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MULTIRATE PERIODIC SYSTEMS AND CONSTRAINED ANALYTIC FUNCTION INTERPOLATION PROBLEMS*

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Abstract. Multirate periodic systems and some related constrained analytic function interpolation problems are studied in this paper. After showing how to convert a general multirate periodic system to an equivalent linear time invariant (LTI) system with a structural constraint, we formulate some analytic function interpolation problems with such a constraint that can find various applications in the study of multirate and periodic systems. Both the solvability conditions and characterization of all solutions are presented to these constrained interpolation problems.

Key words. multirate systems, periodic systems, Nevanlinna–Pick interpolation

AMS subject classifications. 93C55, 93C35, 30E10, 91A28, 41A05

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1. Introduction.

Periodic and multirate systems are finding more and more applications in control, signal processing, communication, econometrics, and numerical mathematics. There are several reasons for this.

• In signal processing, the use of periodic and multirate systems can often lead to the reduction of the required transmission rate, storage space, or computational complexity for a given task, depending on the application [24].

• In large-scale multivariable digital systems, often it is unrealistic, or sometimes impossible, to sample all physical signals uniformly at one single rate. In such situations, one is forced to use multirate sampling.

• Periodic and multirate systems can often achieve objectives that cannot be achieved by single rate systems. Examples include gain margin improvement and simultaneous stabilization [14].

The study of periodic systems can be traced back to [10]. Examples of more recent studies are [2, 15]. The study of multirate systems goes back to the late 1950s. A renewal of research on multirate systems has occurred since 1980 within the signal processing, communication, and control communities. The driving force for studying multirate systems in signal processing comes from the need for sampling rate conversion, subband coding, and their ability to generate wavelets. Multirate signal processing is now one of the most vibrant areas of research within the signal processing community; see the recent book [24] and references therein. In the communication community, blind identification and equalization call for the use of multirate sampling [23]. In the control community, two groups of research stand out: (i) using multirate control to achieve something that otherwise cannot be achieved by single rate control (see, for example, [14]) and (ii) optimal design of multirate controllers [7, 18].

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A standard technique for treating periodic and multirate systems is called lifting in control theory [7, 14] and blocking in signal processing [15, 24]. First, we establish a setup of multirate periodic (MP) systems, which cover many familiar systems as special cases. Using the technique of lifting, an MP system can be converted to an equivalent linear time invariant (LTI) system satisfying a causality constraint that requires the feedthrough term to be block lower triangular. Motivated by this fact we propose and then study the problem of analytic function interpolation with an additional constraint that requires the value of the interpolating function at the origin to be block lower triangular [4, 5]. These constrained analytic function interpolation problems play the same role for multirate systems as their unconstrained counterparts do for single rate systems.

Analytic function interpolation problems have a very rich history in mathematics, and there has been a large volume of literature on this subject; see the recent books [1, 11, 13]. Many successful approaches have been proposed to solve the analytic function interpolation problems since the theory was first proposed at the beginning of the last century. In particular, Sarason [21] encompassed different classical interpolation problems in a representation theorem of operators commuting with special contractions, which was later developed to a general framework on commutant lifting theorem [11, 22]. On the other hand, using the realization method from the system theory, Ball, Gohberg, and Rodman [1] present another systematic way to deal with the interpolation of rational matrix functions. Recently, Foias et al. [12] combined the commutant lifting theorem from operator theory and state-space method from system theory to provide a unified approach for a more general setup of the problems, where they used the concept of operator-valued functions with operator arguments.

The increasing research interest on analytic function interpolation theory is also partly due to its wide applications in a variety of engineering problems such as those in control, circuit theory, and digital filter design [4, 8, 13]. The Nevanlinna–Pick (NP) interpolation theory was first brought into system theory by Youla and Saito, who gave a circuit theoretical proof of the Pick criterion [28]. In the early stage of the development of $\mathcal{H}_\infty$ control theory, the analytic function interpolation theory played a fundamental role [25]. A detailed review of this connection can be found in [13]. Recently, some new methods in high-resolution spectral estimation have been presented based on the NP interpolation with degree constraints [3]. The NP interpolation and Carathéodory–Fejér (CF) interpolation problems are also used extensively in robust model validation and identification [5, 6, 16].

In this paper, we propose a general model of multirate and periodic systems which covers single rate periodic systems and many other multirate systems in the literature as special cases. We also propose and solve some constrained analytic function interpolation problems that play the same role in multirate and periodic systems as the unconstrained counterparts do in single rate systems. That is, our results can be applied directly to multirate and periodic systems for $\mathcal{H}_\infty$ control, robust model validation and identification, etc. We present the necessary and sufficient solvability conditions and the parametrization of all solutions explicitly. The interpolation and distance problems involving analytic function with such structural constraints were first discussed in [13], but explicit solutions to the problem considered in this paper were not given there.

The notation used in this paper is standard. The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The open unit disc of the complex plane is denoted by $\mathbb{D}$. Let

$$p = [\ p_1 \quad \cdots \quad p_l\ ] \quad \text{and} \quad q = [\ q_1 \quad \cdots \quad q_l\ ] \tag{1.1}$$

be two vectors, where p_i and q_i, $i = 1, \dots, l$, are nonnegative integers. Denote

$$|p| = \sum_{i=1}^{l} p_i \quad \text{and} \quad |q| = \sum_{i=1}^{l} q_i. \tag{1.2}$$

For $k = 0, \dots, l$, define

$$\Pi_k(p) = \mathrm{diag}(0_{p_1}, \dots, 0_{p_k}, I_{p_{k+1}}, \dots, I_{p_l}), \tag{1.3}$$

$$\Pi_k(q) = \mathrm{diag}(0_{q_1}, \dots, 0_{q_k}, I_{q_{k+1}}, \dots, I_{q_l}), \tag{1.4}$$

where 0_n denote the $n \times n$ zero matrix and I_n the $n \times n$ unit matrix. Note that $\Pi_0(p) = I_{|p|}$, $\Pi_l(p) = 0_{|p|}$, $\Pi_0(q) = I_{|q|}$, and $\Pi_l(q) = 0_{|q|}$. The set of $|q| \times |p|$ matrices is denoted by $\mathbb{C}^{|q|\times|p|}$, and every such matrix is assumed to have an underlining partition so that its ijth block is q_i by p_j. Hence we have

$$\mathbb{C}^{|q|\times|p|} := \left\{ \begin{bmatrix} M_{11} & \cdots & M_{1l} \\ \vdots & \ddots & \vdots \\ M_{l1} & \cdots & M_{ll} \end{bmatrix} : M_{ij} \in \mathbb{C}^{q_i \times p_j} \right\}.$$

Note that the entry M_{ij} is "empty" if $q_i = 0$ or $p_j = 0$. The block lower triangular subset of $\mathbb{C}^{|q|\times|p|}$, denoted by $\mathcal{T}(q,p)$, consists of all matrices with $M_{ij} = 0, i < j$, and the strictly block lower triangular subset, $\mathcal{T}_s(q,p)$, consists of all matrices with $M_{ij} = 0, i \le j$. Let $\mathcal{H}_\infty^{|q|\times|p|}$ denote the Hardy class of all uniformly bounded analytic functions on $\mathbb{D}$ with values in $\mathbb{C}^{|q|\times|p|}$. For any $G(\lambda) \in \mathcal{H}_\infty^{|q|\times|p|}$, there exist $G_0, G_1, \dots \in \mathbb{C}^{|q|\times|p|}$ such that $G(\lambda) = \sum_{m=0}^{\infty} \lambda^m G_m$ for $\lambda \in \mathbb{D}$.

2. Multirate periodic systems. To introduce the general setup of multirate periodic systems, we need the concept of signals with time-varying dimensions. A signal with time-varying dimensions is defined as

$$x = \{\dots, x(-2), x(-1), |x(0), x(1), x(2), \dots\},$$

where $x(k) \in \mathbb{R}^{p(k)}$ and $p(k)$ is a nonnegative integer for any k. Here the vertical line indicates the position of time zero. Note that when $p(k) = 0$, $x(k) \in \mathbb{R}^{p(k)}$ means that $x(k)$ is always equal to 0. If $p(k)$ is periodic with period l, i.e., $p(k+l) = p(k)$ for any k, we call x a signal with l-periodically time-varying dimensions. Define the l-step shift operator S^l as

$$\begin{aligned} &S^l\{\dots, x(-1), |x(0), x(1), \dots\} \\ &= \{\dots, x(-l-1), |x(-l), x(-l+1), \dots\}. \end{aligned} \tag{2.1}$$

Denote P_k as the truncation operator, i.e.,

$$P_k\{\dots, x(k-1), x(k), x(k+1), \dots\} = \{\dots, x(k-1), x(k), 0, \dots\}. \tag{2.2}$$

Consider the system G_{mp}, shown in Figure 2.1, where the input u with $u(k) \in \mathbb{R}^{p(k)}$ and output y with $y(k) \in \mathbb{R}^{q(k)}$ are signals with l-periodically time-varying dimensions; that is, $p(k)$ and $q(k)$ are periodic with period l. Assume that G_{mp} satisfy the following properties:

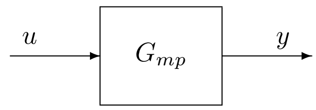

FIG. 2.1. *The multirate periodic system.*

(1) Linearity. The system G_{mp} is a linear operator.
(2) Periodicity. G_{mp} satisfies $G_{mp}S^l = S^l G_{mp}$, where S^l is given by (2.1).
(3) Causality. G_{mp} satisfies $P_k G_{mp}(I - P_k) = 0$, where P_k is given by (2.2).

In this paper, we focus on the systems that satisfy linear, periodic, and causal properties defined above. We call them multirate periodic (MP) systems. The general class of MP systems defined here covers many familiar classes of systems as special cases.

If $p(k) = p_1$ and $q(k) = q_1$ for all $k \in \mathbb{Z}$, then an MP system is a usual l-periodic system, for which there is a vast literature [2]. The multirate feature arises when $p(k)$ and $q(k)$ are truly time-varying. Let l be a multiple of m and n. If

$$u(k) \in \begin{cases} \mathbb{R}^{p_1} & \text{if } m|k \\ \{0\} & \text{otherwise} \end{cases} \quad \text{and} \quad y(k) \in \begin{cases} \mathbb{R}^{q_1} & \text{if } n|k \\ \{0\} & \text{otherwise} \end{cases},$$

then such an MP system is a dual rate system considered in [17]. Let l be a multiple of integers $m_i, i = 1, \dots, s$, and $n_j, j = 1, \dots, t$. If

$$u(k) = \begin{bmatrix} u_1(k) \\ \vdots \\ u_s(k) \end{bmatrix} \quad \text{and} \quad y(k) = \begin{bmatrix} y_1(k) \\ \vdots \\ y_t(k) \end{bmatrix},$$

where

$$u_i(k) \in \begin{cases} \mathbb{R}^{p_i} & \text{if } m_i|k \\ \{0\} & \text{otherwise} \end{cases} \quad \text{and} \quad y_j(k) \in \begin{cases} \mathbb{R}^{q_j} & \text{if } n_j|k \\ \{0\} & \text{otherwise} \end{cases},$$

then such an MP system becomes a general multirate system with uniform synchronized but different sampling in each input or output channel [7, 18, 20, 26]. The study of periodic and multirate systems in such a generality as indicated above, however, has never been done before.

Remark 1. One advantage of modelling a multirate system as a periodic system with periodically time-varying input-output spaces is that it better relates the present study to the rich theory on the usual periodic systems, as surveyed in [2]. Other advantages are its generality (it allows for nonuniform and asynchronous sampling) and its convenience (the treatments using this model take similar forms to those using other models, such as the one discussed in [7, 18]).

A standard method for the analysis of MP systems is to use lifting or blocking. For the MP system shown in Figure 2.1, define a lifting operator L_l on $\bigoplus_{k=-\infty}^{\infty} \mathbb{R}^{p(k)}$ by

$$L_l : \{\cdots | u(0), u(1), \cdots\} \mapsto \left\{ \cdots \left| \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(l-1) \end{bmatrix}, \begin{bmatrix} u(l) \\ u(l+1) \\ \vdots \\ u(2l-1) \end{bmatrix}, \cdots \right. \right\}.$$

L_l on $\bigoplus_{k=-\infty}^{\infty} \mathbb{R}^{q(k)}$ is defined similarly. Then the lifted system $G = L_l G_{mp} L_l^{-1}$ is an LTI system in the sense that $GS^1 = S^1 G$, where S^1 is the unit shift. Hence it has transfer function in the λ-transform ($\lambda = \frac{1}{z}$)

$$G(\lambda) = \begin{bmatrix} G_{11}(\lambda) & \cdots & G_{1l}(\lambda) \\ \vdots & \ddots & \vdots \\ G_{l1}(\lambda) & \cdots & G_{ll}(\lambda) \end{bmatrix}.$$

The LTI system G is not an arbitrary LTI system. Instead, its direct feedthrough term $G(0)$ is subject to a constraint that results from the causality of G_{mp} :

$$G_{ij}(0) = 0_{q(i)\times p(j)} \qquad \text{for } 1 \le i < j \le l;$$

i.e., $G(0)$ is a block lower triangular matrix. Therefore the causality constraint can be represented by

$$G(0) \in \mathcal{T}(q,p),$$

where $p = [\ p(1) \quad \cdots \quad p(l)\ ]$ and $q = [\ q(1) \quad \cdots \quad q(l)\ ]$. Notice that the form of the causality here is simpler than that in [7, 18] due to the new form of the model.

3. Constrained analytic interpolation problems. In this section, we will present some constrained analytic function interpolation problems, which can be viewed as a multirate version of the standard interpolation problems. These constrained interpolation problems have various applications in MP systems as do their unconstrained counterparts in single rate systems. We first present a general case: a constrained tangential NP interpolation problem. Some more useful special cases are then formulated for convenience. In the following sections, we always assume that p, q, $|p|$, and $|q|$ are defined by (1.1)–(1.2).

Problem 1 (constrained tangential NP interpolation). Given $U \in \mathbb{C}^{|p|\times n}$, $Y \in \mathbb{C}^{|q|\times n}$, and $Z \in \mathbb{C}^{n\times n}$ with spectral radius $\rho(Z) < 1$, find (if possible) a function $G(\lambda) = \sum_{m=0}^{\infty} G_m \lambda^m \in \mathcal{H}_\infty^{|q|\times|p|}$ such that

(i) $\|G\|_\infty \le 1$,

(ii) $\sum_{m=0}^{\infty} G_m U Z^m = Y$,

(iii) $G(0) \in \mathcal{T}(q,p)$.

Roughly speaking, the integer n in the problem determines the number of interpolation conditions.

We can also formulate the following interpolation problems with block lower triangular constraints, which are special cases of Problem 1.

Problem 2 (constrained classical NP interpolation). Given a set of complex numbers $\lambda_\alpha \in \mathbb{D}$ along with matrices $U_\alpha \in \mathbb{C}^{|p|\times n}$ and $Y_\alpha \in \mathbb{C}^{|q|\times n}$ for $\alpha = 1, \ldots, s$, find (if possible) a function $G \in \mathcal{H}_\infty^{|q|\times|p|}$ such that

(i) $\|G\|_\infty \le 1$,

(ii) $G(\lambda_\alpha)U_\alpha = Y_\alpha$ for $\alpha = 1, \ldots, s$,

(iii) $G(0) \in \mathcal{T}(q,p)$.

Problem 3 (constrained CF interpolation). Given $U_\beta \in \mathbb{C}^{|p|\times n}$ and $Y_\beta \in \mathbb{C}^{|q|\times n}$, $\beta = 0, \ldots, r-1$, find (if possible) a function $G(\lambda) = \sum_{m=0}^{\infty} G_m \lambda^m \in \mathcal{H}_\infty^{|q|\times|p|}$ such that

(i) $\|G\|_\infty \le 1$,

(ii) $\begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{r-1} \end{bmatrix} = \begin{bmatrix} G_0 & 0 & \cdots & 0 \\ G_1 & G_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{r-1} & G_{r-2} & \cdots & G_0 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{r-1} \end{bmatrix}$,

(iii) $G(0) \in \mathcal{T}(q,p)$.

Problem 4 (constrained simultaneous NP and CF interpolation). Given $U_{1,\beta} \in \mathbb{C}^{|p|\times n}$, $Y_{1,\beta} \in \mathbb{C}^{|q|\times n}$ for $j = 0,\dots,r-1$, and $U_\alpha \in \mathbb{C}^{|p|\times n}$, $Y_\alpha \in \mathbb{C}^{|q|\times n}$, and $\lambda_\alpha \in \mathbb{D}$ for $\alpha = 2,\dots,s$, find (if possible) a function $G(\lambda) = \sum_{m=0}^{\infty} G_m\lambda^m \in \mathcal{H}_\infty^{|q|\times|p|}$ such that

(i) $\|G\|_\infty \le 1$,

(ii) $\begin{bmatrix} Y_{1,0} \\ Y_{1,1} \\ \vdots \\ Y_{1,r-1} \end{bmatrix} = \begin{bmatrix} G_0 & 0 & \cdots & 0 \\ G_1 & G_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ G_{r-1} & G_{r-2} & \cdots & G_0 \end{bmatrix} \begin{bmatrix} U_{1,0} \\ U_{1,1} \\ \vdots \\ U_{1,r-1} \end{bmatrix}$,

(iii) $G(\lambda_\alpha)U_\alpha = Y_\alpha$ for $\alpha = 2,\dots,s$,

(iv) $G(0) \in \mathcal{T}(q,p)$.

Before giving the necessary and sufficient conditions of the above constrained analytic function interpolation problems, we need a result on matrix positive completion.

The matrix positive completion problem is as follows [9]: For a block matrix $B = [B_{ij}]_{i,j=1}^{n}$, given $B_{ij}, |j-i| \le m$, satisfying $B_{ij} = B_{ji}^*$, find the remaining matrices $B_{ij}, |j-i| > m$, such that the block matrix B is positive definite. The matrix positive completion problem was first proposed by Dym and Gohberg [9], who gave the following result.

LEMMA 3.1. *The matrix positive completion problem has a solution if and only if*

$$\begin{bmatrix} B_{ii} & \cdots & B_{i(i+m)} \\ \vdots & & \vdots \\ B_{(i+m)i} & \cdots & B_{(i+m)(i+m)} \end{bmatrix} \ge 0, \qquad i = 1,\dots,n-m. \tag{3.1}$$

Reference [27] gave a detailed discussion of this problem and presented an explicit description of the set of all solutions via a linear fractional map whose coefficients are given in terms of the original data. However, Lemma 3.1 is sufficient for our purpose. We are now in a position to give the main result of this section.

THEOREM 3.2. *There exists a solution to Problem* 1 *if and only if*

$$Q - \tilde{Q} + Y^*\Pi_k(q)Y - U^*\Pi_k(p)U \ge 0 \tag{3.2}$$

for all $k = 1,\dots,l$, *where* Q *and* $\tilde{Q}$ *are, respectively, the unique solutions of Lyapunov equations*

$$Q = Z^*QZ + U^*U, \tag{3.3}$$

$$\tilde{Q} = Z^*\tilde{Q}Z + Y^*Y. \tag{3.4}$$

Here $\Pi_k(p)$ *and* $\Pi_k(q)$ *are defined in* (1.3)–(1.4).

Proof. The structural constraint on the interpolation function G can be viewed as an additional interpolation condition,

$$G(0)I = T,$$

for some $T \in \mathcal{T}(q,p)$. Set $\lambda_0 = 0$, $U_0 = I$, and $Y_0 = T$. By the solvability condition of the standard NP interpolation problem [12], the constrained interpolation problem has a solution if and only if there exists $T \in \mathcal{T}(q,p)$ such that

$$Q_a - \tilde{Q}_a \geq 0, \tag{3.5}$$

where Q_a and $\tilde{Q}_a$ satisfy

$$Q_a = \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix}^* Q_a \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix} + \begin{bmatrix} I \\ U^* \end{bmatrix} \begin{bmatrix} I & U \end{bmatrix}, \tag{3.6}$$

$$\tilde{Q}_a = \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix}^* \tilde{Q}_a \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix} + \begin{bmatrix} T^* \\ Y^* \end{bmatrix} \begin{bmatrix} T & Y \end{bmatrix}. \tag{3.7}$$

It is easy to see from (3.6)–(3.7) that

$$Q_a = \begin{bmatrix} I & U \\ U^* & Q \end{bmatrix} \text{ and } \tilde{Q}_a = \begin{bmatrix} T^*T & T^*Y \\ Y^*T & \tilde{Q} \end{bmatrix}.$$

Substituting Q_a and $\tilde{Q}_a$ into the inequality (3.5), we have

$$\begin{bmatrix} I - T^*T & U - T^*Y \\ U^* - Y^*T & Q - \tilde{Q} \end{bmatrix} \geq 0. \tag{3.8}$$

The left-hand side of (3.8) can be rewritten as

$$\begin{bmatrix} I & U \\ U^* & Q - \tilde{Q} + Y^*Y \end{bmatrix} - \begin{bmatrix} T^* \\ Y^* \end{bmatrix} \begin{bmatrix} T & Y \end{bmatrix}.$$

By the Schur complement, (3.8) is equivalent to

$$\begin{bmatrix} I & U & T^* \\ U^* & Q - \tilde{Q} + Y^*Y & Y^* \\ T & Y & I \end{bmatrix} \geq 0. \tag{3.9}$$

Therefore, the constrained NP interpolation problem has a solution if and only if (3.9) holds for a block lower triangular matrix T. This is a matrix positive completion problem. By Lemma 3.1, there is a block lower triangular T such that (3.9) holds if and only if

$$\begin{bmatrix} \Pi_k(p) & \Pi_k(p)U & 0 \\ U^*\Pi_k(p) & Q - \tilde{Q} + Y^*Y & Y^*[I_q - \Pi_k(q)] \\ 0 & [I_q - \Pi_k(q)]Y & I_q - \Pi_k(q) \end{bmatrix} \geq 0 \tag{3.10}$$

for $k = 0, \ldots, l$. Using the Schur complement twice, we can easily show that (3.10) is equivalent to

$$Q - \tilde{Q} + Y^*\Pi_k(q)Y - U^*\Pi_k(p)U \geq 0 \tag{3.11}$$

for $k = 0, \ldots, l$. We claim that inequality (3.11) when $k = l$ implies the case when $k = 0$. In fact, when $k = 0$, inequality (3.11) gives

$$Q - \tilde{Q} + Y^*Y - U^*U \geq 0. \tag{3.12}$$

Note that inequality (3.12) is equivalent to

$$Z^*(Q - \tilde{Q})Z \geq 0.$$

When $k = l$, inequality (3.11) gives

$$Q - \tilde{Q} \geq 0. \tag{3.13}$$

It is obvious that inequality (3.13) implies (3.12). $\square$

Remark 2. If there is no constraint, then we have $l = 1$. In this case, the condition in Theorem 3.2 becomes $Q - \tilde{Q} \geq 0$, which is a well-known result in the literature [1, 11, 12].

Remark 3. To verify the condition in Theorem 3.2, two Lyapunov equations, (3.3) and (3.4), can be combined into one:

$$Q - \tilde{Q} = Z^*(Q - \tilde{Q})Z + U^*U - Y^*Y.$$

However, Q and $\tilde{Q}$ will be used in the next section.

Q and $\tilde{Q}$ can be given directly from the original data in some special cases. We end this section by providing the explicit formula for these special cases.

COROLLARY 3.3. *There exists a solution to Problem* 2 *if and only if*

$$\left[\frac{U_\alpha^* U_\beta - Y_\alpha^* Y_\beta}{1 - \lambda_\alpha^* \lambda_\beta} + Y_\alpha^* \Pi_k(q) Y_\beta - U_\alpha^* \Pi_k(p) U_\beta\right]_{\alpha,\beta=1}^{s} \geq 0 \tag{3.14}$$

for $k = 1, \ldots, l$.

Proof. Note that Problem 2 can be viewed as a special case of Problem 1 with

$$\begin{aligned} Z &= \operatorname{diag}(\lambda_1 I_n, \ldots, \lambda_s I_n), \\ U &= [\ U_1 \quad \cdots \quad U_s\ ], \\ Y &= [\ Y_1 \quad \cdots \quad Y_s\ ]. \end{aligned}$$

Then it is easy to check that

$$Q = \left[\frac{U_\alpha^* U_\beta}{1 - \bar{\lambda}_\alpha \lambda_\beta}\right]_{\alpha,\beta=1}^{s} \quad \text{and} \quad \tilde{Q} = \left[\frac{Y_\alpha^* Y_\beta}{1 - \bar{\lambda}_\alpha \lambda_\beta}\right]_{\alpha,\beta=1}^{s}$$

are the solution of the Lyapunov equations (3.3) and (3.4), respectively. The result then follows from Theorem 3.2 directly. $\square$

For $V = [\ V_0 \quad \cdots \quad V_{r-1}\ ]$, we use T_V to denote a corresponding lower Toeplitz matrix

$$T_V := \begin{bmatrix} V_0 & 0 & \cdots & 0 \\ V_1 & V_0 & \ddots & \\ \vdots & \vdots & \ddots & 0 \\ V_{r-1} & V_{r-2} & \cdots & V_0 \end{bmatrix}. \tag{3.15}$$

COROLLARY 3.4. *For the data of Problem* 3, *denote*

$$\begin{aligned} U &= [\ U_0 \quad \cdots \quad U_{r-1}\ ], \\ Y &= [\ Y_0 \quad \cdots \quad Y_{r-1}\ ]. \end{aligned}$$

Then there exists a solution to Problem 3 *if and only if*

$$T_U^* \begin{bmatrix} I_{p(r-1)} & 0 \\ 0 & I_p - \Pi_k(p) \end{bmatrix} T_U - T_Y^* \begin{bmatrix} I_{q(r-1)} & 0 \\ 0 & I_q - \Pi_k(q) \end{bmatrix} T_Y \geq 0 \tag{3.16}$$

for all $k = 1, \ldots, l$.

Proof. Note that Problem 3 can be viewed as a special case of Problem 1 with U, Y, and

$$Z = \begin{bmatrix} 0 & I_n & & 0 \\ & 0 & \ddots & \\ & & \ddots & I_n \\ & & & 0 \end{bmatrix}_{rn \times rn}.$$

Hence Q can be computed by

$$\begin{aligned} Q &= \sum_{m=0}^{\infty} Z^{*m} U^* U Z^m = \sum_{m=0}^{r-1} Z^{*m} U^* U Z^m \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ U_0^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & U_0 \end{bmatrix} + \cdots + \begin{bmatrix} U_0^* \\ \vdots \\ U_{r-1}^* \end{bmatrix} \begin{bmatrix} U_0 & \cdots & U_{r-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \cdots & U_0^* \\ \vdots & \vdots & \iddots & \vdots \\ 0 & U_0^* & \cdots & U_{r-2}^* \\ U_0^* & U_1^* & \cdots & U_{r-1}^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & U_0 \\ 0 & \cdots & U_0 & U_1 \\ \vdots & \iddots & \vdots & \vdots \\ U_0 & \cdots & U_{r-2} & U_{r-1} \end{bmatrix}. \end{aligned}$$

Similarly, we have

$$\tilde{Q} = \begin{bmatrix} 0 & 0 & \cdots & Y_0^* \\ \vdots & \vdots & \iddots & \vdots \\ 0 & Y_0^* & \cdots & Y_{r-2}^* \\ Y_0^* & Y_1^* & \cdots & Y_{r-1}^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & Y_0 \\ 0 & \cdots & Y_0 & Y_1 \\ \vdots & \iddots & \vdots & \vdots \\ Y_0 & \cdots & Y_{r-2} & Y_{r-1} \end{bmatrix}.$$

Condition (3.2) then becomes

$$Q - \tilde{Q} + Y^* \Pi_k(q) Y - U^* \Pi_k(p) U \geq 0. \tag{3.17}$$

By pre- and postmultiplying inequality (3.17) by

$$\begin{bmatrix} 0 & \cdots & I_n \\ \vdots & \iddots & \vdots \\ I_n & \cdots & 0 \end{bmatrix}_{rn \times rn},$$

we obtain another equivalent condition

$$T_U^* T_U - T_Y^* T_Y + \begin{bmatrix} Y_{r-1}^* \\ \vdots \\ Y_0^* \end{bmatrix} \Pi_k(q) \begin{bmatrix} Y_{r-1} & \cdots & Y_0 \end{bmatrix} - \begin{bmatrix} U_{r-1}^* \\ \vdots \\ U_0^* \end{bmatrix} \Pi_k(p) \begin{bmatrix} U_{r-1} & \cdots & U_0 \end{bmatrix} \geq 0.$$

This is exactly (3.16) after some simple algebraic manipulations. □

COROLLARY 3.5. *For the data of Problem* 4, *denote*

$$U_1 = \begin{bmatrix} U_{10} & \cdots & U_{1(r-1)} \end{bmatrix}, \quad Y_1 = \begin{bmatrix} Y_{10} & \cdots & Y_{1(r-1)} \end{bmatrix}.$$

Then there exists a solution to Problem 4 *if and only if*

$$\begin{bmatrix} A_{11k} & A_{21k}^* \\ A_{21k} & A_{22k} \end{bmatrix} \geq 0 \tag{3.18}$$

for all $k = 1, \ldots, l$, *where*

$$A_{11k} = T_{U_1}^* \begin{bmatrix} I_{p(r-1)} & 0 \\ 0 & I_p - \Pi_k(p) \end{bmatrix} T_{U_1} - T_{Y_1}^* \begin{bmatrix} I_{q(r-1)} & 0 \\ 0 & I_q - \Pi_k(q) \end{bmatrix} T_{Y_1},$$

$$A_{21k} = \begin{bmatrix} \bar\lambda_2^{r-1} U_2^* & \cdots & \bar\lambda_2 U_2^* & U_2^* \\ \vdots & \cdots & \vdots & \vdots \\ \bar\lambda_s^{r-1} U_s^* & \cdots & \bar\lambda_s U_s^* & U_s^* \end{bmatrix} \begin{bmatrix} I_{p(r-1)} & 0 \\ 0 & I_p - \Pi_k(p) \end{bmatrix} T_{U_1} - \begin{bmatrix} \bar\lambda_2^{r-1} Y_2^* & \cdots & \bar\lambda_2 Y_2^* & Y_2^* \\ \vdots & \cdots & \vdots & \vdots \\ \bar\lambda_s^{r-1} Y_s^* & \cdots & \bar\lambda_s Y_s^* & Y_s^* \end{bmatrix} \begin{bmatrix} I_{q(r-1)} & 0 \\ 0 & I_q - \Pi_k(q) \end{bmatrix} T_{Y_1},$$

$$A_{22k} = \left[\frac{U_\alpha^* U_\beta - Y_\alpha^* Y_\beta}{1 - \bar\lambda_\alpha \lambda_\beta} - U_\alpha^* \Pi_k(p) U_\beta + Y_\alpha^* \Pi_k(q) Y_\beta \right]_{\alpha,\beta=2}^{s}.$$

Proof. Note that Problem 4 can be viewed as a special case of Problem 1 with U, Y, and Z, where

$$U = \begin{bmatrix} U_1 & U_2 & \cdots & U_s \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix}, \quad Z = \operatorname{diag}(Z_1, \lambda_2 I_n, \ldots, \lambda_s I_n),$$

$$Z_1 = \begin{bmatrix} 0 & I_n & & 0 \\ & 0 & \ddots & \\ & & \ddots & I_n \\ & & & 0 \end{bmatrix}_{rn \times rn}.$$

Some simple algebraic manipulations show that

$$Q = \begin{bmatrix} Q_{11} & Q_{21}^* \\ Q_{21} & Q_{22} \end{bmatrix} \quad \text{and} \quad \tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{21}^* \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix}$$

satisfy the Lyapunov equations (3.3) and (3.4), respectively, where

$$Q_{11} = \begin{bmatrix} 0 & \cdots & 0 & U_{10}^* \\ 0 & \cdots & U_{10}^* & U_{11}^* \\ \vdots & \cdots & \vdots & \vdots \\ U_{10}^* & \cdots & U_{1(r-2)}^* & U_{1(r-1)}^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & U_{10} \\ 0 & \cdots & U_{10} & U_{11} \\ \vdots & \cdots & \vdots & \vdots \\ U_{10} & \cdots & U_{1(r-2)} & U_{1(r-1)} \end{bmatrix},$$

$$Q_{21} = \begin{bmatrix} \bar{\lambda}_2^{r-1} U_2^* & \cdots & \bar{\lambda}_2 U_2^* & U_2^* \\ \bar{\lambda}_3^{r-1} U_3^* & \cdots & \bar{\lambda}_3 U_3^* & U_3^* \\ \vdots & \cdots & \vdots & \vdots \\ \bar{\lambda}_s^{r-1} U_s^* & \cdots & \bar{\lambda}_s U_s^* & U_s^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & U_{10} \\ 0 & \cdots & U_{10} & U_{11} \\ \vdots & \cdots & \vdots & \vdots \\ U_{10} & \cdots & U_{1(r-2)} & U_{1(r-1)} \end{bmatrix},$$

$$Q_{22} = \left[\frac{U_\alpha^* U_\beta}{1 - \bar{\lambda}_\alpha \lambda_\beta} \right]_{\alpha,\beta=2}^{s},$$

$$\tilde{Q}_{11} = \begin{bmatrix} 0 & \cdots & 0 & Y_{10}^* \\ 0 & \cdots & Y_{10}^* & Y_{11}^* \\ \vdots & \cdots & \vdots & \vdots \\ Y_{10}^* & \cdots & Y_{1(r-2)}^* & Y_{1(r-1)}^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & Y_{10} \\ 0 & \cdots & Y_{10} & Y_{11} \\ \vdots & \cdots & \vdots & \vdots \\ Y_{10} & \cdots & Y_{1(r-2)} & Y_{1(r-1)} \end{bmatrix},$$

$$\tilde{Q}_{21} = \begin{bmatrix} \bar{\lambda}_2^{r-1} Y_2^* & \cdots & \bar{\lambda}_2 Y_2^* & Y_2^* \\ \bar{\lambda}_3^{r-1} Y_3^* & \cdots & \bar{\lambda}_3 Y_3^* & Y_3^* \\ \vdots & \cdots & \vdots & \vdots \\ \bar{\lambda}_s^{r-1} Y_s^* & \cdots & \bar{\lambda}_s Y_s^* & Y_s^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & Y_{10} \\ 0 & \cdots & Y_{10} & Y_{11} \\ \vdots & \cdots & \vdots & \vdots \\ Y_{10} & \cdots & Y_{1(r-2)} & Y_{1(r-1)} \end{bmatrix},$$

$$\tilde{Q}_{22} = \left[\frac{Y_\alpha^* Y_\beta}{1 - \bar{\lambda}_\alpha \lambda_\beta} \right]_{\alpha,\beta=2}^{s}.$$

Condition (3.2) then becomes

$$Q - \tilde{Q} + Y^* \Pi_k(q) Y - U^* \Pi_k(p) U \geq 0. \tag{3.19}$$

By pre- and postmultiplying inequality (3.19) by

$$\begin{bmatrix} 0 & \cdots & I_n & \\ \vdots & \iddots & \vdots & 0 \\ I_n & \cdots & 0 & \\ & 0 & & I_{(s-1)n} \end{bmatrix},$$

we obtain condition (3.18) after some direct operator manipulations. □

Remark 4. Corollaries 3.3 and 3.4 can be directly used for robust model validation of multirate systems following the method for LTI systems studied in [6, 16].

4. Parametrization of all solutions. In this section, we characterize all solutions G to Problem 1 when the solvability condition is satisfied. We will consider only the generic case when $Q - \tilde{Q} > 0$. The unlikely case when $Q - \tilde{Q}$ is singular is technically more involved.

Since the characterization for the unconstrained case has been given in [12], our strategy in solving the constrained problem is then to choose, if possible, from this characterization all those solutions that satisfy the structural constraint. The same notation is used as in previous sections and more notation is needed. Given an operator Δ and two operator matrices

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix},$$

the linear fractional transformation associated with Λ and Δ is denoted by

$$\mathcal{F}(\Lambda, \Delta) = \Lambda_{11} + \Lambda_{12}\Delta(I - \Lambda_{22}\Delta)^{-1}\Lambda_{21},$$

and the star product of Λ and Γ is defined as

$$\Lambda \star \Gamma = \begin{bmatrix} \Lambda_{11} + \Lambda_{12}\Gamma_{11}(I - \Lambda_{22}\Gamma_{11})^{-1}\Lambda_{21} & \Lambda_{12}(I - \Gamma_{11}\Lambda_{22})^{-1}\Gamma_{12} \\ \Gamma_{21}(I - \Lambda_{22}\Gamma_{11})^{-1}\Lambda_{21} & \Gamma_{21}(I - \Lambda_{22}\Gamma_{11})^{-1}\Lambda_{22}\Gamma_{12} + \Gamma_{22} \end{bmatrix}.$$

Here we assume that the operator manipulations are all compatible. With these definitions, we have

$$\mathcal{F}(\Lambda, \mathcal{F}(\Gamma, \Delta)) = \mathcal{F}(\Lambda \star \Gamma, \Delta).$$

The following lemma from [19] will be used later.

LEMMA 4.1. *For $M \in \mathbb{C}^{|q|\times|p|}$, the following statements are equivalent:*

(1) *There exists $T \in \mathcal{T}(q,p)$ such that $\|M + T\| \le 1$.*

(2) *There exists*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with $P_{11} \in \mathcal{T}(q,p)$, $P_{12} \in \mathcal{T}(q,q)$ invertible, $P_{21} \in \mathcal{T}(p,p)$ invertible, and $P_{22} \in \mathcal{T}_s(p,q)$ such that

$$\begin{bmatrix} M + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary.

A way to find P from M was given in [19]. Recall that an operator valued function Θ is said to be two-sided inner if Θ is an inner function and $\Theta(e^{jw})$ is almost everywhere unitary. For U, Y, and Z in Problem 1, assume that $Q - \tilde{Q} > 0$, where Q and $\tilde{Q}$ are defined by (3.3) and (3.4), respectively. By [12, Theorem III 7.2], there exist matrices $C \in \mathbb{C}^{n\times|p|}$ and $E \in \mathbb{C}^{|p|\times|p|}$ such that the state space model $\{Z, C, U, E\}$ is controllable and observable and the transfer function

$$\Theta(\lambda) := E + \lambda U(I - \lambda Z)^{-1}C \tag{4.1}$$

is two-sided inner in $\mathcal{H}^{|p|\times|p|}$. It follows from QR factorization that there is a special $\Theta(\lambda)$ such that $E^* \in \mathcal{T}(p,p)$ and (4.1) holds. By Cholesky factorization, there exist $N \in \mathcal{T}(q,q)$ and $S \in \mathcal{T}(p,p)$ such that

$$N^*N = [I + Y(Q - \tilde{Q})^{-1}Y^*]^{-1}, \tag{4.2}$$

$$S^*S = [I + C^*\tilde{Q}(Q - \tilde{Q})^{-1}QC]^{-1}. \tag{4.3}$$

Let $A_0 = (Q - Z^*\tilde{Q}Z)^{-1}Z^*(Q - \tilde{Q})$. It is shown in [12, Proposition V 1.7] that A_0 is stable. Define

$$\Phi(\lambda) = \begin{bmatrix} \Phi_{11}(\lambda) & \Phi_{12}(\lambda) \\ \Phi_{21}(\lambda) & \Phi_{22}(\lambda) \end{bmatrix},$$

where

$$\begin{aligned} \Phi_{11}(\lambda) &= Y(I - \lambda A_0)^{-1}(Q - Z^*\tilde{Q}Z)^{-1}U^*, \\ \Phi_{12}(\lambda) &= N^{-1} - \lambda Y A_0(I - \lambda A_0)^{-1}(Q - \tilde{Q})^{-1}Y^*N^{-1}, \\ \Phi_{21}(\lambda) &= S\Theta^*(\lambda) - S^{-1}C^*Q(I - \lambda A_0)^{-1}(Q - \tilde{Q})^{-1}\tilde{Q}C\Theta^*(\lambda), \\ \Phi_{22}(\lambda) &= -\lambda S^{-1}C^*Q(I - \lambda A_0)^{-1}(Q - \tilde{Q})^{-1}Y^*N^{-1}. \end{aligned}$$

The set of all $G(\lambda)$ solving the unconstrained interpolation problem is then given by

$$G(\lambda) = \mathcal{F}(\Phi(\lambda), V(\lambda)),$$

where $V(\lambda)$ is a contractive analytic function in $\mathcal{H}_\infty^{|q|\times|p|}$. Obviously, the set of all solutions to the constrained interpolation problem is

$$\{G(\lambda) = \mathcal{F}(\Phi(\lambda), V(\lambda)) : G(0) \in \mathcal{T}(q,p)\}. \tag{4.4}$$

It is easy to check that

$$\begin{aligned} \Phi_{11}(0) &= Y(Q - Z^*\tilde{Q}Z)^{-1}U^*, \\ \Phi_{12}(0) &= N^{-1}, \\ \Phi_{21}(0) &= SE^* - S^{-1}C^*Q(Q - \tilde{Q})^{-1}\tilde{Q}CE^* \\ &= S[I - C^*Q(Q - \tilde{Q})^{-1}\tilde{Q}C]E^* \\ &= S^{-1}E^*, \\ \Phi_{22}(0) &= 0. \end{aligned}$$

Now assume condition (3.2) in Theorem 3.2 is satisfied. Then there is a contractive analytic function $V(\lambda)$ in $\mathcal{H}_\infty^{|q|\times|p|}$ such that

$$G(0) = \Phi_{11}(0) + N^{-1}V(0)S^{-1}E^* \in \mathcal{T}(q,p).$$

That is,

$$\| - N\Phi_{11}(0)(S^{-1}E^*)^{-1} + N\hat{G}(0)(S^{-1}E^*)^{-1}\| \leq 1.$$

By Lemma 4.1, there exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with $P_{11} \in \mathcal{T}(q,p)$, $P_{12} \in \mathcal{T}(q,q)$ invertible, $P_{21} \in \mathcal{T}(p,p)$ invertible, and $P_{22} \in \mathcal{T}_s(p,q)$ such that

$$B := \begin{bmatrix} -N\Phi_{11}(0)(S^{-1}E^*)^{-1} + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary. Define $\Psi = \Phi \star B$. It is easy to check that $\Psi_{11} \in \mathcal{T}(q,p)$, $\Psi_{12} \in \mathcal{T}(q,q)$, $\Psi_{21} \in \mathcal{T}(p,p)$, $\Psi_{22} \in \mathcal{T}_s(p,q)$, and both Ψ_{12} and Ψ_{21} are invertible. By setting a bijective map $R = \mathcal{F}(B,V)$, we have

$$G(\lambda) = \mathcal{F}(\Phi, V) = \mathcal{F}(\Phi, \mathcal{F}(B,R)) = \mathcal{F}(\Phi \star B, R) = \mathcal{F}(\Psi, R).$$

Note that $G(0) \in \mathcal{T}(q,p)$ if and only if $R(0) \in \mathcal{T}(q,p)$. Hence the set (4.4) can be rewritten as

$$\{G(\lambda) = \mathcal{F}(\Psi, R) : R \in \mathcal{H}_\infty^{|q|\times|p|} \text{ with } R(0) \in \mathcal{T}(q,p) \text{ and } \|R\| \leq 1\}.$$

This gives us the main result of this section, the following theorem.

THEOREM 4.2. *For Problem* 1*, assume that* $Q - \tilde{Q} > 0$ *and the solvability condition* (3.2) *holds. Then the set of all interpolants* $G(\lambda)$ *is given by*

$$G(\lambda) = \mathcal{F}(\Psi(\lambda), R(\lambda)),$$

where R *is a contractive analytic function with* $R(0) \in \mathcal{T}(q,p)$.

5. Conclusion. In this paper, we study the MP systems and some related analytic function interpolation problems. We show that each MP system has an equivalent LTI system with a causality constraint which can be represented by a set of block lower triangular matrices. We then study some analytic function interpolation problems with such a constraint. The necessary and sufficient solvability conditions are given using the result of the positive matrix completion problem. Finally, all the solutions are presented in terms of linear fractional transformation.

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