one nonminimum phase zeros then the effect imposed by plant unstable poles will also depend on how the zero directions are aligned. We illustrate this point by examining two limiting cases. Suppose again that there is only one unstable pole p whose direction contains v. In the first case we assume that all the zero directions are mutually orthogonal, by which we mean that $\cos \angle(\eta_i, \eta_j) = 0$ for $i \neq j$. It is easy to show from (2.9) that

$$L_i^{-1}(s) = I + \frac{2\mathrm{Re}z_i}{\overline{z}_i} \frac{s}{z_i - s} \eta_i \eta_i^H,$$

and hence in this case

$$L^{-1}(s) = I + \sum_{i=1}^{k} \frac{2\mathrm{Re}z_i}{\overline{z}_i} \frac{s}{z_i - s} \eta_i \eta_i^H.$$

This leads to

$$\begin{aligned} \|(I - L^{-1}(p))v\|^2 &= \left\| \sum_{i=1}^{k} \frac{2\mathrm{Re}z_i}{\overline{z}_i} (\eta_i^H v) \frac{p}{z_i - p} \eta_i \right\|^2 \\ &= \sum_{i=1}^{k} \frac{4(\mathrm{Re}z_i)^2 |p|^2}{|z_i|^2 |z_i - p|^2} |\eta_i^H v|^2. \end{aligned}$$

By combining this expression with (3.5) and using (3.7), we have

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \left|\frac{\overline{z_i}+p}{z_i-p}\right|^2 \cos^2 \angle(\eta_i, v). \tag{3.8}$$

In the second case, we assume that the zero directions are all parallel, i.e., $\cos \angle(\eta_i, \eta_j) = 1$ for $i \neq j$. With no loss of generality we may assume that $\eta_i = \eta$ for all $i = 1, \cdots, k$. In this case, one can write

$$L(s) = [\ \eta \quad U\ ] \begin{bmatrix} \prod_{i=1}^{k} \left(\frac{\overline{z}_i}{z_i} \frac{z_i - s}{\overline{z}_i + s}\right) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta^H \\ U^H \end{bmatrix},$$

where U is some matrix which together with η forms a unitary matrix. It follows that

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta, v)$$

$$+ \frac{2\mathrm{Re}(p)}{|p|^2} \left| \prod_{i=1}^{k} \left(\frac{z_i}{\overline{z}_i} \frac{\overline{z}_i + p}{z_i - p}\right) - 1 \right|^2 \cos^2 \angle(\eta, v). \tag{3.9}$$

A comparison of (3.8) and (3.9) shows that J^* may differ drastically due to different alignments between η_1 and η_2.

4 Effects of Time Delays

We now consider plants with measurement time delays, by which we mean that the plant transfer function matrix can be expressed as

$$P_d(s) = \Lambda(s) P(s), \tag{4.1}$$

where $P(s)$ is a real rational transfer function matrix, and

$$\Lambda(s) := \mathrm{diag}(e^{-T_1 s}, \cdots, e^{-T_m s})$$

Here $T_i > 0$ represent the delay constants in the different channels, and m is the number of plant outputs. Let u be a given input whose Laplace transform is $\hat{u}$. We denote the tracking error with respect to P_d by

$$J_d := \left\|(I + P_d K)^{-1} \hat{u}\right\|_2^2,$$

and correspondingly, the minimal tracking error by

$$J_d^* := \inf_{K \in \mathcal{K}} J_d.$$

As in the preceding development, J^* will denote the minimal tracking error with respect to P.

For purpose of illustration we shall consider the case that P is stable. In this case the right and left coprime factorizations of P_d are given by $N_d = \Lambda N = \Lambda P = P_d$, $\tilde{N}_d = \Lambda \tilde{N} = \Lambda P = P_d$, and $M = \tilde{M} = I$. Let the input be the unit step signal given by (3.1). Then, it follows that

$$\begin{aligned} J_d^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - P_d Q) \frac{v}{s} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - \Lambda P Q) \frac{v}{s} \right\|_2^2 . \end{aligned}$$

Clearly, for J_d^* to be meaningful, it is necessary that $(I - P_d(0)Q(0))v = (I - P(0)Q(0))v = 0$, thus necessitating Assumption 3.1. The following theorem gives an explicit formula for J_d^*.

Theorem 4.1 *Let u be given by (3.1), and suppose that P is stable. In addition, suppose that Assumption 2.1 and Assumption 3.1 hold. Then,*

$$J_d^* = \sum_{i=1}^{m} T_i |v_i|^2 + J^*. \tag{4.2}$$

Since for $T_i > 0$ we have $J_d^* > J^*$, Theorem 5.1 shows, and indeed attests to the inuition that the tracking performance is generally more difficult to attain with respect to delay systems. It is interesting to note that the delay units affect the tracking error in much the same way as nonminimum phase zeros will, in the sense that they do so in an additive fashion.

5 Conclusion

This is the first part of a two-part series devoted to the study of optimal tracking performance issues pertaining to finite dimensional, linear, time invariant feedback control systems. We have examined, based upon an integral square criterion, specifically the minimal tracking error between the output and input signals of a system in response to step signals. We presented explicit formulae for the minimal error. These formulae demonstrate explicitly how plant properties such as nonminimum phase zeros, unstable poles, and time delays in a multivariable system may degrade tracking performance, thus leading to new insights into the optimal tracking problem, and more generally, insights lending a concrete support to the previously known assertions regarding fundamental limitations on performance achievable via feedback control. Our results indicate that in a multivariable system tracking performance can be seriously limited to an extent determined by both the location and directions of plant nonminimum phase zeros, and it partially depends upon how the input and zero directions are aligned. When the plant is also unstable, this adverse effect can be more acute. Our result shows that for plants that are both nonminimum phase and unstable, a close coupling between the nonminimum phase zeros and unstable poles can be particularly detrimental to tracking performance. This is determined not only by the closeness of pole and zero locations, but also in how the pole and zero directions are aligned.

Part II of this series contains results with respect to other typical signals and will be presented elsewhere. Examples of such signals include sinusoidal and ramp inputs. The approach as well as the results presented herein can be extended to several other related problems. An immediate extension of interest is the tracking performance of discrete-time systems. Additionally, while the present work is restricted fully to right invertible plants, it is of interest to formulate and solve similar problems pertaining to plants that are not right invertible. The latter issue is of importance not only by itself, but also because it has a far reaching implication toward sampled-data systems. These problems are currently under investigation by the authors.

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