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Linear periodically time-varying discrete-time systems: Aliasing and LTI approximations[1]

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Abstract

Linear periodically time-varying (LPTV) systems are abundant in control and signal processing; examples include multirate sampled-data control systems and multirate filter-bank systems. In this paper, several ways are proposed to quantify aliasing effect in discrete-time LPTV systems; these are associated with optimal time-invariant approximations of LPTV systems using operator norms. © 1997 Elsevier Science B.V.

Keywords: Periodic systems; Multirate systems; Optimization; Aliasing; Discrete-time systems

1. Introduction

Examples of linear periodically time-varying (LPTV) systems are abundant: In control, multirate sampled-data systems are designed to exploit their cost advantage in digital implementation [6,5]; in signal processing, multirate filter banks, which are typically LPTV, are designed for efficient coding and transmission of digital signals [11].

Different from linear time-invariant (LTI) systems, aliasing exists in LPTV systems; this may cause adverse effect for robustness against high-frequency uncertainties in periodic control systems [7] and for perfect reconstruction in multirate filter banks [11]. The *first question* in this paper is therefore:

How to quantify aliasing effect in LPTV systems?

If aliasing is negligible in an LPTV system to be controlled, one can then approximate it by an LTI system with little error. Control design can be then based on the LTI model; this has several advantages: First, robust control design for LTI systems is thoroughly studied and there are now many techniques applicable; second, the controller designed in this way normally is LTI too and so is easier to implement than an LPTV controller, resulting from design based on the original LPTV system. Hence the *second question* in this paper is:

How to optimally approximate an LPTV system by an LTI one?

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Throughout the paper, we will focus on discrete-time MIMO (multi-input multi-output) systems. The two questions are related as follows. Since LTI systems form a subspace within LPTV systems, we consider the following distance problem:

Given an LPTV system, compute its distance to the subspace of LTI systems.

The distance, to be measured by norms, is a measure of how time varying the LPTV system is and hence can be used to quantify aliasing; the LTI system achieving the distance is the optimal LTI approximation to the given LPTV system.

Two norms will be used for LPTV systems: the Hilbert–Schmidt norm or $\mathscr{H}_2$ norm and the ℓ_2-induced norm or $\mathscr{H}_\infty$ norm.

LPTV systems have no transfer functions in general; however, there are two ways to describe their frequency responses using matrices: The first one is based on a time-domain technique called lifting in control [8] or blocking in signal processing [11]; the second one is a frequency-domain technique, also independently used in control [7] and signal processing [10]. Though the two techniques are essentially related [9], here we adopt the latter for better insight in the frequency domain.

Briefly, the paper is organized as follows. In the next section we discuss a frequency-domain representation for LPTV systems, which is relevant to our studies later. Section 3 studies the distance problem using Hilbert–Schmidt norm and gives complete solutions. Section 4 looks at the distance problem using ℓ_2-induced norm and only partial solutions are obtained. In Section 5 we show the relevance of the work here to an example of LPTV systems in signal processing, namely, the multirate filter-bank system used in, e.g., subband coding. Finally, we conclude in Section 5.

2. Frequency-response matrices

We begin with the definition of frequency-response matrices from [7, 10]. (A continuous-time analog of this was introduced in [2].) It is convenient to define the exponential signal of frequency f:

$$e_f(k) := \mathrm{e}^{\mathrm{j}2\pi fk}.$$

If H were LTI and stable, the output of H due to this input $e_f(k)$ would be $\hat{H}(f)e_f(k)$, $\hat{H}(f)$ being the frequency response.

Now let H denote an LPTV system of period m. If the input to H is again $e_f(k)$, the corresponding output is now a linear combination of the complex exponentials of frequencies [10]

$$f, f+\frac{1}{m}, \ldots, f+\frac{m-1}{m}.$$

Define the m-dimensional subspace

$$\mathscr{S}_f := \operatorname{span}\{e_f, e_{f+1/m}, \ldots, e_{f+(m-1)/m}\}.$$

It follows [10] that $\mathscr{S}_f$ is an invariant subspace for H. Thus, an input to H of the form

$$u(k) = \sum_{n=0}^{m-1} u_n e_{f+n/m}(k),$$

which is represented by the m-dimensional vector

$$\hat{u} := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix}$$

for the obvious choice of basis functions in $\mathscr{S}_f$, will produce an output of the form

$$y(k) = \sum_{n=0}^{m-1} y_n e_{f+n/m}(k),$$

which is represented by the vector

$$\hat{y} := \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{m-1} \end{bmatrix}$$

This induces an $m \times m$ matrix, denoted $\hat{H}_{\mathrm{FR}}(f)$, relating $\hat{u}$ to $\hat{y}$:

$$\hat{y} = \hat{H}_{\mathrm{FR}}(f)\hat{u}.$$

This matrix is called the alias component matrix in [10] in view of its prior occurrence in the literature on multirate filter banks, and is a generalization of frequency-response function. The matrix $\hat{H}_{\mathrm{FR}}$ is called the *frequency-response matrix* for the LPTV system H from now on. Note that in the definition of $\hat{H}_{\mathrm{FR}}(f)$, f ranges over the interval $-1/2m \leqslant f \leqslant 1/2m$.

This frequency-response matrix has an equivalent interpretation as follows. Let the input and output of H be u and y. Denote the Fourier transform of u by $\hat{u}(f)$, a periodic function with period 1 (in f). Chop one period of $\hat{u}$ into m pieces and form a vector:

$$\begin{bmatrix} \hat{u}(f) \\ \hat{u}\left(f - \frac{1}{m}\right) \\ \vdots \\ \hat{u}\left(f - \frac{m-1}{m}\right) \end{bmatrix}, \quad -\frac{1}{2m} \leqslant f \leqslant \frac{1}{2m}.$$

Similarly, for the Fourier transform $\hat{y}(f)$. It follows that the two vectors are related exactly by the frequency-response matrix:

$$\begin{bmatrix} \hat{y}(f) \\ \hat{y}\left(f - \frac{1}{m}\right) \\ \vdots \\ \hat{y}\left(f - \frac{m-1}{m}\right) \end{bmatrix} = \hat{H}_{\mathrm{FR}} \begin{bmatrix} \hat{u}(f) \\ \hat{u}\left(f - \frac{1}{m}\right) \\ \vdots \\ \hat{u}\left(f - \frac{m-1}{m}\right) \end{bmatrix}, \quad -\frac{1}{2m} \leqslant f \leqslant \frac{1}{2m}. \tag{1}$$

Now we look at how to compute frequency-response matrices. We start with a causal, MIMO, LPTV system H with period m. H is characterized by its impulse response matrix $h(k,l)$ as follows:

$$y(k) = \sum_{l=0}^{k} h(k,l)u(l). \tag{2}$$

Periodicity is equivalent to the condition

$$h(k+m, l+m) = h(k,l), \quad \forall k, l,$$

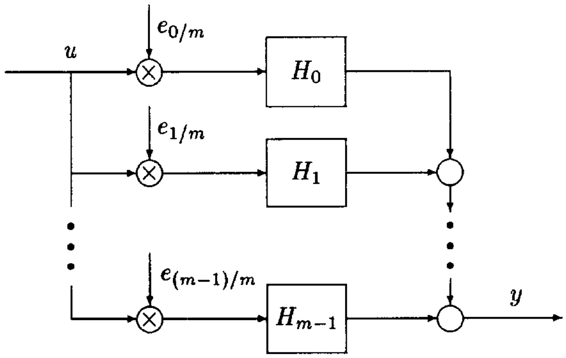

Fig. 1. Decomposition of the LPTV system H.

and causality is equivalent to

$$h(k,l) = 0, \quad \text{whenever } l > k.$$

Define $\tau = k - l$ to get $h(k,l) = h(\tau + l, l)$. It follows that for any fixed τ, $h(\tau + l, l)$ is m-periodic in l and hence has a discrete Fourier series:

$$h(\tau + l, l) = \sum_{n=0}^{m-1} h_n(\tau) e_{n/m}(l), \tag{3}$$

$$h_n(\tau) = \frac{1}{m} \sum_{l=0}^{m-1} h(\tau + l, l) e_{-n/m}(l). \tag{4}$$

Here $h_n(\tau) = 0$ for $\tau < 0$ due to causality. Substitute (3) into (2) to get that y can be expressed as a sum of convolutions of h_n with $e_{n/m}u$:

$$y = \sum_{n=0}^{m-1} h_n * [e_{n/m} u]. \tag{5}$$

Let H_n be the causal, LTI system with impulse response matrix $h_n(k)$. Eq. (5) represents a time-domain decomposition of the LPTV H as depicted in Fig. 1: The input $u(k)$ is channeled into m different subsystems numbered $0, 1, \ldots, m-1$; at the nth subsystem $u(k)$ is first modulated by the exponential function $e_{n/m}(k)$ and then passed through the LTI system H_n; the sum of the outputs of H_n forms y. The m LTI systems H_n are called the *components* of the LPTV system H; they uniquely characterize H.

Based on the decomposition in (5), it is easy to get that the LPTV system H becomes LTI iff $h_n = 0$ for $n = 1, 2, \ldots, m-1$. To generalize this, let $m = m_1 m_2$ with m_1 and m_2 both positive integers. How to test m_1 periodicity of H based on its m LTI component systems H_n?

Theorem 1. *Assume a causal, MIMO system H is LPTV with period m; its associated component systems are denoted $H_n^{(m)}$: $n = 0, 1, \ldots, m-1$. Let positive integers m_1 and m_2 satisfy $m = m_1 m_2$. Then H is LPTV with period m_1 iff*

$$H_n^{(m)} = 0, \quad n \neq 0, m_2, 2m_2, \ldots, (m_1 - 1)m_2;$$

in this case, the m_1 component systems of H associated with periodicity m_1 are given by

$$H_n^{(m_1)} = H_{nm_2}^{(m)}, \quad n = 0, 1, \ldots, m_1 - 1.$$

The decomposition in (5) or in Fig. 1 gives a way to compute the frequency-response matrix for H. Take Fourier transform of both sides of (5) to get

$$\hat{y}(f) = \sum_{n=0}^{m-1} \hat{H}_n(f)\hat{u}\left(f - \frac{n}{m}\right) = [\hat{H}_0(f)\ \hat{H}_1(f)\ \cdots\ \hat{H}_{m-1}(f)] \begin{bmatrix} \hat{u}(f) \\ \hat{u}\left(f - \frac{1}{m}\right) \\ \vdots \\ \hat{u}\left(f - \frac{m-1}{m}\right) \end{bmatrix}.$$

Replacing the frequency f in the above equation by $f - 1/m, \ldots, f - (m-1)/m$, respectively, and noting the periodicity of $\hat{u}$ $[\hat{u}(f \pm 1) = \hat{u}(f)]$, one can get a matrix equation in the form of (1), where the frequency-response matrix is

$$\hat{H}_{\mathrm{FR}}(f) = \begin{bmatrix} \hat{H}_0(f) & \hat{H}_1(f) & \cdots & \hat{H}_{m-1}(f) \\ \hat{H}_{m-1}\left(f - \frac{1}{m}\right) & \hat{H}_0\left(f - \frac{1}{m}\right) & \cdots & \hat{H}_{m-2}\left(f - \frac{1}{m}\right) \\ \vdots & \vdots & & \vdots \\ \hat{H}_1\left(f - \frac{m-1}{m}\right) & \hat{H}_2\left(f - \frac{m-1}{m}\right) & \cdots & \hat{H}_0\left(f - \frac{m-1}{m}\right) \end{bmatrix}, \quad -\frac{1}{2m} \leqslant f \leqslant \frac{1}{2m}.$$

(This representation is also given in [14].) The frequency-response matrix is completely characterized by the m transfer functions $\hat{H}_0, \hat{H}_1, \ldots, \hat{H}_{m-1}$. Note the row-wise circular structure coupled with the frequency shift.

As a special case, if H is LTI, the frequency-response matrix is diagonal:

$$\hat{H}_{\mathrm{FR}}(f) = \begin{bmatrix} \hat{H}_0(f) & 0 & \cdots & 0 \\ 0 & \hat{H}_0\left(f - \frac{1}{m}\right) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \hat{H}_0\left(f - \frac{m-1}{m}\right) \end{bmatrix}, \quad -\frac{1}{2m} \leqslant f \leqslant \frac{1}{2m}.$$

(A condition for time invariance was also obtained in [12] in the time domain using state-space models.)

As an example, consider the state-space model with input u, output y, and state vector x:

$$x(k+1) = A(k)x(k) + B(k)u(k), \qquad y(k) = C(k)x(k) + D(k)u(k).$$

Assume that $A(k)$, $B(k)$, $C(k)$, and $D(k)$ are LPTV with period 2; write

$$A(k) = \begin{cases} A_0 & \text{if } k \text{ is even}, \\ A_1 & \text{if } k \text{ is odd}, \end{cases}$$

and similarly for $B(k)$, $C(k)$, and $D(k)$. This system is LPTV with period 2. Its component systems H_0 and H_1 can be computed from definitions; they are given by their transfer functions $\hat{H}_0(z)$ and $\hat{H}_1(z)$: First define two functions

$$\hat{G}_0(z) = D_0 + (C_0A_1 + zC_1)(z^{-2}I - A_0A_1)^{-1}B_0, \qquad \hat{G}_1(z) = D_1 + (C_1A_0 + zC_0)(z^{-2}I - A_1A_0)^{-1}B_1;$$

then

$$\hat{H}_0(z) = \tfrac{1}{2}[\hat{G}_0(z) + \hat{G}_1(z)], \qquad \hat{H}_1(z) = \tfrac{1}{2}[\hat{G}_0(z) - \hat{G}_1(z)].$$

Note that in general the orders of the LTI systems H_0 and H_1 exceed the dimension in the matrix $A(k)$ but are finite. Of course, these formulas can be generalized to LPTV state-space systems with a general period m.

Based on the frequency-response matrices, we shall study two optimal approximation problems involving LPTV and LTI systems; the quantities used to measure degree of closeness of two LPTV systems are the Hilbert–Schmidt norm and the ℓ_2-induced norm.

3. Using the Hilbert–Schmidt norm

Any MIMO, causal, LPTV system H is completely determined by its impulse response matrix $h(k,l)$ for $0 \leqslant k < \infty$ and $0 \leqslant l < m-1$. We say H is *stable* if

$$\sum_{l=0}^{m-1} \sum_{k=0}^{\infty} \text{trace}\,[h(k,l)'h(k,l)] < \infty.$$

All causal, stable, LPTV systems with period m form a Hilbert space with the Hilbert–Schmidt norm:

$$\|H\|_{\text{HS}} = \left(\frac{1}{m} \sum_{l=0}^{m-1} \sum_{k=0}^{\infty} \text{trace}\,[h(k,l)'h(k,l)] \right)^{1/2}. \tag{6}$$

(H can be regarded as a Hilbert–Schmidt operator if one restricts the input to be defined on the time set $[0,1,\ldots,m-1]$.) Several ways to evaluate this norm are given below (some of these are also given in [14]):

1. In terms of the component functions $h_n(\tau)$, $n = 0,1,\ldots,m-1$, we have

$$\|H\|_{\text{HS}}^2 = \sum_{n=0}^{m-1} \sum_{\tau=0}^{\infty} \text{trace}\,[h_n(\tau)'h_n(\tau)]. \tag{7}$$

Proof. Rearrange the summations in (6) to get

$$\begin{aligned} \|H\|_{\text{HS}}^2 &= \frac{1}{m} \sum_{\tau=0}^{\infty} \sum_{l=0}^{m-1} \text{trace}\,[h(\tau+l,l)'h(\tau+l,l)] \\ &= \frac{1}{m} \sum_{\tau=0}^{\infty} \text{trace} \begin{bmatrix} h(\tau,0) \\ h(\tau+1,1) \\ \vdots \\ h(\tau+m-1,m-1) \end{bmatrix}' \begin{bmatrix} h(\tau,0) \\ h(\tau+1,1) \\ \vdots \\ h(\tau+m-1,m-1) \end{bmatrix}. \end{aligned} \tag{8}$$

The DFT involved in (3) implies

$$\begin{bmatrix} h(\tau,0) \\ h(\tau+1,1) \\ \vdots \\ h(\tau+m-1,m-1) \end{bmatrix} = \begin{bmatrix} I & I & \cdots & I \\ I & e^{j2\pi/m}I & \cdots & e^{j2\pi(m-1)/m}I \\ \vdots & \vdots & & \vdots \\ I & e^{j2\pi(m-1)/m}I & \cdots & e^{j2\pi(m-1)^2/m}I \end{bmatrix} \begin{bmatrix} h_0(\tau) \\ h_1(\tau) \\ \vdots \\ h_{m-1}(\tau) \end{bmatrix}.$$

Note that the DFT matrix W appears here. Substituting this into (8) and using the property $W^*W = mI$, we get

$$\|H\|_{\text{HS}}^2 = \sum_{\tau=0}^{\infty} \text{trace} \begin{bmatrix} h_0(\tau) \\ h_1(\tau) \\ \vdots \\ h_{m-1}(\tau) \end{bmatrix}' \begin{bmatrix} h_0(\tau) \\ h_1(\tau) \\ \vdots \\ h_{m-1}(\tau) \end{bmatrix} = \sum_{n=0}^{m-1} \sum_{\tau=0}^{\infty} \text{trace}\,[h_n(\tau)'h_n(\tau)]. \qquad \square$$

2. In terms of the frequency responses $\hat{H}_n(f)$ of the LTI component systems H_n, $n = 0, 1, \ldots, m-1$, we have

$$\|H\|_{\mathrm{HS}}^2 = \sum_{n=0}^{m-1} \|\hat{H}_n\|_2^2 = \sum_{n=0}^{m-1} \int_{-1/2}^{1/2} \mathrm{trace}\,[\hat{H}_n(f)^* \hat{H}_n(f)]\,\mathrm{d}f. \tag{9}$$

Here $\|\hat{H}_n\|_2$ denotes the $\mathscr{H}_2$ norm of the LTI system. This result follows from (7) by Parseval's equality.

3. In terms of the frequency-response matrix $\hat{H}_{\mathrm{FR}}(f)$, we have

$$\|H\|_{\mathrm{HS}}^2 = \|\hat{H}_{\mathrm{FR}}\|_2^2 := \int_{-1/2m}^{1/2m} \mathrm{trace}\,[\hat{H}_{\mathrm{FR}}(f)^* \hat{H}_{\mathrm{FR}}(f)]\,\mathrm{d}f.$$

This follows readily from (9) and the definitions of $\hat{H}_{\mathrm{FR}}(f)$.

The distance problem to be studied in this section is: Given a causal, stable, LPTV system H, what is the distance, measured by the Hilbert–Schmidt norm, from H to the subspace of causal, stable, LTI systems? This problem is written:

$$\mu := \min_{\mathrm{LTI}\,G} \|H - G\|_{\mathrm{HS}}.$$

The LTI system G_{opt} achieving this minimum is the optimal LTI approximation to the LPTV H. The quantity μ can be used as a measure of aliasing in the LPTV system, or relatively to the size of H, one can use the quantity $v := \mu/\|H\|_{\mathrm{HS}}$ $(0 \leqslant v \leqslant 1)$ to measure aliasing; in this case, $v = 0$ means H is already LTI and $v = 1$ means H is anti-LTI, i.e., the optimal LTI approximation $G_{\mathrm{opt}} = 0$.

Theorem 2. *The optimal LTI approximation to the LPTV system H is $G_{\mathrm{opt}} = H_0$ and*

$$\mu = \left(\sum_{n=1}^{m-1} \|\hat{H}_n\|_2^2 \right)^{1/2}.$$

Proof. Let G be any LTI, causal, stable system with compatible input and output dimensions with H. The component systems G_n for G as an LPTV system with period m are given by

$$G_n = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } 1 \leqslant n \leqslant m-1. \end{cases}$$

By property 2 above we have

$$\|H - G\|_{\mathrm{HS}}^2 = \sum_{n=0}^{m-1} \|\hat{H}_n - \hat{G}_n\|_2^2 = \|\hat{H}_0 - \hat{G}\|_2^2 + \sum_{n=1}^{m-1} \|\hat{H}_n\|_2^2.$$

Clearly, this quantity is minimized by taking $\hat{G} = \hat{H}_0$ and hence the results. □

This theorem also suggests that one can *orthogonally* decompose an LPTV system H into $H = H_{\mathrm{LTI}} + H_{\mathrm{LTV}}$, where the LTI component is $H_{\mathrm{LTI}} = G_{\mathrm{opt}}$ and the anti-LTI (aliasing) component is H_{LTV}. Then

$$\|H\|_{\mathrm{HS}}^2 = \|H_{\mathrm{LTI}}\|_{\mathrm{HS}}^2 + \|H_{\mathrm{LTV}}\|_{\mathrm{HS}}^2.$$

To generalize Theorem 2, we can also consider approximating an LPTV system with period m by an LPTV system with period m_1, where m is a multiple of m_1. Let H be LPTV with period m as before and write $m = m_1 m_2$ with positive integers m_1 and m_2. The problem is as follows:

$$\rho := \min\{\|H - G\|_{\mathrm{HS}}\colon\ G \text{ is LPTV with period } m_1\}.$$

Theorem 3. *Given the LPTV system H with period m as above, the optimal LPTV approximation $G^{\rm opt}$ with period m_1 is given by the component systems:*

$$G_n^{\rm opt} = H_{nm_2}, \qquad n = 0, 1, \ldots, m_1 - 1.$$

Moreover,

$$\rho = \left(\sum \|\hat{H}_n\|_2^2\right)^{1/2},$$

where the sum is over all integers n with $0 \leqslant n \leqslant m-1$ and $n \neq 0, m_2, 2m_2, \ldots, (m_1-1)m_2$.

This generalization is relevant if one would like to reduce the number of modeling parameters in LPTV systems by reducing the periodicity number; and the quantity ρ can be used as an indicator for error incurred in this approximation.

4. Using the ℓ_2-induced norm

The second norm we use for approximation is the ℓ_2-induced norm. For a causal, MIMO, LPTV system H with period m, let u and y be the input and output vectors, respectively. The ℓ_2-induced norm of H is defined as

$$\|H\| := \sup\{\|y\|_2 \colon \|u\|_2 = 1\}.$$

If this is finite, we say in this section that H is *stable*.

It is proved in [10,9] that the ℓ_2-induced norm of an LPTV system H equals the ∞-norm of the frequency-response matrix:

$$\|H\| = \|\hat{H}_{\rm FR}\|_\infty = \max_{-1/2m \leqslant f \leqslant 1/2m} \sigma_{\max}[\hat{H}_{\rm FR}(f)],$$

where $\sigma_{\max}$ denotes the maximum singular value. This gives a way to evaluate the ℓ_2-induced norm in terms of the frequency-response matrices.

The distance problem in this section is as follows: Given a stable, LPTV system H, compute the distance, measured by the ℓ_2-induced norm, to the subspace of LTI systems:

$$\mu_\infty := \inf_{\text{LTI } G} \|H - G\|.$$

Again, the minimizing LTI system, $G_{\rm opt}$, can be used as an approximation to H and the quantity μ_∞ as a measure of aliasing in H. These are useful for robust control using unstructured uncertainty models: Using the LTI $G_{\rm opt}$ as the nominal model, the size of the modeling error is given by μ_∞.

Theorem 4. *Assume H is causal, stable, and LPTV with period 2. The optimal LTI approximation is $G_{\rm opt} = H_0$ and*

$$\mu_\infty = \|\hat{H}_1\|_\infty := \max_{-1/4 \leqslant f \leqslant 1/4} \sigma_{\max}[\hat{H}_1(f)].$$

Proof. For any LTI G, the frequency-response matrix for $H - G$ is

$$\hat{H}_{\rm FR}(f) - \hat{G}_{\rm FR}(f) = \begin{bmatrix} \hat{H}_0(f) - \hat{G}_0(f) & \hat{H}_1(f) \\ \hat{H}_1(f - \frac{1}{2}) & \hat{H}_0(f - \frac{1}{2}) - \hat{G}_0(f - \frac{1}{2}) \end{bmatrix}, \quad -\tfrac{1}{4} \leqslant f \leqslant \tfrac{1}{4}.$$

In this matrix we note that both $\hat{H}_1(f)$ and $\hat{H}_1(f-\frac{1}{2})$, $-\frac{1}{4}\leqslant f\leqslant\frac{1}{4}$, appear as submatrices; hence for any LTI G, we have

$$\|H-G\| = \|\hat{H}_{\mathrm{FR}}(f)-\hat{G}_{\mathrm{FR}}(f)\|_\infty \geqslant \|\hat{H}_1\|_\infty.$$

So $\mu_\infty \geqslant \|\hat{H}_1\|_\infty$. It can be verified that setting $G=H_0$, we get $\|H-G\| = \|\hat{H}_1\|_\infty$ and therefore $G_{\mathrm{opt}}=H_0$ and $\mu_\infty = \|\hat{H}_1\|_\infty$. □

Though in this special case the optimal approximation equals that using the Hilbert–Schmidt norm, the distance μ_∞ is different.

The proof does not work for general LPTV systems with period m; in this case, only lower and upper bounds for μ_∞ are obtained:

$$\mu_\infty \geqslant \|[\hat{H}_1(f)\ \ \hat{H}_2(f)\ \ \cdots\ \ \hat{H}_{m-1}(f)]\|_\infty,$$

$$\mu_\infty \leqslant \left\| \begin{bmatrix} 0 & \hat{H}_1(f) & \cdots & \hat{H}_{m-1}(f) \\ \hat{H}_{m-1}\left(f-\dfrac{1}{m}\right) & 0 & \cdots & \hat{H}_{m-2}\left(f-\dfrac{1}{m}\right) \\ \vdots & \vdots & & \vdots \\ \hat{H}_1\left(f-\dfrac{m-1}{m}\right) & \hat{H}_2\left(f-\dfrac{m-1}{m}\right) & \cdots & 0 \end{bmatrix} \right\|_\infty .$$

If $m=2$, the lower and upper bounds reduce to the same (and hence the results in Theorem 4); however, it is not the case in general.

In [3], LTI approximations of periodic systems were also studied, but they are not optimal in the sense of the ℓ_2-induced norm.

5. Application to multirate signal processing

The results in the preceding sections have applications in quantifying aliasing in multirate digital signal processing systems.

Consider the multirate filter bank in Fig. 2. In this discrete-time system, G_0, G_1, F_0, and F_1 are LTI filters, $\downarrow 2$ denotes the downsampler (decimator) by a factor of two, and $\uparrow 2$ denotes the upsampler (expander) by the same factor. This multirate signal processing system is important in, e.g., subband coding, and has been studied a great deal; see the recent book [11] and the references therein. Typically, the overall system $H: x \mapsto y$ is required to reconstruct $x(k)$, ideally with at most a time-delay error. So *perfect reconstruction* is said to be achieved [11] if for some integer $m\geqslant 0$, $y(k)=x(k-m)$. That is, the desired system from $x(k)$ to $y(k)$ is LTI with transfer function

$$\hat{H}_{\mathrm{d}}(z) = z^{-m}.$$

Note that in general the system H is LPTV with period 2 because of the presence of the down and upsamplers.

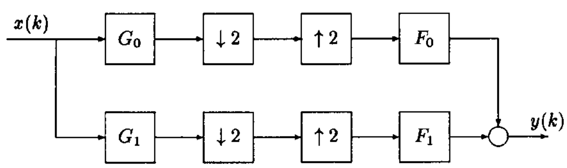

Fig. 2. Multirate filter bank.

Compared with the ideal system H_{d}, the LPTV system H in general suffers three types of distortion due to aliasing, magnitude and phase errors. Aliasing distortion can be quantified using the norms introduced earlier. It is known that the input and output of the filter bank system H are related in the frequency domain as follows [11]:

$$\hat{y}(z) = \tfrac{1}{2}[\hat{G}_0(z)\hat{F}_0(z) + \hat{G}_1(z)\hat{F}_1(z)]\hat{x}(z) + \tfrac{1}{2}[\hat{G}_0(-z)\hat{F}_0(z) + \hat{G}_1(-z)\hat{F}_1(z)]\hat{x}(-z).$$

Hence, the component systems for H are given by

$$\hat{H}_0(f) = \tfrac{1}{2}[\hat{G}_0(f)\hat{F}_0(f) + \hat{G}_1(f)\hat{F}_1(f)],$$
$$\hat{H}_1(f) = \tfrac{1}{2}[\hat{G}_0(f-\tfrac{1}{2})\hat{F}_0(f) + \hat{G}_1(f-\tfrac{1}{2})\hat{F}_1(f)].$$

In this case, two quantities can be used to measure the degree of aliasing:

$$\mu = \left(\int_{-1/2}^{1/2} |\hat{H}_1(f)|^2\,\mathrm{d}f\right)^{1/2}, \qquad \mu_\infty = \max_{|f|\leqslant 1/4} |\hat{H}_1(f)|.$$

Their interpretations are as follows: μ is the distance from this LPTV system H to the space of LTI systems via the Hilbert–Schmidt norm; and μ_∞ the distance via the ℓ_2-induced norm. (Of course, the optimal LTI approximation to H is H_0 using both norms.)

In [4], design of the synthesis filters for optimal reconstruction is developed based on $\mathscr{H}_\infty$ optimization. Note that for a given set of analysis filters, the designed optimal synthesis filters do not remove aliasing completely but keep it at a small level for optimal overall performance.

6. Concluding remarks

In Section 4, the distance problem is solved only when LPTV systems are of period 2; in the general case we obtained only lower and upper bounds for the minimum distance. Though no closed-form solutions are obtained in the general case, it is possible to compute the optimal solution within any desired accuracy via numerical optimization, because it is easy to see that the associated optimization problem is convex in nature.

In Section 2 we discussed frequency-response matrices for discrete-time LPTV systems. Frequency responses of continuous-time LPTV systems, which include sampled-data control systems as special cases, and their computation are among the recent developments in sampled-data control theory, see, e.g., [2, 1, 13]. How to quantify aliasing and compute optimal LTI approximations in sampled-data systems could be addressed using the ideas of this paper and the ground work in [2, 1, 13].

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