Now, choose the coordinates transformation as follows:

$$\begin{aligned}\xi_1 &= h = x_1^{-1}x_3 - 1\\ \xi_2 &= L_f h = 1 - x_3 x_4^{-1}\\ \xi_3 &= \lambda_1 = x_2^{-1}x_4 \in SC^{\infty}(M)\\ \xi_4 &= \mu_1 = x_1^{-1}x_2^{-1}x_3x_4 \ln x_3 + x_2^{-1}x_4 \ln x_4. \end{aligned} \tag{24}$$

Writing system (23) in ξ coordinates results in

$$\begin{aligned}\dot{\xi}_1 &= \xi_2,\\ \dot{\xi}_2 &= (1+\xi_1)^{-1}(\xi_2 - 1) + 1 - \xi_2 + (1-\xi_2)\xi_3^{-1}\\ &\quad - (1-\xi_2)^2 - ((1+\xi_1)^{-1}(1-\xi_2) + 1 - \xi_2)u\\ \dot{\xi}_3 &= 1 - \xi_3^2 + \xi_2\xi_3\\ \dot{\xi}_4 &= \xi_2\xi_4 - \xi_3\xi_4 + \xi_3^{-1}\xi_4 + \xi_2(\xi_1+2)^{-1}\xi_4 + 1\\ &\quad + \xi_2\xi_3(\xi_1+2)^{-1}\ln(1-\xi_2) - \xi_1\xi_3 + \xi_2\xi_3 - \xi_3. \end{aligned} \tag{25}$$

The zero dynamics is

$$\begin{aligned}\dot{\xi}_3 &= 1 - \xi_3^2\\ \dot{\xi}_4 &= -\xi_3\xi_4 + \xi_3^{-1}\xi_4 + 1 - \xi_3 \end{aligned} \tag{26}$$

which is of cascade decomposition form.

V. Conclusions

It has been shown that the zero dynamics of a nonlinear system possessing symmetries has many special properties. The zero dynamics maintains symmetries if it exists, and a semiglobal zero dynamics always exists. We can obtain this kind of semiglobal zero dynamics by "moving" any known local zero dynamics along the Φ-orbits. The zero dynamics has a cascade decomposition according to the vector relative degree and the dimension of the symmetry *Lie* group. This kind of decomposition is also called triangular decomposition, which is widely used in the study of stability in the literature (for example, refer to [5]).

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Limitations on Maximal Tracking Accuracy

Jie Chen, Li Qiu, and Onur Toker

***Abstract*—This paper studies optimal tracking performance issues pertaining to finite-dimensional, linear, time-invariant feedback control systems. The problem under consideration amounts to determining the minimal tracking error between the output and reference signals of a feedback system, attainable by all possible stabilizing compensators. An integral square error criterion is used as a measure for the tracking error, and explicit expressions are derived for this minimal tracking error with respect to step reference signals. It is shown that plant nonminimum phase zeros have a negative effect on a feedback system's ability to reduce the tracking error, and that in a multivariable system this effect results in a way depending on not only the zero locations, but also the zero directions. It is also shown that if unity feedback structure is used for tracking purposes, plant nonminimum phase zeros and unstable poles can together play a particularly detrimental role in the achievable tracking performance, especially when the zeros and poles are nearby and their directions are closely aligned. On the other hand, if a two-parameter controller structure is used, the achievable tracking performance depends only on the plant nonminimum phase zeros.**

***Index Terms*—Nonminimum phase zeros, optimal tracking, performance limitation, two-parameter control, unstable poles.**

I. Introduction

In this paper, we study optimal tracking performance issues pertaining to finite-dimensional, linear, time-invariant feedback control systems. The problem under consideration amounts to determining the maximal tracking accuracy, or the minimal tracking error between the output and the reference signals of a feedback system, attainable by all possible stabilizing compensators under either a unity feedback structure or a two-parameter structure. Here, the tracking error is defined in the $\mathcal{L}_2$ sense using an integral square error criterion, and the reference signals under consideration are step signals.

We are interested in the intrinsic limit on the tracking performance achievable via feedback. For this purpose, we adopt a frequency domain approach and derive explicit formulas for the minimal tracking error. Although the same problem has been examined previously in [9] with respect to single-input–single-output (SISO) stable plants, we

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consider multi-input–multi-output (MIMO) unstable systems. Our investigation leads to several new discoveries unique to the latter. A rather interesting consequence is that in a multivariable system the minimal tracking error depends not only on the location of the plant nonminimum phase zeros, but also on how the input signal may interact with the zeros. This interaction can be precisely characterized by the angles between the input and zero directions. It becomes clear that although for SISO systems it is impossible to achieve perfect tracking in the presence of plant nonminimum phase zeros, this can be accomplished in MIMO systems for reference signals that are properly aligned with zero directions, specifically when the input and zero directions are orthogonal. Additionally, as another interesting feature, our result shows that in a unity feedback configuration, a close coupling between plant nonminimum phase zeros and unstable poles can be particularly detrimental. Here, the coupling is determined not only by the closeness of pole and zero locations, but also in how the pole and zero directions are aligned. This phenomenon was unknown previously with respect to tracking performance. Nevertheless, our study further reveals that the adverse "coupling" effect can be overcome by use of two-parameter feedback schemes, and that the tracking performance achievable by a two-parameter controller depends only on the nonminimum phase zeros, in exactly the same way as it does when the unity feedback is used for a stable plant. From a broad perspective, our contribution is related to and reinforces a number of well-known results regarding cheap LQR control [8], servomechanism problems [10], and $\mathcal{H}_2$ optimal tracking control [9], and it also bears a close relationship to performance studies facilitated by Bode- and Poisson-type integrals (see, e.g., [1], [2], [6], and [11]), as well as those under $\mathcal{H}_\infty$ criteria [7], [13].

The notation used throughout this paper is fairly standard. For any complex number z, we denote its complex conjugate by $\bar{z}$. For any vector u, we denote its conjugate transpose by u^H. For any signal $u(t)$, we denote its Laplace transform by $\hat{u}(s)$. The conjugate transpose of a matrix A is denoted by A^H. If A is a Hermitian matrix, its largest and smallest eigenvalues are written as $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. All vectors and matrices involved in the sequel are assumed to have compatible dimensions, and for simplicity, their dimensions are omitted. Let the open right half plane be denoted by $\mathbb{C}_+ := \{s: \mathrm{Re}(s) > 0\}$. Moreover, let $\|\cdot\|$ denote the Euclidean vector norm and $\|\cdot\|_2$ the norm in the space $\mathcal{L}_2$. It is well known [5] that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspaces of $\mathcal{L}_2$, and they constitute orthogonal complements in $\mathcal{L}_2$. The space $\mathcal{L}_2$ is a Hilbert space with an inner product defined by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} f^H(j\omega)g(j\omega)\,d\omega.$$

For any $f \in \mathcal{H}_2^\perp$ and $g \in \mathcal{H}_2$, we have $\langle f, g\rangle = 0$. This simple fact furnishes the main mechanism in our development.

Partial and related results of this paper were presented previously in [3] and [4].

II. Preliminaries

Let us first consider the finite-dimensional, linear, time-invariant unity feedback system depicted in Fig. 1. In this setup, P denotes the plant model and K denotes the compensator, whose transfer function matrices are $P(s)$ and $K(s)$, respectively.[1] The signals r, y, and e represent, respectively, the reference input, the system output, and the error signal between y and r. We shall assume throughout the

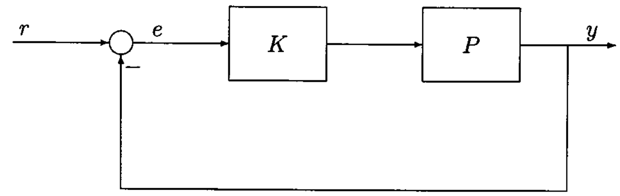

Fig. 1. The unity feedback system.

paper that the system is initially at rest. For a given input signal r, the tracking error of the system is defined as

$$J := \int_0^\infty \|e(t)\|^2\,dt.$$

Let the system sensitivity function be defined by $S(s) := (I + P(s)K(s))^{-1}$. Because $\hat{e}(s) = S(s)\hat{r}(s)$, it follows from the well-known Parseval identity that

$$J = \|S\hat{r}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|S(j\omega)\hat{r}(j\omega)\|^2\,d\omega. \tag{2.1}$$

In this paper, we shall consider a step reference signal of the form

$$r(t) = \begin{cases} v, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.2}$$

where v is a constant unitary vector: $\|v\| = 1$. The subspace spanned by v will be called the input direction. This signal may be viewed as a generalized unit step signal, whose Laplace transform is given by $\hat{r}(s) = v/s$.

Let $\mathbb{R}\mathcal{H}_\infty$ denote the set of all stable, proper, rational transfer function matrices. Let also the right and left coprime factorizations of P be given by

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N} \tag{2.3}$$

where $N, M, \tilde{N}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$ and satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \tag{2.4}$$

for some $X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$. It is well known that every stabilizing compensator K can be described via the so-called Youla parameterization [5]

$$\begin{aligned}\mathcal{K} := \{K: K &= -(Y - MQ)(X - NQ)^{-1} \\ &= -(\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}), Q \in \mathbb{R}\mathcal{H}_\infty\}.\end{aligned} \tag{2.5}$$

In particular, when P is stable, then we can select $N = \tilde{N} = P$, $\tilde{X} = M = I$, $X = \tilde{M} = I$, $Y = 0$, and $\tilde{Y} = 0$. As a result, the parameterization (2.5) reduces to

$$\mathcal{K} = \{K: K = Q(I - PQ)^{-1} = (I - QP)^{-1}Q,\ Q \in \mathbb{R}\mathcal{H}_\infty\}. \tag{2.6}$$

The minimal tracking error attainable by all possible stabilizing compensators, accordingly, is

$$J^* := \inf_{K \in \mathcal{K}} J.$$

It is well known that for J^* to be finite for all v, a necessary and sufficient condition is that $N(0)$ is right invertible. This will be assumed throughout.

We shall be particularly interested in the behavior of nonminimum phase plants, by which we mean that $P(s)$ has zeros in the open right half plane. Suppose that $P(s)$ is right-invertible. It is well known (see, e.g., [2] and [11]) that any $z \in \mathbb{C}_+$ is a nonminimum phase zero of P if and only if $\eta^H N(z) = 0$ for some unitary vector η. The vector η is called an (output) zero vector associated with the zero z, and the

[1] In the sequel, we shall use the same symbol to denote a system and its transfer function, and whenever convenient, to omit the dependence on the frequency variable s.

subspace it spans is called an (output) zero direction associated with z. Similarly, any $p \in \mathbb{C}_+$ is an unstable pole of P if and only if $\tilde{M}(p)w = 0$ for some unitary vector w. Likewise, the vector w is called an (input) pole vector associated with the pole p, and the subspace it spans is called an (input) pole direction associated with p.

Let $z_i \in \mathbb{C}_+$, $i = 1, \cdots, k$, be the nonminimum phase zeros of P. It is well known [2] that $P(s)$ can be factorized as

$$P(s) = L(s)P_m(s) = \left(\prod_{i=1}^{k} L_i(s)\right) P_m(s)$$
$$L(s) := \prod_{i=1}^{k} L_i(s) \tag{2.7}$$

where $P_m(s)$ represents the minimum phase part of $P(s)$, and $L(s)$ is an all-pass factor. A specific factorization admits the form of

$$L_i(s) = [\eta_i \; U_i] \begin{bmatrix} \dfrac{\overline{z}_i}{z_i} \dfrac{z_i - s}{\overline{z}_i + s} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta_i^H \\ U_i^H \end{bmatrix}. \tag{2.8}$$

Here, the unitary vector η_i can be sequentially determined from the zero direction vectors of P, and U_i is a matrix whose columns, together with η_i, form an orthonormal basis of the corresponding Euclidean space, i.e., $\eta_i \eta_i^H + U_i U_i^H = I$. It is immediately clear from (2.8) that $L_i(s)$ is indeed all pass, and that $L_i(0) = I$. It is useful to point out that when P is stable, then P_m will be an *outer factor*, and (2.7) amounts to an inner–outer factorization of P[5]. Under this circumstance, a right inverse of P_m analytic in $\mathbb{C}_+$ exists. Clearly, if P is nonminimum phase and left invertible, then a similar factorization exists and is of the form $P(s) = \tilde{P}_m(s)\tilde{L}(s)$, with $\tilde{P}_m$ being the minimum phase part and $\tilde{L}$ the all-pass factor, which is clear by examining $P^T(s)$. Additionally, because the nonminimum phase zeros of $N(s)$ coincide with those of $P(s)$, $N(s)$ can be similarly factorized.

Finally, we define the angle between (the subspaces spanned by) two unitary vectors u, v to be

$$\angle(u, v) := \arccos |u^H v|.$$

III. Tracking Limitation of Unity Feedback

A. Stable Plants

We shall examine first stable plants, for which the parameterization (2.6) results in $S = I - PQ$, and hence, the minimal tracking error can be expressed as

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - PQ)\frac{v}{s} \right\|_2^2. \tag{3.1}$$

It is clear that in order for J^* to be finite, Q must be selected such that $(I - P(0)Q(0))v = 0$. Our following result gives an explicit expression of J^*.

Theorem 3.1: Let r be given by (2.2), and suppose that $P(s)$ is stable. Also, suppose that $P(s)$ is factorized as in (2.7). Then

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i, v). \tag{3.2}$$

Proof: According to (2.7) and (3.1), we can first write J^* as

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - LP_mQ)\frac{v}{s} \right\|_2^2.$$

Because $L_i(s)$ is all pass, it follows that

$$\begin{aligned} J^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(L_1^{-1} - \left(\prod_{i=2}^{k} L_i\right) P_m Q \right) \frac{v}{s} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(L_1^{-1} - I\right) \frac{v}{s} + \left(I - \left(\prod_{i=2}^{k} L_i\right) P_m Q \right) \frac{v}{s} \right\|_2^2. \end{aligned}$$

Note that $L_1^{-1}(0) = I$, which implies that $(L_1^{-1} - I)v/s \in \mathcal{H}_2^\perp$. On the other hand, because Q is to be selected so that $(I - P(0)Q(0))v = 0$, we have $(I - (\Pi_{i=2}^k L_i)P_m Q)v/s \in \mathcal{H}_2$. Hence

$$J^* = \left\| (L_1^{-1} - I)\frac{v}{s} \right\|_2^2 + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(I - \left(\prod_{i=2}^{k} L_i\right) P_m Q \right) \frac{v}{s} \right\|_2^2. \tag{3.3}$$

By a repeated use of (3.3), we obtain

$$\begin{aligned} J^* &= \sum_{i=1}^{k} \left\| (L_i^{-1} - I)\frac{v}{s} \right\|_2^2 + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - P_m Q)\frac{v}{s} \right\|_2^2 \\ &= \sum_{i=1}^{k} \left\| (I - L_i)\frac{v}{s} \right\|_2^2 + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - P_m Q)\frac{v}{s} \right\|_2^2. \end{aligned}$$

Because P_m is an outer matrix function, we have

$$\inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - P_m Q)\frac{v}{s} \right\|_2^2 = 0. \tag{3.4}$$

Next, it follows from (2.8) that

$$\left\| (I - L_i)\frac{v}{s} \right\|_2^2 = \left| \frac{2\mathrm{Re}(z_i)}{z_i} \right|^2 \left\| \frac{1}{\overline{z}_i + s} \right\|_2^2 \cdot |\eta_i^H v|^2.$$

A straightforward calculation gives rise to

$$\left\| \frac{1}{\overline{z}_i + s} \right\|_2^2 = \frac{1}{2\mathrm{Re}(z_i)}.$$

The proof can now be completed by observing that $|\eta_i^H v|^2 = \cos^2 \angle(\eta_i, v)$. ■

On the one hand, Theorem 3.1 is similar to its counterpart for SISO systems, which shows that in general perfect tracking cannot be achieved with respect to nonminimum phase plants. A distinguishing feature about this result, on the other hand, shows that the tracking accuracy depends not only on the zero locations, but also on the zero and input directions, and that this dependence is fully captured by the angles between the vectors v and η_i. To further illustrate, consider the case that P has only one right half plane zero z with a zero vector η, to which (3.2) reduces to

$$J^* = \frac{2\mathrm{Re}(z)}{|z|^2} \cos^2 \angle(\eta, v).$$

It is clear from this expression that perfect tracking is still possible, provided the input direction is orthogonal to the zero direction. Spatial properties of this kind have no analog in SISO systems. Note that for any nonminimum phase zero z_i with a zero vector $\tilde{\eta}_i$, we always have

$$J^* \geq \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\tilde{\eta}_i, v).$$

Because the tracking error J^* depends on the input direction, it is of interest to examine its largest and smallest possible value achievable among all possible v. This process amounts to determining

$$J^*_{\max} := \max_{\|v\|=1} J^*, \qquad J^*_{\min} := \min_{\|v\|=1} J^*$$

which can be readily characterized as follows. Note first that J^* can be alternatively written as

$$J^* = v^H \left(\sum_{i=1}^{k} \frac{2\mathrm{Re} z_i}{|z_i|^2} \eta_i \eta_i^H \right) v.$$

This process suggests that

$$J^*_{\max} = \lambda_{\max} \left(\sum_{i=1}^{k} \frac{2\mathrm{Re} z_i}{|z_i|^2} \eta_i \eta_i^H \right)$$
$$J^*_{\min} = \lambda_{\min} \left(\sum_{i=1}^{k} \frac{2\mathrm{Re} z_i}{|z_i|^2} \eta_i \eta_i^H \right)$$

and that the least and the most desirable input signal directions coincide with those of the eigenvectors corresponding to the largest and smallest eigenvalues. Let the dimension of v be m. We can see that whenever $k < m$, $J^*_{\min} = 0$, and hence, perfect tracking is possible in this situation when v is appropriately aligned. Furthermore, it is not difficult to see that $J^*_{\max}$ depends on the angles between the vectors η_i. Indeed, a simple calculation for the case $k = 2$ reveals that

$$J^*_{\max} = \frac{\mathrm{Re} z_1}{|z_1|^2} + \frac{\mathrm{Re} z_2}{|z_2|^2} + \sqrt{\left(\frac{\mathrm{Re} z_1}{|z_1|^2} + \frac{\mathrm{Re} z_2}{|z_2|^2} \right)^2 - 4 \frac{\mathrm{Re} z_1}{|z_1|^2} \cdot \frac{\mathrm{Re} z_2}{|z_2|^2} \sin^2 \angle(\eta_1, \eta_2)}.$$

When $m = k = 2$, we also obtain

$$J^*_{\min} = \frac{\mathrm{Re} z_1}{|z_1|^2} + \frac{\mathrm{Re} z_2}{|z_2|^2} - \sqrt{\left(\frac{\mathrm{Re} z_1}{|z_1|^2} + \frac{\mathrm{Re} z_2}{|z_2|^2} \right)^2 - 4 \frac{\mathrm{Re} z_1}{|z_1|^2} \cdot \frac{\mathrm{Re} z_2}{|z_2|^2} \sin^2 \angle(\eta_1, \eta_2)}.$$

Both of these expressions exhibit an explicit dependence of $J^*_{\max}$ and $J^*_{\min}$ on the angle between η_1 and η_2.

B. Unstable Plants

More generally, Theorem 3.1 can be extended to unstable plants. For technical reasons, we shall assume that the plant transfer function P does not have a right half plane zero and pole at the same location, and that P has only *simple* poles in $\mathbb{C}_+$. Our next result demonstrates that plant unstable poles, when coupled with its nonminimum phase zeros, may have a severe negative effect on the achievable tracking performance.

Theorem 3.2: Let r be given by (2.2) and $N(s)$ be factorized as in (2.7). Then

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i, v) + v^H H v \tag{3.5}$$

where

$$H = \sum_{i,\, j \in \mathbb{I}} \frac{4\mathrm{Re}(p_i)\mathrm{Re}(p_j)}{(\overline{p}_i + p_j) p_i \overline{p}_j \overline{b}_i b_j} (I - L^{-1}(p_i))^H (I - L^{-1}(p_j))$$
$$b_i := \prod_{\substack{j \in \mathbb{I} \\ j \neq i}} \frac{\overline{p}_j}{p_j} \frac{p_j - p_i}{\overline{p}_j + p_i}$$

and $\mathbb{I}$ is an index set defined by $\mathbb{I} := \{i \colon \tilde{M}(p_i)v = 0\}$.

Proof: First, using the doubly coprime factorizations (2.3) and Youla parameterization (2.5) yields $S = (X - NQ)\tilde{M}$. Hence

$$J^* = \inf_{Q \in \mathcal{RH}_\infty} \left\| (X\tilde{M} - NQ\tilde{M}) \frac{v}{s} \right\|_2^2.$$

Because N can be factorized as in (2.7), it follows that

$$\begin{aligned} J^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (X\tilde{M} - LN_mQ\tilde{M}) \frac{v}{s} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (L^{-1}X\tilde{M} - N_mQ\tilde{M}) \frac{v}{s} \right\|_2^2. \end{aligned} \tag{3.6}$$

We claim that $L^{-1}X\tilde{M} = L^{-1} + R_1$ for some $R_1 \in \mathbb{R}\mathcal{H}_\infty$. Indeed, it follows from (2.4) that $\tilde{M}X - \tilde{N}Y = I$. Premultiplying this equation by $\tilde{M}^{-1}$ and postmultiplying it by $\tilde{M}$ leads to $X\tilde{M} - PY\tilde{M} = I$, which implies that $PY\tilde{M} \in \mathbb{R}\mathcal{H}_\infty$, and accordingly $P_mY\tilde{M} \in \mathbb{R}\mathcal{H}_\infty$. Consequently, it follows that $R_1 = L^{-1}(X\tilde{M} - I) = P_mY\tilde{M} \in \mathbb{R}\mathcal{H}_\infty$. It then follows that

$$\begin{aligned} J^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (L^{-1} + R_1 - N_mQ\tilde{M}) \frac{v}{s} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (L^{-1} - I) \frac{v}{s} + (I + R_1 - N_mQ\tilde{M}) \frac{v}{s} \right\|_2^2. \end{aligned}$$

As in the proof for Theorem 3.1, we have $(L^{-1} - I)v/s \in \mathcal{H}_2^\perp$, and $(I + R_1 - N_mQ\tilde{M})v/s \in \mathcal{H}_2$. Hence

$$J^* = \left\| (L^{-1} - I) \frac{v}{s} \right\|_2^2 + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I + R_1 - N_mQ\tilde{M}) \frac{v}{s} \right\|_2^2. \tag{3.7}$$

Denote

$$J_1^* := \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I + R_1 - N_mQ\tilde{M}) \frac{v}{s} \right\|_2^2.$$

In light of (3.7), it suffices to show that $J_1^* = v^H H v$. Toward this end, define $f(s) := \tilde{M}(s)v$. Because $f(p_i) = 0$ for any $i \in \mathbb{I}$, and because $f(s)$ is left invertible, it admits a factorization $f(s) = g(s)b(s)$, where $g(s)$ is a minimum phase part and $b(s) \in \mathbb{R}\mathcal{H}_\infty$ is a scalar all-pass factor that can be formed as

$$b(s) = \prod_{i \in \mathbb{I}} \frac{\overline{p}_i}{p_i} \frac{p_i - s}{\overline{p}_i + s}.$$

Moreover, because $f(s) \in \mathbb{R}\mathcal{H}_\infty$, $g(s)$ is a co-outer factor [5] of $f(s)$ whose left inverse is analytic in $\mathbb{C}_+$. It now follows that

$$\begin{aligned} J_1^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| ((I + R_1)v - N_mQgb) \frac{1}{s} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(\frac{(I + R_1)v}{b} - N_mQg \right) \frac{1}{s} \right\|_2^2. \end{aligned}$$

Based on a partial fraction procedure, we may write

$$\frac{I + R_1(s)}{b} = \sum_{i \in \mathbb{I}} \left(\frac{p_i}{\overline{p}_i} \frac{\overline{p}_i + s}{p_i - s} \right) \frac{I + R_1(p_i)}{b_i} + R_2(s)$$

where $R_2 \in \mathbb{R}\mathcal{H}_\infty$. Furthermore, it is easy to verify that $R_1(p_i)v = -L^{-1}(p_i)v$ for any $i \in \mathbb{I}$. Therefore

$$\begin{aligned} J_1^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(\sum_{i \in \mathbb{I}} \left(\frac{p_i}{\overline{p}_i} \frac{\overline{p}_i + s}{p_i - s} \right) \frac{I - L^{-1}(p_i)}{b_i} v \right. \right. \\ &\qquad \left. \left. + R_2(s)v - N_mQg \right) \frac{1}{s} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(\sum_{i \in \mathbb{I}} \left(\frac{p_i}{\overline{p}_i} \frac{\overline{p}_i + s}{p_i - s} - 1 \right) \frac{I - L^{-1}(p_i)}{b_i} v \right. \right. \end{aligned}$$

$$+\ R_2(s)v + \frac{I - L^{-1}(p_i)}{b_i}v - N_m Q g\Bigg) \frac{1}{s}\Bigg\|_2^2$$
$$= \left\| \sum_{i \in \mathbb{I}} \frac{1}{p_i - s}\left(\frac{2\mathrm{Re}p_i(I - L^{-1}(p_i))}{\overline{p}_i b_i} v\right)\right\|_2^2$$
$$+ \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(R_2(s)v + \frac{I - L^{-1}(p_i)}{b_i}v - N_m Q g\right)\frac{1}{s}\right\|_2^2.$$

Because N_m is right invertible and g left invertible, we have

$$\inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(R_2(s)v + \frac{I - L^{-1}(p_i)}{b_i}v - N_m Q g\right)\frac{1}{s}\right\|_2^2 = 0.$$

Consequently

$$J^* = v^H \Bigg(\sum_{i,j \in \mathbb{I}} \frac{4\mathrm{Re}(p_i)\mathrm{Re}(p_j)}{p_i \overline{p}_j \overline{b}_i b_j}(I - L^{-1}(p_i))^H$$
$$\cdot\ (I - L^{-1}(p_j)) \left\langle \frac{1}{p_i - s}, \frac{1}{p_j - s}\right\rangle\Bigg) v.$$

Let Γ be a positively oriented closed curve that encircles $s = p_j$. Then, by invoking Cauchy's theorem, we obtain

$$\left\langle \frac{1}{p_i - s}, \frac{1}{p_j - s}\right\rangle = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{d(j\omega)}{(\overline{p}_i + j\omega)(p_j - j\omega)}$$
$$= \frac{1}{2\pi j}\int_{\Gamma} \frac{ds}{(s + \overline{p}_i)(s - p_j)} = \frac{1}{\overline{p}_i + p_j}.$$

This process establishes that $J_1^* = v^H H v$, and, hence, completes the proof. ∎

It is clear from Theorem 3.2 that for an unstable, nonminimum phase plant, its unstable poles will in general worsen the optimal tracking performance. It is interesting to note, however, that they affect the tracking performance in a rather intricate way. First, such poles will have an effect on J^* only when the plant is also nonminimum phase. Indeed, if the plant is minimum phase, then $H = 0$, for in this case $L = I$; this can also be seen from the fact that $S = (X - NQ)\tilde{M}$. Second, even for a nonminimum phase plant, they will have an effect only when the input direction coincides certain pole directions. The latter property appears rather intriguing, and it points to yet another conceptual difference between MIMO and SISO systems. Although in a multivariable system, plant unstable poles may affect the tracking performance and it depends on the alignment between the pole and input directions, in a SISO system they always do. Indeed, in the latter situation, $\mathbb{I} = \{1, 2, \cdots, l\}$, and (3.5) reduces to

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} + \sum_{i,j=1}^{l} \frac{4\mathrm{Re}(p_i); \mathrm{Re}(p_j)}{(\overline{p_i} + p_j)p_i \overline{p}_j \overline{b}_i b_j}$$
$$\cdot\ (1 - L^{-1}(p_i))^H (1 - L^{-1}(p_j)).$$

A strong indication exitsts from Theorem 3.2 that nonminimum phase zeros and unstable poles can particularly limit the tracking performance when they are close to each other. To better observe this property, it is instructive to examine (3.5) with respect to certain special cases. Consider the case in which only one unstable pole p exists whose direction coincides with the input direction. Under this circumstance

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i, v)$$
$$+ \frac{2\mathrm{Re}(p)}{|p|^2} \|(I - L^{-1}(p))v\|^2. \tag{3.8}$$

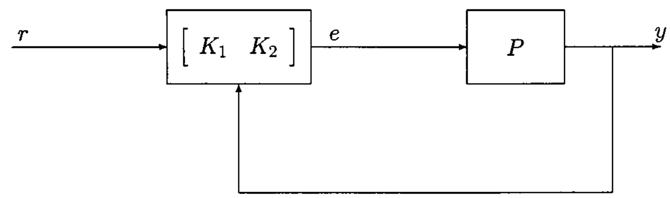

Fig. 2. Two-parameter control scheme.

Additionally, suppose that the plant has only one nonminimum phase zero z with a zero vector η. Then, it follows from (3.8) that

$$J^* = \frac{2\mathrm{Re}(z)}{|z|^2}\left(1 + \frac{4\mathrm{Re}(z)\mathrm{Re}(p)}{|z - p|^2}\right)\cos^2 \angle(\eta, v). \tag{3.9}$$

Because

$$\frac{4\mathrm{Re}(z)\mathrm{Re}(p)}{|z - p|^2} = \left|\frac{\overline{p} + z}{p - z}\right|^2 - 1$$

we may rewrite (3.9) as

$$J^* = \frac{2\mathrm{Re}(z)}{|z|^2}\left|\frac{\overline{z} + p}{z - p}\right|^2 \cos^2 \angle(\eta,\, v).$$

It is clear from this result that closely located plant nonminimum phase zeros and unstable poles can be detrimental. Nevertheless, we can also see that the extent to which the zero and pole may affect J^* depends on how the zero and pole directions are aligned, noticing that in this case the input direction and the pole direction are identical.

IV. Tracking Limitation of Two-Parameter Feedback

Two-parameter or two-degree-of-freedom systems represent the most general feedback configuration in a linear control scheme. A generic structure for this class of systems is shown in Fig. 2, with $\hat{e} = K_1\hat{r} + K_2\hat{y}$. Let the right and left coprime factorizations of P be given by (2.3). Then, the set of all stabilizing two-parameter compensators is [12]

$$\mathcal{K}_2 := \{K\colon K = [K_1\ K_2] = (\tilde{X} - R\tilde{N})^{-1}$$
$$\cdot\ (Q\tilde{Y} - R\tilde{M}),\ Q \in \mathbb{R}\mathcal{H}_\infty,\ R \in \mathbb{R}\mathcal{H}_\infty\}. \tag{4.1}$$

The tracking error and its minimal version are, respectively

$$J := \int_0^\infty \|r - y\|^2\ dt, \qquad J^* := \inf_{K \in \mathcal{K}_2} J.$$

Because $\hat{y} = NQ\hat{r}$ [12], where $Q \in \mathbb{R}\mathcal{H}_\infty$ is the parameter in (4.1), it follows that $J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \|(I - NQ)\hat{r}\|_2^2$. The following result is thus clear.

Theorem 4.1: Let r be given by (2.2) and $P(s)$ be factorized as in (2.7) and (2.8). Then

$$J^* = \sum_{i=1}^{k} \frac{2\mathrm{Re}(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i,\, v). \tag{4.2}$$

Theorem 4.1 makes it clear that in a two-parameter scheme the plant unstable poles do not affect the tracking performance, hence demonstrating a distinctive advantage of two-parameter compensators. This feature falls under the general statement in [12, p. 148] that in a two-parameter system the achievable performance between an external input and the plant output is not limited by the plant unstable poles. For this reason, for a stable plant, the optimal tracking performance achievable by both one-parameter and two-parameter compensators coincide.

V. Effect of Time Delays

Consider again the unity feedback structure shown in Fig. 1. Assume that the plant has measurement time delays, by which we mean that the plant transfer function matrix can be expressed as

$$P_d(s) = \Lambda(s)P(s)$$
$$\Lambda(s) := \mathrm{diag}(e^{-T_1 s}, \cdots, e^{-T_m s}) \tag{5.1}$$

where $P(s)$ is a real rational transfer function matrix, $T_i > 0$ represent the delay constants in the different channels, and m is the number of plant outputs. Denote the tracking error with respect to P_d by J_d, and correspondingly, the minimal error by J_d^*

$$J_d := \|(I + P_d K)^{-1}\hat{u}\|_2^2, \qquad J_d^* := \inf_{K \in \mathcal{K}} J_d.$$

Furthermore, denote the minimal tracking error with respect to P by J^*.

Consider the case in which P is stable. In this case, the right and left coprime factorizations of P_d are given by $N_d = \Lambda N = \Lambda P = P_d$, $\tilde{N}_d = \Lambda \tilde{N} = \Lambda P = P_d$, and $M = \tilde{M} = I$. Let the input be the unit step signal given by (2.2). Then, it follows that

$$J_d^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\|(I - P_d Q)\frac{v}{s}\right\|_2^2 = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\|(I - \Lambda P Q)\frac{v}{s}\right\|_2^2. \tag{5.2}$$

The following theorem gives an explicit formula for J_d^*. It shows how time delays may affect the tracking performance, attesting to the intuition that the tracking performance is generally more difficult to attain with respect to delay systems.

Theorem 5.1: Let r be given by (2.2), and suppose that P is stable. Then

$$J_d^* = \sum_{i=1}^{m} T_i |v_i|^2 + J^*. \tag{5.3}$$

Proof: First, because Λ is all pass, it follows from (5.2) that

$$J_d^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\|(\Lambda^{-1} - PQ)\frac{v}{s}\right\|_2^2 = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\|(\Lambda^{-1} - I)\frac{v}{s} + (I - PQ)\frac{v}{s}\right\|_2^2.$$

It is clear that $(\Lambda^{-1} - I)v/s \in \mathcal{H}_2^\perp$, and that $(I - PQ)v/s \in \mathcal{H}_2$. Hence

$$J_d^* = \left\|(\Lambda^{-1} - I)\frac{v}{s}\right\|_2^2 + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\|(I - PQ)\frac{v}{s}\right\|_2^2 = \left\|(\Lambda^{-1} - I)\frac{v}{s}\right\|_2^2 + J^*.$$

To complete the proof, we first note that

$$\left\|(\Lambda^{-1} - I)\frac{v}{s}\right\|_2^2 = \sum_{i=1}^{m} |v_i|^2 \left\|\frac{e^{T_i s} - 1}{s}\right\|_2^2.$$

Next, according to the Parseval identity, we have

$$\left\|\frac{e^{T_i s} - 1}{s}\right\|_2^2 = \int_0^{T_i} dt = T_i.$$

This completes the proof. ■

VI. Conclusion

Our results in this paper indicate that in a MIMO system tracking performance can be seriously limited to an extent determined by both the location and directions of plant nonminimum phase zeros, and it partially depends on how the input and zero directions are aligned. When the plant is also unstable, this adverse effect can be more acute. In general, the following statements can be made from this work.

- Nonminimum phase zeros will limit the tracking performance to a significant extent only when they are close to the imaginary axis. The effect of nonminimum phase zeros is determined by zero locations and the mutual orientation between zero and input signal directions.
- Unstable poles will affect the tracking performance only when the plant is also nonminimum phase, and when the input and certain pole directions are completely aligned. When this is the case, approximate unstable pole-zero cancelation can lead to particularly poor tracking performance.
- Time delays in the plant will degrade the tracking performance, in much the same way as nonminimum phase zeros do.
- In a two-parameter control scheme, only nonminimum phase zeros will affect the tracking performance.

These results shed new light on tracking performance issues, and, more generally, lend new insight into the study of fundamental limitation of feedback control.

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