Proceedings of the American Control Conference
Arlington, VA June 25-27, 2001

Model Validation of Multirate Systems from Time-Domain Experimental Data[1]

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Abstract

The model validation problem using time-domain experimental data is studied for multirate linear fractional uncertain models in this paper. As a technical tool, the Carathéodory-Fejér interpolation problem with a nest operator constraint is first investigated. This problem is itself of interest mathematically and has potential applications in addressing other problems in control, signal processing, and circuit theory. A necessary and sufficient solvability condition for this interpolation problem is given. The validation tests are then given based on this condition and the lifting technique. Tractable convex optimization methods can be used to solve the validation problems.

1 Introduction

Recently, much attention has been paid on multirate systems due to its wide applications in control, communication, signal processing, econometrics and numerical mathematics. Multirate signal processing is now one of the most vibrant areas of research in signal processing, see recent book [18, 19] and references therein. The driving force for studying multirate systems in signal processing comes from the need of sampling rate conversion, subband coding, and their ability to generate wavelets. In control community, two groups of research stand out: using multirate control to achieve what single rate control cannot as well as the limitation of doing this [7] and the optimal design of multirate controllers [3, 14]. In communication community, multirate sampling is used for blind system identification and equalization [8]. We also notice the cross discipline fertilization between signal processing and control in using $\mathcal{H}_\infty$ optimization to design filter banks [4, 20].

In this paper, we will study the control-oriented model validation problems pertaining to the general multirate systems. There has recently been considerable research devoted to robust or control-oriented model validation [10, 16]. However, the research on model validation of multirate systems is almost nonexisting. Due to the wide applications of multirate systems, their model validation problem should receive comparable attention to those of single rate systems.

Model validation is a very important step in the process of control system modeling. Generally, the model validation problem is to examine if the sets of experimental data are consistent with the model of the plant. Our confidence in the model set is increased if the model is consistent with the data. A model is said to be invalidated when the validation test fails. Since a model is either invalidated or not invalidated, it is actually more accurate to call the validation procedure as model invalidation.

The approaches taken vary with the different formulation of model validation, depending on the type of the model, the assumption of the noise, the physical data available, and the identification technique. Ljung [9] discusses model validation in the traditional identification setting. More recently, motivated by the considerable research on control-oriented system identification, much attention has been paid on validation of uncertain models consisting of a nominal model and a norm bounded modeling uncertainty [1, 11, 16]. Such uncertainty models are the starting point for robust control. The first study of model validation for linear fractional transformation (LFT) model-sets was carried out by Smith and Doyle [16]. Chen [1] considered the general validation problems of linear fractional uncertain models in frequency domain and reduced it to the Nevanlinna-Pick interpolation problem, which can be solved by standard convex optimization methods. Based on the Carathéodory-Fejér (CF) interpolation problem, a purely time-domain formulation for models with an additive uncertainty is presented in [11]. It is shown that the problem can be solved as a convex program involving linear matrix inequalities (LMI). The time domain validation approach in a more general setup which is for LFT uncertain model-sets is studied in [2]. The similar setup is also used to consider the validation problem in a sampled-data framework [15, 17].

[1]This work is supported by the Hong Kong Reseach Grants Council. It is completed when the authors were visiting the National Laboratory of Industrial Control Technology, Zhejiang University, China.

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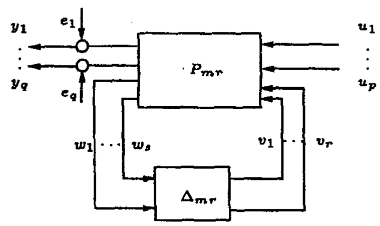

Figure 1: A general multirate LFT uncertain model.

In this paper, we extend the results in [2] and [11] to multirate systems. The setup is shown in Fig . 1, where P_{mr} and Δ_{mr} are both multirate systems, and they together form a multirate uncertain system model with P_{mr} fixed and Δ_{mr} unknown. The model validation problem considered in this paper is as follows. Given P_{mr}, an uncertainty set which Δ_{mr} belongs to, a set of time domain experimental data on u_i and y_i, and a set $\mathcal{E}$ of noise signals, find out if there exists a Δ_{mr} in the uncertainty set such that the experimental data can be reproduced with P_{mr} and Δ_{mr} together with the noises $\mathcal{E}$. As a technical tool, we first propose and study a CF interpolation problem with a nest operator constraint. This problem is itself of interest mathematically and has potential applications in addressing other problems in control, signal processing, and circuit theory. A necessary and sufficient solvability condition for this constrained CF interpolation problem is given. Then the validation tests are presented based on the above condition.

The paper is organized as follows. The next section introduces some basic facts about the general multirate systems and shows how to convert a multirate system to an equivalent LTI system with a causality constraint. Section III addresses the tangential CF interpolation problem with nest operator constraint, which are the main tool to obtain the model validation test for general multirate systems. Section IV provides the necessary and sufficient validation conditions for the time-domain experimental data. Section V concludes the paper.

2 General Multirate Systems

The setup of a general MIMO multirate system is shown in Fig. 2. Here u_i, $i = 1,2,\dots,p$, are input signals whose sampling intervals are $m_i h$ respectively, and y_j, $j = 1,2,\dots,q$ are output signals whose sampling intervals are $n_j h$ respectively, where h is a real number called base sampling interval and m_i, n_j are natural numbers (positive integers). Such systems can result from discretizing continuous time systems using samplers of different rates or they can be found in their own right. We will assume that all signals in the system are synchronized at time 0, i.e., the time 0 instances of all signals occur at the same time. In this paper, we will focus on those multirate systems that satisfy certain causal, linear, shift invariance properties which are to be defined below.

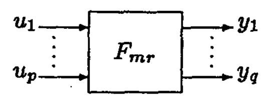

Figure 2: A general multirate system.

Since we need to deal with signals with different rates, it is more convenient and clearer to associate each signal explicitly with its sampling interval. Let $\ell^r(\tau)$ denote the space of $\mathbb{R}^r$ valued sequences:

$$\ell^r(\tau) = \{\{\dots,x(-\tau),|x(0),x(\tau),\dots\} : x(k\tau) \in \mathbb{R}^r\}.$$

The system in Fig. 2 is a map from $\oplus_{i=1}^p \ell(m_i h)$ to $\oplus_{j=1}^q \ell(n_j h)$. It is said to be linear if this map is a linear map.

Let $l \in \mathbb{N}$ be a multiple of m_i and n_j, $i = 1,2,\dots,p$, $j = 1,2,\dots,q$. Let $\bar{m}_i = l/m_i$ and $\bar{n}_j = l/n_j$. Denote the sets $\{m_i\}$ and $\{n_j\}$ by M and N respectively and the sets $\{\bar{m}_i\}$ and $\{\bar{n}_j\}$ by $\bar{M}$ and $\bar{N}$ respectively. Let $S : \ell^r(\tau) \longrightarrow \ell^r(\tau)$ be the forward shift operator, i.e.,

$$\begin{aligned} S\{\dots,x(-\tau),|x(0),x(\tau),\dots\} \\ = \{\dots,x(-2\tau),|x(-\tau),x(0),x(\tau),\dots\}. \end{aligned}$$

Define

$$\begin{aligned} S_{\bar{M}} &= \operatorname{diag}\{S^{\bar{m}_1},\dots,S^{\bar{m}_p}\}, \\ S_{\bar{N}} &= \operatorname{diag}\{S^{\bar{n}_1},\dots,S^{\bar{n}_q}\}. \end{aligned}$$

Then the multirate system in Fig. 2 is said to be $(\bar{M},\bar{N})$-shift invariant or lh periodic in real time if $F_{mr}S_{\bar{M}} = S_{\bar{N}}F_{mr}$. Now let $P_t : \ell^r(\tau) \longrightarrow \ell^r(\tau)$ be the truncation operator, i.e.,

$$\begin{aligned} P_t\{\dots,x((k-1)\tau),x(k\tau),x((k+1)\tau),\dots\} \\ = \{\dots,x((k-1)\tau),x(k\tau),0,\dots\} \end{aligned}$$

if $k\tau \le t < (k+1)\tau$. Extend this definition to spaces $\oplus_{i=1}^p \ell(m_i h)$ and $\oplus_{j=1}^q \ell(n_j h)$ in an obvious way. Then the multirate system is said to be causal if

$$P_t u = P_t v \Rightarrow P_t F_{mr} u = P_t F_{mr} v$$

for all $t \in \mathbb{R}$. In this paper, we will concentrate on causal linear $(\bar{M},\bar{N})$-shift invariant systems. Such general multirate system covers many familiar classes of systems as special cases. If m_i, n_j, l are all the same, then this is an LTI single rate system. If m_i, n_j are all the same but l is a multiple of them, then it is a single rate l-periodic system [13]. If $p = q = 1$, this becomes the SISO dual rate system studied in [4]. If m_i are the same and n_j are the same, then this becomes the MIMO dual rate system studied in [12]. For systems resulted from discretizing LTI continuous time systems using multirate sample and hold schemes in [3, 14], l turns out to be the least common multiple of m_i and n_j. The study of multirate systems in such a generality as indicated above, however, has never been done before.

A standard way for the analysis of such systems is to use lifting or blocking. Define a lifting operator $L_r : \ell(\tau) \to \ell^r(r\tau)$ by

$$L_r \{\dots | x(0), x(\tau), \dots\} \to \left\{ \dots \left| \begin{bmatrix} x(0) \\ \vdots \\ x((r-1)\tau) \end{bmatrix}, \begin{bmatrix} x(r\tau) \\ \vdots \\ x((2r-1)\tau) \end{bmatrix}, \dots \right. \right\}$$

and let

$$\begin{aligned} L_{\bar{M}} &= \operatorname{diag}\{L_{\bar{m}_1}, \dots, L_{\bar{m}_p}\}, \\ L_{\bar{N}} &= \operatorname{diag}\{L_{\bar{n}_1}, \dots, L_{\bar{n}_q}\}. \end{aligned}$$

Then the lifted system $F = L_{\bar{N}} F_{mr} L_{\bar{M}}^{-1}$ is an LTI system in the sense that $FS = SF$. Hence it has transfer function $\hat{F}$ in λ-transform. However, F is not an arbitrary LTI system, instead its direct feedthrough term $\hat{F}(0)$ is subject to a constraint that is resulted from the causality of F_{mr}. This constraint is best described using the language of nests and nest operators [12, 14].

Let $\mathcal{X}$ be a finite dimensional vector space. A nest in $\mathcal{X}$, denoted $\{\mathcal{X}_k\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the non-increasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{l-1} \supseteq \mathcal{X}_l = \{0\}.$$

Let $\mathcal{U}, \mathcal{Y}$ be finite dimensional vector spaces. Denote by $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ the set of linear operators $\mathcal{U} \to \mathcal{Y}$. Assume that $\mathcal{U}$ and $\mathcal{Y}$ are equipped respectively with nest $\{\mathcal{U}_k\}$ and $\{\mathcal{Y}_k\}$ which have the same number of subspaces, say, $l+1$ as above. A linear map $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is said to be a nest operator if

$$T\mathcal{U}_k \subseteq \mathcal{Y}_k, \ k = 0, 1, \dots, l. \tag{1}$$

The set of all nest operators (with given nests) is denoted $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. If we decompose the spaces $\mathcal{U}$ and $\mathcal{Y}$ in the following way:

$$\mathcal{U} = (\mathcal{U}_0 \ominus \mathcal{U}_1) \oplus (\mathcal{U}_1 \ominus \mathcal{U}_2) \oplus \cdots \oplus (\mathcal{U}_{l-1} \ominus \mathcal{U}_l) \tag{2}$$

$$\mathcal{Y} = (\mathcal{Y}_0 \ominus \mathcal{Y}_1) \oplus (\mathcal{Y}_1 \ominus \mathcal{Y}_2) \oplus \cdots \oplus (\mathcal{Y}_{l-1} \ominus \mathcal{Y}_l) \tag{3}$$

then a nest operator $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ has the following block lower triangular form

$$T = \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ T_{21} & T_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ T_{l1} & T_{l2} & \cdots & T_{ll} \end{bmatrix}. \tag{4}$$

Write $\underline{u} = L_{\bar{M}} u, \quad \underline{y} = L_{\bar{N}} y$. Then

$$\begin{aligned} \underline{u}(0) &= [u_1(0) \cdots u_1((\bar{m}_1 - 1)m_1 h) \cdots \\ &\quad u_p(0) \cdots u_p((\bar{m}_p - 1)m_p h)]^T, \\ \underline{y}(0) &= [y_1(0) \cdots y_1((\bar{n}_1 - 1)n_1 h) \cdots \\ &\quad y_q(0) \cdots y_q((\bar{n}_q - 1)n_q h)]^T. \end{aligned}$$

Define for $k = 0, 1, \dots, l$,

$$\begin{aligned} \mathcal{U}_k &= \{\underline{u}(0) : u_i(rm_i h) = 0 \text{ if } rm_i h < kh\} \\ \mathcal{Y}_k &= \{\underline{y}(0) : y_j(rn_j h) = 0 \text{ if } rn_j h < kh\}. \end{aligned}$$

Then the lifted plant F will have

$$\hat{F}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}). \tag{5}$$

Now we see that each multirate system has an equivalent single rate LTI system satisfying a causality constraint. This causality constraint is characterized by a nest operator constraint as in (5) on its transfer function.

3 Mathematical Preparations

Let $\mathbb{D}$ be the open unit disc. Denote $\mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ the Hardy class of all uniformly bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Denote by $\mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ the set of functions $\hat{G} \in \mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ satisfying $\hat{G}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. The purpose of this section is to adress the CF interpolation problem using functions in $\mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. Before going into this problem, we need to state a result on matrix positive completion.

3.1 Matrix Positive Completion

The matrix positive completion problem is as follows [5]: Given $B_{ij}, |j - i| \le q$, satisfying $B_{ij} = B_{ji}^*$, find the remaining matrices $B_{ij}, |j - i| > q$, such that the block matrix $B = [B_{ij}]_{i,j=1}^n$ is positive definite. The matrix positive problem was first proposed by Dym and Gohberg [5], who gave the following result:

Lemma 1 *The matrix positive completion problem has a solution if and only if*

$$\begin{bmatrix} B_{ii} & \cdots & B_{i,i+q} \\ \vdots & & \vdots \\ B_{i+q,i} & \cdots & B_{i+q,i+q} \end{bmatrix} \ge 0, \ i = 1, \dots, n - q. \tag{6}$$

3.2 Carathéodory-Fejér Interpolation with Nest Operator Constraint

Let $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ be finite dimensional Hilbert spaces. The Hilbert space direct sum of n copies of $\mathcal{X}$ will be denoted by $\mathcal{X}^n$. Assume that $\mathcal{U}$ and $\mathcal{Y}$ are equipped respectively with nests $\{\mathcal{U}_k\}$ and $\{\mathcal{Y}_k\}$. Let U_i and Y_i, $i = 0, 1, \dots, n$, be linear operators from $\mathcal{X}$ to $\mathcal{U}$ and from $\mathcal{X}$ to $\mathcal{Y}$ respectively. Denote

$$U = \begin{bmatrix} U_0 \\ \vdots \\ U_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} Y_0 \\ \vdots \\ Y_n \end{bmatrix}. \tag{7}$$

The Toeplitz matrix generated by U is defined as

$$T_U := \begin{bmatrix} U_0 & 0 & \cdots & 0 \\ U_1 & U_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ U_n & U_{n-1} & \cdots & U_0 \end{bmatrix}. \tag{8}$$

The Toeplitz matrix generated by Y is defined similarly. The tangential CF interpolation problem with constraint $\mathcal{N}(\{\mathcal{U}_k\},\{\mathcal{Y}_k\})$ for the data U, Y is to find (if possible) a function $\hat{G}(\lambda)=\sum_{i=0}^{\infty}G_i\lambda^i$ in $\mathcal{H}_\infty(\{\mathcal{U}_k\},\{\mathcal{Y}_k\})$ such that $\|\hat{G}\|_\infty<1$ and

$$Y = T_G U$$

where

$$T_G=\begin{bmatrix} G_0 & 0 & \cdots & 0 \\ G_1 & G_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ G_n & G_{n-1} & \cdots & G_0 \end{bmatrix}.$$

Note that the dimension of T_G depends on an integer n, to simplify the notation, however, we choose to ignore this dependence. In fact, this does not cause any confusion if we always assume that all the matrix operations are compatible.

Theorem 1 *There exists a solution to the CF interpolation problem with constraint $\mathcal{N}(\{\mathcal{U}_k\},\{\mathcal{Y}_k\})$ for the data U,Y if and only if*

$$T_U^*\Pi_{\mathcal{U}^n\oplus\mathcal{U}_k^\perp}T_U - T_Y^*\Pi_{\mathcal{Y}^n\oplus\mathcal{Y}_k^\perp}T_Y \geq 0 \tag{9}$$

for all $k=1,\ldots,l$.

Proof: The nest operator constraint on the interpolation function $\hat{G}$ can be considered as an additional interpolation condition

$$T_G\begin{bmatrix}0\\ \vdots\\ 0\\ I\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ T\end{bmatrix}$$

for some $T\in\mathcal{N}(\{\mathcal{U}_k\},\{\mathcal{Y}_k\})$. By the solvability condition of the standard CF interpolation problem [6], the constrained CF interpolation problem has a solution if and only if

$$\hat{U}^*\hat{U}-\hat{Y}^*\hat{Y}\geq 0 \tag{10}$$

where

$$\hat{U} = \begin{bmatrix} 0 & U_0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & U_{n-1} & 0 & U_{n-2} & \cdots & 0 & 0\\ I & U_n & 0 & U_{n-1} & \cdots & 0 & U_0 \end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix} 0 & Y_0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & Y_{n-1} & 0 & Y_{n-2} & \cdots & 0 & 0\\ T & Y_n & 0 & Y_{n-1} & \cdots & 0 & Y_0 \end{bmatrix}.$$

The zero columns in $\widehat{U}$ and $\widehat{Y}$ do not contribute anything to inequality (10). Hence (10) is equivalent to

$$\bar{U}^*\bar{U}-\bar{Y}^*\bar{Y}\geq 0 \tag{11}$$

where

$$\bar{U} = \begin{bmatrix} 0 & U_0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & U_{n-1} & U_{n-2} & \cdots & 0\\ I_p & U_n & U_{n-1} & \cdots & U_0 \end{bmatrix}$$

$$\bar{Y} = \begin{bmatrix} 0 & Y_0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & Y_{n-1} & Y_{n-2} & \cdots & 0\\ T & Y_n & Y_{n-1} & \cdots & Y_0 \end{bmatrix}.$$

Notice that the submatrices of $\bar{U}$ and $\bar{Y}$ formed by removing the first block column are block Toplitze matrices and are equal to T_U and T_Y respectively. It follows from Schur complement that (11) is equivalent to

$$\begin{bmatrix} I & U_n \ \cdots \ U_0 & 0 \quad T^* \\ U_n^* & & \\ \vdots & T_U^*T_U & T_Y^* \\ U_0^* & & \\ 0 & & \\ T & T_Y & I \end{bmatrix}\geq 0 \tag{12}$$

for some $T\in\mathcal{N}(\{\mathcal{U}_k\},\{\mathcal{Y}_k\})$. If we decompose the spaces $\mathcal{U}$ and $\mathcal{Y}$ as in (2), then a nest operator $T\in\mathcal{N}(\{\mathcal{U}_k\},\{\mathcal{Y}_k\})$ has a block lower triangular form shown in (4). Therefore, the constrained CF interpolation problem has a solution if and only if (12) holds for a block lower triangular matrix T. This is a matrix positive completion problem. By Lemma 1, such a T exists if and only if

$$\begin{bmatrix} I & \Pi_{\mathcal{U}_k}U_n\cdots\Pi_{\mathcal{U}_k}U_0 & 0\\ (\Pi_{\mathcal{U}_k}U_n)^* & & \\ \vdots & T_U^*T_U & (\Pi_{\mathcal{Y}^n\oplus\mathcal{Y}_k^\perp}T_Y)^*\\ (\Pi_{\mathcal{U}_k}U_0)^* & & \\ 0 & \Pi_{\mathcal{Y}^n\oplus\mathcal{Y}_k^\perp}T_Y & I \end{bmatrix}\geq 0 \tag{13}$$

for $k=0,1,\ldots,l$. Here $\Pi_{\mathcal{U}_k}$ and $\Pi_{\mathcal{Y}^n\oplus\mathcal{Y}_k^\perp}$ are operators from $\mathcal{U}$ to $\mathcal{U}_k$ and from $\mathcal{Y}^{n+1}$ to $\mathcal{Y}^n\oplus\mathcal{Y}_k^\perp$ respectively. Using Schur complement twice, we see that (13) is equivalent to

$$T_U^*T_U-\begin{bmatrix}(\Pi_{\mathcal{U}_k}U_n)^*\\ \vdots\\ (\Pi_{\mathcal{U}_k}U_0)^*\end{bmatrix}\begin{bmatrix}\Pi_{\mathcal{U}_k}U_n & \cdots & \Pi_{\mathcal{U}_k}U_0\end{bmatrix} - T_Y^*\Pi_{\mathcal{Y}^n\oplus\mathcal{Y}_k^\perp}T_Y\geq 0 \tag{14}$$

for $k=0,1,\ldots,l$. Note that (14) is exactly (9). Finally, notice that (9) when $k=0$ is implied by (9) when $k=l$. This completes the proof. ∎

The solvability condition for the standard CF interpolation problem without constraint is recovered when $l=1$.

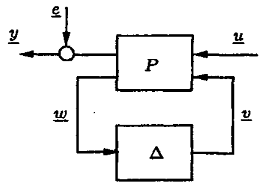

Figure 3: The equivalent LTI uncertain model.

4 Time-Domain Validation for Multirate LFT Uncertain Model

In robust control theory, many problems can be treated in a unified framework using LFT machinery. In fact, additive, multiplicative and coprime factor uncertainty descriptions can all be represented as an LFT on the uncertainty, with a suitable choice of the coefficient matrix [21]. In this section, we will give the validation tests for multirate LFT uncertain models.

Suppose we have an uncertain multirate system shown in Fig. 1. Here, u_i, $i = 1, \dots, p$, are input signals whose sampling intervals are $m_i'h$ and y_j, $j = 1, \dots, q$, are output signals whose sampling intervals are $n_j'h$. Also v_i, $i = 1, \dots, r$, and w_j, $j = 1, \dots, s$, are the auxiliary signals whose sampling intervals are m_ih and n_jh respectively. Assume that both P_{mr} and Δ_{mr} are lh periodic in real time for some integer l. As discussed in Section II, we can then convert the above multirate LFT uncertain system to an equivalent single rate LTI system with a causality constraint. Let $\bar{m}_i' = l/m_i'$, $\bar{n}_j' = l/n_j'$, $\bar{m}_i = l/m_i$, $\bar{n}_j = l/n_j$. And let $\underline{y} = L_{\bar{N}'}y$, $\underline{u} = L_{\bar{M}'}u$, $\underline{v} = L_{\bar{M}}v$, $\underline{w} = L_{\bar{N}}w$, and

$$\begin{aligned} P &= \begin{bmatrix} L_{\bar{N}'} & 0 \\ 0 & L_{\bar{N}} \end{bmatrix} P_{mr} \begin{bmatrix} L_{\bar{M}'} & 0 \\ 0 & L_{\bar{M}} \end{bmatrix}^{-1} \\ \Delta &= L_{\bar{N}} \Delta_{mr} \left(L_{\bar{M}}\right)^{-1} \end{aligned}$$

where $L_{\bar{N}'}, L_{\bar{M}'}, L_{\bar{M}}, L_{\bar{N}}$ are appropriately defined as in Section II. Then the multirate uncertain system in Fig. 1 is converted to an equivalent LTI uncertain system as shown in Fig. 3. We know that such equivalent LTI system satisfies a causality constraint. Denote

$$\begin{aligned} \underline{v}(0) &= [v_1(0)^T, \dots, v_1((\bar{m}_1 - 1)mh)^T, \dots, \\ &\quad v_r(0)^T, \dots, v_r((\bar{m}_r - 1)m_rh)^T]^T \\ \underline{w}(0) &= [w_1(0)^T, \dots, w_1((\bar{n}_1 - 1)n_1h)^T, \dots, \\ &\quad w_s(0)^T, \dots, w_s((\bar{n}_s - 1)n_sh)^T]^T \\ \underline{u}(0) &= [u_1(0)^T, \dots, u_1((\bar{m}_1' - 1)m_1'h)^T, \dots, \\ &\quad u_p(0)^T, \dots, u_p((\bar{m}_p' - 1)m_p'h)^T]^T \\ \underline{y}(0) &= [y_1(0)^T, \dots, y_1((\bar{n}_1' - 1)n_1'h)^T, \dots, \\ &\quad y_q(0)^T, \dots, y_q((\bar{n}_q' - 1)n_q'h)^T]^T. \end{aligned}$$

Define for $k = 0, 1, \dots, l$,

$$\begin{aligned} \mathcal{V}_k &= \{\underline{v}(0) : v_i(rm_ih) = 0 \text{ if } rm_ih < kh\} \\ \mathcal{W}_k &= \{\underline{w}(0) : w_j(rn_jh) = 0 \text{ if } rn_jh < kh\} \\ \mathcal{U}_k &= \{\underline{u}(0) : u_i(rm_i'h) = 0 \text{ if } rm_i'h < kh\} \\ \mathcal{Y}_k &= \{\underline{y}(0) : y_j(rn_j'h) = 0 \text{ if } rn_j'h < kh\}. \end{aligned}$$

Then $\hat{P}$ satisfies $\hat{P}(0) \in \mathcal{N}(\{\mathcal{U}_k \oplus \mathcal{V}_k\}, \{\mathcal{Y}_k \oplus \mathcal{W}_k\})$ and $\hat{\Delta}$ satisfies $\hat{\Delta}(0) \in \mathcal{N}(\{\mathcal{W}_k\}, \{\mathcal{V}_k\})$. From now on, we will only consider the equivalent LTI system shown in Fig. 3 with such constraints.

Assume that an uncertain model of the lifted LTI equivalence of a multirate system is represented by the lower LFT $F_l(\hat{P}, \hat{\Delta})$, where the nominal model $\hat{P} \in \mathcal{H}_\infty$ is given and satisfies $\hat{P}_{22} \in \mathcal{H}_\infty(\{\mathcal{V}_k\}, \{\mathcal{W}_k\})$ and $\|\hat{P}_{22}\|_\infty \le \frac{1}{\gamma}$, and $\hat{\Delta}$ is the uncertainty satisfying $\|\hat{\Delta}\|_\infty \le \gamma$. Several time domain experiments are carried out so that several input/output pairs of the lifted system are collected

$$\underline{U} := \begin{bmatrix} \underline{U}_0 \\ \vdots \\ \underline{U}_n \end{bmatrix}, \ \underline{Y} := \begin{bmatrix} \underline{Y}_0 \\ \vdots \\ \underline{Y}_n \end{bmatrix}.$$

The model validation problem is to test whether the uncertain model is consistent with the experiments data, i.e., whether there exists an $\hat{\Delta} \in \mathcal{H}_\infty(\mathcal{N}\{\mathcal{W}_k\}, \{\mathcal{V}_k\}))$ with $\|\hat{\Delta}\|_\infty \le \gamma$ such that the following holds

$$\underline{Y} = T_{P_{11}}\underline{U} + T_{P_{12}}\underline{V} + \underline{E} \tag{15}$$

$$\underline{W} = T_{P_{21}}\underline{U} + T_{P_{22}}\underline{V} \tag{16}$$

$$\underline{V} = T_\Delta \underline{W} \tag{17}$$

for some $\underline{E} \in \mathcal{E}$, where $\mathcal{E}$ is a compact convex set representing a bound on the error and

$$\underline{V} := \begin{bmatrix} \underline{V}_0 \\ \vdots \\ \underline{V}_n \end{bmatrix}, \ \underline{W} := \begin{bmatrix} \underline{W}_0 \\ \vdots \\ \underline{W}_n \end{bmatrix}.$$

Theorem 2 *For data $\underline{U}$ and $\underline{Y}$, define*

$$\Omega_{\underline{V}} = \{\underline{V} : T_{\underline{Y}} = T_{P_{11}}T_{\underline{U}} + T_{P_{12}}T_{\underline{V}} + T_{\underline{E}},\ \underline{E} \in \mathcal{E}\}.$$

The uncertain model (15-17) is not invalidated if and only if there exists a $\underline{V} \in \Omega_{\underline{V}}$ such that $H_k(\underline{V}) \ge 0$ for $k = 1, \dots, l$, where

$$\begin{aligned} H_k(\underline{V}) &= \begin{bmatrix} H_{k11}(\underline{V}) & H_{k12}(\underline{V}) \\ H_{k21}(\underline{V}) & I \end{bmatrix} \\ H_{k11}(\underline{V}) &= \left(T_{P_{21}}T_{\underline{U}}\right)^T \left(\Pi_{\mathcal{W}^n \oplus \mathcal{W}_k^\perp}\right) \left(T_{P_{21}}T_{\underline{U}}\right) \\ &\quad + \left(T_{P_{21}}T_{\underline{U}}\right)^T \left(\Pi_{\mathcal{W}^n \oplus \mathcal{W}_k^\perp}\right) \left(T_{P_{22}}T_{\underline{V}}\right) \\ &\quad + \left(T_{P_{22}}T_{\underline{V}}\right)^T \left(\Pi_{\mathcal{W}^n \oplus \mathcal{W}_k^\perp}\right) \left(T_{P_{21}}T_{\underline{U}}\right) \\ H_{k12}(\underline{V}) &= T_{\underline{V}}^T Q_k^{\frac{1}{2}} \\ H_{k21}(\underline{V}) &= Q_k^{\frac{1}{2}} T_{\underline{V}} \\ Q_k &= \frac{1}{\gamma^2}\Pi_{\mathcal{V}^n \oplus \mathcal{V}_k^\perp} - T_{P_{22}}^T (\Pi_{\mathcal{W}^n \oplus \mathcal{W}_k^\perp}) T_{P_{22}}. \end{aligned}$$

Proof: First we show $Q_k \geq 0$ so that $Q_k^{\frac{1}{2}}$ is well-defined. We know that $\hat{P}_{22}(0) \in \mathcal{N}(\{\mathcal{V}_k\}, \{\mathcal{W}_k\})$ from $\hat{P}(0) \in \mathcal{N}(\{\mathcal{U}_k \oplus \mathcal{V}_k\}, \{\mathcal{Y}_k \oplus \mathcal{W}_k\})$. Thus $\gamma \hat{P}_{22} \in \mathcal{H}_\infty(\{\mathcal{V}_k\}, \{\mathcal{W}_k\})$ since $\|\gamma \hat{P}_{22}\|_\infty \leq 1$. Recall that $\hat{P}_{22}(\lambda) = \sum_{i=0}^{\infty} P_{22i}\lambda^i$. Setting

$$U = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } Y = \gamma \begin{bmatrix} P_{220} \\ P_{221} \\ \vdots \\ P_{22n} \end{bmatrix}.$$

Note that $T_{\gamma P_{22}} U = Y$, it follows from Theorem 1 that

$$\Pi_{\mathcal{V}^n \oplus \mathcal{V}_k^\perp} - \gamma^2 T_{P_{22}}^T \Pi_{\mathcal{W}^n \oplus \mathcal{W}_k^\perp} T_{P_{22}} \geq 0$$

for all $k = 1, \ldots, l$. Therefore $Q_k \geq 0$ for all $k = 1, \ldots, l$, and $Q_k^{\frac{1}{2}}$ is well-defined. By Theorem 1, there exists an $\hat{\Delta} \in \mathcal{H}_\infty(\mathcal{N}(\{\mathcal{W}_k\}, \{\mathcal{V}_k\}))$ with $\|\hat{\Delta}(\lambda)\|_\infty \leq \gamma$ in (17) if and only if

$$\gamma^2 T_{\underline{W}}^T \Pi_{\mathcal{W}^n \oplus \mathcal{W}_k^\perp} T_{\underline{W}} \leq T_{\underline{V}}^T \Pi_{\mathcal{V}^n \oplus \mathcal{V}_k^\perp} T_{\underline{V}} \tag{18}$$

for all $k = 1, \ldots, l$. Substituting (16) into (18) yields

$$H_{k11}(\underline{V}) - T_{\underline{V}}^T Q_k T_{\underline{V}} \geq 0. \tag{19}$$

Since $Q_k \geq 0$, it follows by Schur complement that (19) is equivalent to $H_k(\underline{V}) \geq 0$. Hence, the uncertain model is not invalidated if and only if there exists a $\underline{V} \in \Omega_{\underline{V}}$ such that $H_k(\underline{V}) \geq 0$ for $k = 1, \ldots, l$. ∎

The conditions in Theorem 2 are the well-known LMI feasibility conditions which is numerically feasible.

5 Conclusion

The model validation for general multirate systems is studied in this paper. Based on the solutions to the constrained CF interpolation problem, the time domain validation test is presented for the general multirate LFT uncertain models. These tests can be carried out by solving feasibility problems involving LMIs.

References

[1] J. Chen. "Frequency-domain tests for validation of linear fractional uncertain models". *IEEE Trans. Automat. Contr.*, 42:748–760, 1997.

[2] J. Chen and S. Wang. "Validation of linear fractional uncertain models: solutions via matrix inequalities". *IEEE Trans. Automat. Contr.*, 41:844–849, 1996.

[3] T. Chen and L. Qiu. "$\mathcal{H}_\infty$ design of general multirate sampled-data control systems". *Automatica*, 30:1139–1152, 1994.

[4] T. Chen, L. Qiu, and E. Bai. "General mulrirate building blocks and their application in nonumiform filter banks". *IEEE Trans. on Circuits and Systems, Part II.* 45:948–958, 1998.

[5] H. Dym and I. Gohberg. "Extensions of band matrices with band inverses". *Linear Algebra and its Applications*, 36:1–24, 1981.

[6] C. Foias and A. E. Frazho. *The Commutant Lifting Approach to Interpolation Problems.* Birkhäuser, 1990.

[7] P. P. Khargonekar, K. Poolla, and A. Tannenbaum. "Robust control of linear time-invariant plants using periodic compensation". *IEEE Trans. Automat. Contr.*, 30:1088–1096, 1985.

[8] H. Liu, G. Xu, L. Tong, and T. Kailath. "Recent developments in blind channel equalization: From cyclostationarity to subspaces". *Signal Processing*, 50:83–99, 1996.

[9] L. Ljung. *System Identification, Theory for the Users.* Prentice-Hall, 1987.

[10] P. M. Mäkilä, J. R. Partington, and T. K. Gustafsson. "Worst-case control-relevant identification". *Automatica*, 31:1799–1819, 1995.

[11] K. Poolla, P. P. Khargonekar, A. Tikku, J. Krause, and K. M. Nagpal. "A time-domain approach to model validation". *IEEE Trans. Automat. Contr.*, 39:1088–1096, 1994.

[12] L. Qiu and T. Chen. "$\mathcal{H}_2$-optimal design of multirate sampled-data systems". *IEEE Trans. Automat. Contr.*, 39:2506–2511, 1994.

[13] L. Qiu and T. Chen. "Contractive completion of block matrices and its application to $\mathcal{H}_\infty$ control of periodic systems". In P. L. I. Gohberg and P. N. Shivakumar, editors, *Applications of Operator Theory*, pages 2506–2511. Birkhäuser, 1996.

[14] L. Qiu and T. Chen. "Multirate sampled-data systems : all $\mathcal{H}_\infty$ suboptimal controllers and the minimum entropy controllers". *IEEE Trans. Automat. Contr.*, 44:537–550, 1999.

[15] S. Rangan and K. Poolla. "Time domain validation for sampled-data uncertainty models". *IEEE Trans. Automat. Contr.*, 41:980–991, 1996.

[16] R. S. Smith and J. C. Doyle. "Model validation: A connection between robust control and identification". *IEEE Trans. Automat. Contr.*, 37:942–952, 1992.

[17] R. S. Smith and G. Dullerud. "Continuous-time control model validation using finite experimental data". *IEEE Trans. Automat. Contr.*, 41:1094–1105, 1996.

[18] G. Strang and T. Q. Nguyen. *Wavelets and Filter Banks.* Wellesley-Cambridge Press, 1996.

[19] P. Vaidyanathan. *Multirate Systems and Filter Banks.* Prentice-Hall, 1993.

[20] Y. Yamamoto and P. P. Khargonekar. "From sampled-data control to signal processing". In Y. Yamamoto and S. Hara, editors, *Learning, Control and Hybrid Systems*, pages 108–126. Springer, 1998.

[21] K. Zhou and J. C. Doyle. *Essentials of Robust Control.* Prentice-Hall, 1998.