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Multirate Periodic Systems, ν-Gap Metric and Robust Stabilization[1]

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Abstract

In this paper, we proposed a new model for multirate and periodic systems using the concept of periodicly time-varying input-output spaces. We then study the robust stabilization problem of multirate periodic systems with the ν-gap metric uncertainty. Both the optimal robust stability margin and suboptimal observer-form controller are presented explicitly by the tool of constrained Nehari extension problem.

1 Introduction

Multirate and periodic systems are finding more and more applications in control, communication, signal processing, econometrics and numerical mathematics. The reason may be due to their power in modeling physical systems with inherent features like periodic behavior changes, seasonal operating environment, nonuniform information exchange pattern, multirate sampling, etc., or due to the fact that they can often achieve objectives that cannot be achieved by single rate LTI systems.

The study of periodic systems can be traced back to [6]. Examples of more recent studies are [11, 10, 20], as well as the works of an Italian school [1] and a survey of some computational aspects [23]. The study of multirate systems goes back to late 1950's, see for example [13]. A renaissance of research on multirate systems has occured since 1980 in signal processing, communication and control communities. The driving force for studying multirate systems in signal processing comes from the need for sampling rate conversion, subband coding, and their ability to generate wavelets. Multirate signal processing is now one of the most vibrant areas of research in the signal processing community, see recent book [22] and references therein. In communication systems, blind identification and equalization call for using multirate sampling [21]. In the control community, two groups of research stand out: using multirate control to achieve what single rate control cannot as well as the limitation of doing this [14, 15], and the optimal design of multirate controllers [4, 8, 18]. We also notice the cross discipline fertilization between signal processing and control in using $\mathcal{H}_\infty$ optimization to design filter banks [3].

Recently, there has been considerable research devoted to the problem of robust stabilization [9, 12, 26]. For LTI systems with gap and ν-gap metric uncertainty, it is now well-known that both the optimal robustness bound and the suboptimal controller can be easily obtained without the so-called γ-iteration and the suboptimal controller is an observer form. In this paper, we will extend these results to multirate periodic systems. The robust stabilization problem for discrete-time periodic uncertain systems described by the normalized coprime factorization was studied in [29]. This study is based on periodic Riccati equations, which cannot be straightforwardly extended to general multirate systems. A related study was presented in [12] which provides a method to design a strictly proper controller for the discrete-time, normalized left-coprime factorization robust stabilization problem. A general study for robust stabilization of LTI discrete-time systems with normalized stable factor perturbation is given in [26].

Using the concept of periodic time-varying input-output spaces [10], we propose a new model for multirate and periodic systems, called multirate periodic (MP) systems. The advantages of this setup are its generality: it allows for nonuniform and asynchronous sampling, and its convenience: it better relates back to the rich theory on the usual periodic systems. The ν-gap metric is defined for MP systems. We then study the robust stabilization problem for MP systems with the ν-gap metric uncertainty. Both the optimal robust stability margin and suboptimal observer-form controller are presented explicitly by the tool of constrained Nehari extension problem.

The paper is organized as follows. In section 2, we

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give the general setup on MP systems and the lifting technique. We will see that a general MP system can be converted to an LTI system with a structural constraint due to the causality requirement. In section 3, we introduce the ν-gap metric to MP systems and show that a robust stabilization problem of an MP system with the ν-gap metric uncertainty can be converted to a constrained $\mathcal{H}_\infty$ optimization problem. The optimal robust stability margin and an observer-based suboptimal controller are presented explicitly in section 4. Finally, this paper is concluded in section 5.

2 Setup of MP systems

In this paper, we model an MP system by a discrete time system, shown in Fig. 1, with periodic time-varying input and output spaces. Precisely, we assume that the input sequence $u = \{u(k)\}_{k=-\infty}^{\infty}$ takes values in $\bigoplus_{k=-\infty}^{\infty}\mathcal{U}(k)$, i.e., $u(k) \in \mathcal{U}(k)$, and the output sequence $y = \{y(k)\}_{k=-\infty}^{\infty}$ takes values in $\bigoplus_{k=-\infty}^{\infty}\mathcal{Y}(k)$, i.e., $y(k) \in \mathcal{Y}(k)$, where $\mathcal{U}(k)$ and $\mathcal{Y}(k)$ are M-periodic time-varying vector spaces, i.e., they satisfy $\mathcal{U}(k+M) = \mathcal{U}(k)$ and $\mathcal{Y}(k+M) = \mathcal{Y}(k)$. We further make the following assumptions:

1. Linearity. The system G_{mp} is a linear operator from $\bigoplus_{k=-\infty}^{\infty}\mathcal{U}(k)$ to $\bigoplus_{k=-\infty}^{\infty}\mathcal{Y}(k)$.

2. Periodicity. Let $\mathcal{X}(k)$ be vector space valued M-periodic functions. Define the M-step shift operator S^M on $\bigoplus_{k=-\infty}^{\infty}\mathcal{X}(k)$ as

$$S^M\{\ldots,x(-1),|x(0),x(1),\ldots\} = \{\ldots,x(-M-1),|x(-M),x(-M+1),\ldots\}.$$

   Then G_{mp} satisfies $G_{mp}S^M = S^M G_{mp}$. Notices that when $M > 1$, the 1-step shift S^1 is generally not defined.

3. Causality. Let P_k be a projection operator on $\bigoplus_{k=-\infty}^{\infty}\mathcal{X}(k)$ defined as

$$\begin{aligned} &P_k\{\ldots,x(k-1),x(k),x(k+1),\ldots\} \\ &= \{\ldots,x(k-1),x(k),0,\ldots\}. \end{aligned}$$

   Then G_{mp} satisfies $P_k G_{mp}(I-P_k) = 0$.

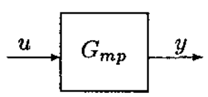

Figure 1: A general periodic and multirate system

The general class of MP systems defined here covers many familiar classes of systems as special cases. An MP system with $\mathcal{U}(k) = \mathcal{U}$ and $\mathcal{Y}(k) = \mathcal{Y}$ for all $k \in \mathbb{Z}$ is a usual M-periodic system, for which there is a vast literature [1]. The multirate feature arises when $\mathcal{U}(k)$ and $\mathcal{Y}(k)$ are truly time-varying. If

$$\mathcal{U}(k) = \begin{cases} \mathcal{U} \text{ if } m|k \\ \{0\} \text{ otherwise} \end{cases}, \mathcal{Y}(k) = \begin{cases} \mathcal{Y} \text{ if } n|k \\ \{0\} \text{ otherwise} \end{cases},$$

and M is a multiple of m and n, then such an MP system is a dual rate system considered in [16]. Let M be a multiple of integers $m_i, i = 1,\ldots,p$, and $n_j, j = 1,\ldots,q$. If

$$\mathcal{U}_i(k) = \begin{cases} \mathcal{U}_i \text{ if } m_i|k \\ \{0\} \text{ otherwise} \end{cases}, \mathcal{Y}_j(k) = \begin{cases} \mathcal{Y}_j \text{ if } n_j|k \\ \{0\} \text{ otherwise} \end{cases}$$

and

$$\mathcal{U}(k) = \bigoplus_{i=1}^{p}\mathcal{U}_i(k), \qquad \mathcal{Y}(k) = \bigoplus_{j=1}^{q}\mathcal{Y}_j(k),$$

then such an MP system becomes a general multirate system with uniform synchronized but different sampling in each input or output channel [4, 18, 25]. One advantage of modeling a multirate system as a periodic system with periodically varying input output spaces is that it better relates back to the rich theory on the usual periodic system, as surveyed in [1]. Other advantages are its generality: it allows for nonuniform and asynchronous sampling, and its convenience: the treatments using this model take similar forms than those using other models, such as the one in [4, 18].

A standard way for the analysis of such systems is to use lifting or blocking. Let $\mathcal{X}_l(r) = \bigoplus_{k=l+rM}^{(r+1)M+l-1}\mathcal{X}(k)$. Define a lifting operator $L_l : \bigoplus_{k=-\infty}^{\infty}\mathcal{X}(k) \to \bigoplus_{r=-\infty}^{\infty}\mathcal{X}_l(r)$ by

$$L_l : \{\cdots|x(0),x(1),\cdots\} \mapsto \left\{\cdots\left|\begin{bmatrix} x(l) \\ x(l+1) \\ \vdots \\ x(M+l-1)\end{bmatrix}, \begin{bmatrix} x(M+l) \\ x(M+l+1) \\ \vdots \\ x(2M+l-1)\end{bmatrix}, \cdots\right.\right\}.$$

Then the lifted systems $G_l = L_l G_{mp} L_l^{-1}$ are LTI systems in the sense that $G_l S^1 = S^1 G_l$, where S^1 is the unit shift on $\bigoplus_{r=-\infty}^{\infty}\mathcal{X}_l(r)$. Hence they have transfer functions in the λ-transform ($\lambda = \frac{1}{z}$) :

$$\hat{G}_l(\lambda) = \begin{bmatrix} \hat{G}_{l,11}(\lambda) & \cdots & \hat{G}_{l,1M}(\lambda) \\ \vdots & \ddots & \vdots \\ \hat{G}_{l,M1}(\lambda) & \cdots & \hat{G}_{l,MM}(\lambda) \end{bmatrix}.$$

Assume that $\dim\mathcal{U}(k) = p(k)$ and $\dim\mathcal{Y}(k) = q(k)$. Then $\hat{G}_l$ takes values in the set of $\sum_{k=l}^{M+l-1} q(k) \times \sum_{k=l}^{M+l-1} p(k)$ complex matrices. The LTI system G_l is not an arbitrary LTI system, instead its direct feedthrough term $\hat{G}_l(0)$ is subject to a constraint that results from the causality of G_{mp} :

$$\hat{G}_{l,ij}(0) = 0 \quad \text{for } i < j,$$

i.e., $\hat{G}_l(0)$ is a block lower triangular matrix. Notice that the form of the causality here is simpler than that in [4, 18] due to the new form of the model. It can be easily shown that $\hat{G}_l$ are not all independent. Any one of the $G_l, l = 0, \ldots, M-1$, can be defined as the LTI equivalent of the MP system G_{mp}. In the rest of this paper, we choose G_0 as the LTI equivalent of the MP system G_{mp} without loss of generality.

3 ν-gap Metric of MP Systems

The first issue in robust control is the description of the uncertainty. The most natural way to describe system uncertainty is by using a metric in the set of all systems under consideration and an uncertain system is then simply a ball defined by this metric centered at a nominal system with certain radius. In this paper, the ν-gap metric is used for MP systems due to the fact that the ν-gap metric is more advantageous over other metrics as shown in [24]. We will see that the treatment in [27] can help us to generalize the definition of the ν-gap metric to MP systems.

Given two M-periodic MP systems G_{mp} and $\tilde{G}_{mp}$, the graphs of G_{mp} and $\tilde{G}_{mp}$ are defined as

$$\mathcal{G}(G_{mp}) = \left\{ \begin{bmatrix} u \\ G_{mp}u \end{bmatrix}; u \in \ell_+^2 \text{ and } G_{mp}u \in \ell_+^2 \right\},$$

$$\mathcal{G}(\tilde{G}_{mp}) = \left\{ \begin{bmatrix} \tilde{u} \\ \tilde{G}_{mp}\tilde{u} \end{bmatrix}; \tilde{u} \in \ell_+^2 \text{ and } \tilde{G}_{mp}\tilde{u} \in \ell_+^2 \right\},$$

respectively. Here ℓ_+^2 means the direct sum $\oplus_{k=0}^{\infty}\mathcal{X}(k)$ with $\sum_{n=0}^{\infty} x^2(k+nM) < \infty$ for any $k = 0, 1, \ldots, M-1$. Clearly $\mathcal{G}(G_{mp})$ and $\mathcal{G}(\tilde{G}_{mp})$ are subspaces of $\ell_+^2 \oplus \ell_+^2$. A subspace $\mathcal{G}$ of $\ell_+^2 \oplus \ell_+^2$ is said to be M-shift-invariant if $S^M\mathcal{G} \subset \mathcal{G}$. It is easy to see that the graph of G_{mp} is M-shift-invariant. A subgraph of an M-periodic MP system is defined as an M-shift-invariant subspace of its graph. We denote the set of all subgraphs as $\mathcal{S}_\mathcal{G}(G_{mp})$. To define the $\nu-$gap between two MP systems, we need the notion of the index of a subgraph $\mathcal{V}$ with respect to $\mathcal{G}(G_{mp})$, defined as [27]

$$\text{ind}(\mathcal{V}) := \dim\left(\mathcal{G}(G_{mp}) \ominus \mathcal{V}\right).$$

The $\nu-$gap between two plants G_{mp} and $\tilde{G}_{mp}$ is then defined by

$$\delta_\nu(G_{mp}, \tilde{G}_{mp}) = \inf_{\substack{\mathcal{V} \in \mathcal{S}_\mathcal{G}(G_{mp}) \\ \tilde{\mathcal{V}} \in \mathcal{S}_\mathcal{G}(\tilde{G}_{mp}) \\ \text{ind}(\mathcal{V}) = \text{ind}(\tilde{\mathcal{V}})}} \|\Pi_\mathcal{V} - \Pi_{\tilde{\mathcal{V}}}\|$$

where $\Pi_\mathcal{V}$ and $\Pi_{\tilde{\mathcal{V}}}$ are the orthogonal projections from $\ell_+^2 \oplus \ell_+^2$ onto $\mathcal{V}$ and $\tilde{\mathcal{V}}$ respectively. The ν-gap metric ball centered at G_{mp} with radius r is defined by

$$\mathcal{B}_\nu(G_{mp}, r) = \{\tilde{G}_{mp} : \delta_\nu(G_{mp}, \tilde{G}_{mp}) < r\}.$$

By the following lemma, the ν-gap between two M-periodic MP systems can be computed from that between their equivalent LTI systems, for which efficient methods are available [24].

Lemma 1 Let G_{mp} and $\tilde{G}_{mp}$ be two M-periodic MP systems and their equivalent LTI systems be G and $\tilde{G}$ respectively, that is

$$G = L_0 G_{mp} L_0^{-1}, \quad \tilde{G} = L_0 \tilde{G}_{mp} L_0^{-1}.$$

Then we have $\delta_\nu(G_{mp}, \tilde{G}_{mp}) = \delta_\nu(G, \tilde{G})$.

Proof: Note that $\mathcal{V}$ a subgraph of G_{mp} if and only if $\mathcal{V}_L = \begin{bmatrix} L_0 & 0 \\ 0 & L_0 \end{bmatrix} \mathcal{V}$ is a subgraph of G. Similar result holds for a subgraph $\tilde{\mathcal{V}}$ of $\tilde{G}_{mp}$. Denote

$$\tilde{\mathcal{V}}_L = \begin{bmatrix} L_0 & 0 \\ 0 & L_0 \end{bmatrix} \tilde{\mathcal{V}}.$$

Since the lifting operator L_0 is unitary, we then have

$$\begin{aligned} \delta_\nu(G_{mp}, \tilde{G}_{mp}) &= \inf \|\Pi_\mathcal{V} - \Pi_{\tilde{\mathcal{V}}}\| \\ &= \inf \left\|\Pi_{\mathcal{V}_L} - \Pi_{\tilde{\mathcal{V}}_L}\right\| = \delta_\nu(G, \tilde{G}). \end{aligned}$$

∎

In the following, we discuss the robust stabilization problem for MP uncertain systems with the ν-gap metric. First, some notation is needed. Define the set of block matrices:

$$\mathcal{M}(\mathbb{R}^{m\times n}) := \left\{ T = \begin{bmatrix} T_{11} & \cdots & T_{1M} \\ \vdots & & \vdots \\ T_{M1} & \cdots & T_{MM} \end{bmatrix} \in \mathbb{R}^{m\times n} \right\}.$$

The block lower triangular subset of $\mathcal{M}(\mathbb{R}^{m\times n})$, denoted by $\mathcal{T}(\mathbb{R}^{m\times n})$, consists of all matrices with $T_{ij} = 0$, $i < j$, and the strictly block lower triangular subset, $\mathcal{T}_s(\mathbb{R}^{m\times n})$, consists of matrices with $T_{ij} = 0$, $i \leq j$.

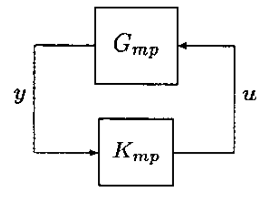

Figure 2: A general MP feedback control system.

Consider the feedback system shown in Fig. 2. Here we assume that G_{mp} and K_{mp} are M-periodic MP systems with M-periodic time-varying signal spaces $\oplus_{k=-\infty}^{\infty}\mathcal{U}(k)$ and $\oplus_{k=-\infty}^{\infty}\mathcal{Y}(k)$. Assume that $\dim\mathcal{U}(k) = p(k)$ and $\dim\mathcal{Y}(k) = q(k)$. Denote $p =$

$\sum_{k=0}^{M-1} p(k)$ and $q = \sum_{k=0}^{M-1} q(k)$. Let $G = L_0 G_{mp} L_0^{-1}$ and $K = L_0 K_{mp} L_0^{-1}$, then G and K are LTI and hence have transfer functions $\hat{G}(\lambda)$ and $\hat{K}(\lambda)$ respectively. Due to causality constraint, $\hat{G}(0)$ and $\hat{K}(0)$ are block lower triangular, that is, $\hat{G}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $\hat{K} \in \mathcal{T}(\mathbb{R}^{p\times q})$. For fixed G_{mp} and K_{mp}, the stability robustness of the feedback system is given by the following lemma:

Lemma 2 ([24, 19]) For a given plant G_{mp} and a given stabilizing controller K_{mp}, let G and K be the LTI equivalence of G_{mp} and K_{mp} respectively. For any positive real numbers r_1 and r_2, the feedback system with plant $\tilde{G}_{mp}$ and controller $\tilde{K}_{mp}$ is stable for all $\tilde{G}_{mp} \in \mathcal{B}_\nu(G_{mp}, r_1)$ and all $\tilde{K}_{mp} \in \mathcal{B}_\nu(K_{mp}, r_2)$ if and only if

$$\arcsin r_1 + \arcsin r_2 + \arccos b_{G,K} \le \frac{1}{2}\pi$$

where

$$b_{G,K} = \left\| \begin{bmatrix} I \\ \hat{G} \end{bmatrix} (I - \hat{K}\hat{G})^{-1} \begin{bmatrix} I & -\hat{K} \end{bmatrix} \right\|_\infty^{-1}.$$

The proof is straightforward by slightly modifying the procedure given in [24]. The quantity $b_{G,K}$ is defined as the robust stability margin. The robust stabilization problem is to find the optimal robust stability margin

$$b_{opt} = \sup_{K,\ \hat{K}(0)\in\mathcal{T}(\mathbb{R}^{p\times q})} b_{G,K} \qquad (1)$$

for a given G and also find a K with $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$, called a suboptimal controller, such that $b_{G,K} \ge \gamma$ for any $\gamma < b_{opt}$.

Hence our robust stabilization problem becomes a special discrete-time $\mathcal{H}_\infty$ optimal control problem. Since the causality of G_{mp} and K_{mp} is equivalent to that $\hat{G}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$, we need to respect the structural constraint $\hat{K}(0)$ and possibly to utilize the structural constraint $\hat{G}(0)$ in solving the special discrete-time $\mathcal{H}_\infty$ optimal control problem. The continuous-time counterpart of such an $\mathcal{H}_\infty$ optimal control problem (without causality constraint) has been explicitly solved in [9].

4 Robust Stabilization of MP Systems

Now we return to the robust stabilization problem stated in section 3: Given a nominal LTI model G resulted from the lifting of G_{mp}, find the optimal robust stability margin b_{opt} defined in (1) and a suboptimal controller K with $\hat{K}(0) \in \mathcal{T}(\mathbb{R}^{p\times q})$ such that $b_{G,K} \ge \gamma$ for any given $\gamma < b_{opt}$. To solve this problem, we need some assumption and notation. Assume that G has a stabilizable and detectable state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{T}(\mathbb{R}^{q\times p})$. Let X and Y be the stabilizing solutions of Riccati equations

$$X = A^*XA + C^*C - (A^*XB + C^*D) \cdot(B^*XB + I + D^*D)^{-1}(B^*XA + D^*C) \qquad (2)$$

$$Y = AYA^* + BB^* - (AYC^* + BD^*) \cdot(CYC^* + I + DD^*)^{-1}(CYA^* + DB^*). \qquad (3)$$

Denote

$$F = -(B^*XB + I + D^*D)^{-1}(B^*XA + D^*C) \qquad (4)$$

$$L = -(AYC^* + BD^*)(CYC^* + I + DD^*)^{-1}. \qquad (5)$$

Here $(A+BF)$ and $(A+LC)$ are stable since X and Y are stablilizing solutions. The following equation [2, 9] gives a relationship between $A+BF$, $A+LC$, X and Y , which will be used later.

$$(A+LC)(I+YX) = (I+YX)(A+BF). \qquad (6)$$

Using Cholesky factorization, we can get constant matrix $S \in \mathcal{T}(\mathbb{R}^{q\times q})$ satisfying [4],

$$SS^* = CYC^* + I + DD^*. \qquad (7)$$

Denote

$$\alpha = (1-\gamma^2)^{\frac{1}{2}}. \qquad (8)$$

and

$$W = \alpha^2 I + (\alpha^2 - 1)YX. \qquad (9)$$

Let $N_1 \in \mathcal{T}(\mathbb{R}^{q\times q})$ be a constant matrix satisfying

$$N_1 N_1^* = I + S^{-1}C(I+YX)W^{-1}YC^*S^{*-1}. \qquad (10)$$

Choose matrix $N_2 = \begin{bmatrix} N_{2,11} & N_{2,12} \\ N_{2,21} & N_{2,22} \end{bmatrix}$ with $N_{2,11} \in \mathcal{T}(\mathbb{R}^{p\times p})$, $N_{2,12} \in \mathcal{T}(\mathbb{R}^{p\times q})$, $N_{2,21} \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $N_{2,22} \in \mathcal{T}(\mathbb{R}^{q\times q})$ satisfying

$$N_2^* N_2 = \begin{bmatrix} I + B^*X(I+YX)W^{-1}B & -B^*X(I+YX)W^{-1}L \\ -L^*X(I+YX)W^{-1}B & I + L^*X(I+YX)W^{-1}L \end{bmatrix}. \qquad (11)$$

We know that there are normalized left coprime factorizations $G = \tilde{M}^{-1}\tilde{N}$ with $\tilde{M}(0) \in \mathcal{T}(\mathbb{R}^{q\times q})$ and $\tilde{N}(0) \in \mathcal{T}(\mathbb{R}^{q\times p})$. One particular realization of such factorization is as follows:

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & B+LD & L \\ \hline S^{-1}C & S^{-1}D & S^{-1} \end{array}\right]. \qquad (12)$$

Now we are ready to present the main results of this paper.

Theorem 1 Given a lifted LTI plant $\hat{G}(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{T}(\mathbb{R}^{q\times p})$, let X and Y be the stabilizing solutions of Riccati equations (2) and (3), and let F, L, S be defined as in (4)-(7). Then the optimal robust stabilization margin is b_{opt}, where

$$b_{opt}^2 = \sup_{K, \hat{K}(0)\in\mathcal{T}(\mathbb{R}^{p\times q})} b_{G,K}^2 =$$
$$1 - \max_r \left\| \begin{bmatrix} \Pi_{\mathcal{U}_r} & 0 & 0 \\ 0 & \Pi_{\mathcal{Y}_r} & 0 \\ 0 & 0 & I \end{bmatrix} \Gamma \begin{bmatrix} I - \Pi_{\mathcal{Y}_r} & 0 \\ 0 & I \end{bmatrix} \right\|^2,$$

$$\mathcal{U}_r = \mathcal{U}(0) \oplus \cdots \oplus \mathcal{U}(r) \tag{13}$$
$$\mathcal{Y}_r = \mathcal{Y}(0) \oplus \cdots \oplus \mathcal{Y}(r) \tag{14}$$

$$\Gamma = \begin{bmatrix} -D^*S^{*-1} & -(B^* + D^*L^*)(X^{-1}+Y)^{-\frac{1}{2}} \\ S^{*-1} & L^*(X^{-1}+Y)^{-\frac{1}{2}} \\ Y^{\frac{1}{2}}C^*S^{*-1} & Y^{\frac{1}{2}}(A+LC)^*(X^{-1}+Y)^{-\frac{1}{2}} \end{bmatrix}. \tag{15}$$

Theorem 1 tells us the optimal robust stability margin. The next theorem provides us an observer-form suboptimal controller.

Theorem 2 Given a lifted LTI plant $\hat{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{T}(\mathbb{R}^{q\times p})$ and $\gamma < b_{opt}$, let $X, Y, F, L, S, W, \alpha, N_1$ and N_2 be defined as in (2)(5) and (7)-(11). Then a suboptimal controller K exists if and only if there exists a constant matrix $R_0 \in \mathcal{M}(\mathbb{R}^{(p+q)\times q})$ with $\|R_0\|_\infty \le 1$ such that $E_1 \in \mathcal{T}(\mathbb{R}^{p\times q})$ and $E_2 \in \mathcal{T}(\mathbb{R}^{q\times q})$, where

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \alpha^{-1}(N_2^*N_2)^{-1} \begin{bmatrix} -D^* \\ I \end{bmatrix} S^{*-1}N_1 - N_2^{-1}R_0. \tag{16}$$

Furthermore, if such R_0 is found, a suboptimal controller K is given by

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & H \end{array}\right] \tag{17}$$

where

$$A_K = A + LC + (B+LD)(F - HC - HDF)W^{-1} \tag{18}$$

$$B_K = BH + LDH - L \tag{19}$$
$$C_K = (F - HC - HDF)W^{-1} \tag{20}$$
$$H = E_1E_2^{-1}. \tag{21}$$

This controller can be written in the following general observer form

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) \\ &\quad + L[C\hat{x}(k) + Du(k) - y(k)]\end{aligned} \tag{22}$$

$$u(k) = F_K\hat{x}(k) + \bar{H}[C\hat{x}(k) + Du(k) - y(k)] \tag{23}$$

where $\bar{H} = -(I - HD)^{-1}H$ and $F_K = (I - HD)^{-1}(C_K + HC)$.

The proof of these two theorems is omitted here, please refer to [5] for detail.

Remark 1 If there is no causality constraint, we can simply take $R_0 = 0$. Assume the plant G is strictly proper, it can be shown that

$$H = B^*X(W + BB^*X)^{-1}L.$$

Then the controller is given by

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L[C\hat{x}(k) - y(k)]$$
$$u(k) = [(F - HC)W^{-1} + HC]\hat{x}(k)$$
$$-H[C\hat{x}(k) - y(k)].$$

This is exactly the same as the controller of (2.5)-(2.6) in [12].

Remark 2 The problem to design a strictly proper suboptimal controller for an LTI strictly proper plant studied in [12] is a special case of Theorem 2. Actually, if there exists R_0 such that $E_1 = 0$, then the suboptimal controller is given by

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) \\ &\quad + L[C\hat{x}(k) + Du(k) - y(k)] \\ u(k) &= FW^{-1}\hat{x}(k).\end{aligned}$$

The above controller is the same as Theorem 5 of [12].

Remark 3 The extra burden to design a robust controller for an MP system is to solve a contractive matrix completion problem. A unique central solution can be obtained following the method in [17]. In this way, we can get a unique central controller.

Remark 4 For the general $\mathcal{H}_\infty$ optimization problem of MP systems, a two-step design procedure is given in [25]: the first step is to compute the feedthrough term of the controller satisfying the causality constraint and the next step is to solve a high order $\mathcal{H}_\infty$ optimization problem. For the robust stabilization problem with the ν-gap metric, however, Theorem 2 presents us a method to obtain a general observer form controller with the same order as the plant.

5 Conclusion

In this paper, we present a state space solution to the robust stabilization problem of discrete-time periodic and multirate systems. First, we give a general setup of MP systems and show how the robust stabilization problem of multirate systems with the ν-gap metric uncertainty can be converted to a constrained $\mathcal{H}_\infty$ optimal control problem. The optimal robust stabilization margin is explicitly computed and an observer form suboptimal controller is presented. The computational burden is to solve two Riccati equations and an contractive matrix completion problem.

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