Multirate Sampled-Data Systems: All $\mathcal{H}_\infty$ Suboptimal Controllers and the Minimum Entropy Controller *

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Abstract

For a general multirate SD (sampled-data) system, we characterize explicitly the set of all causal, stabilizing controllers that achieve a certain $\mathcal{H}_\infty$ norm bound; moreover, we give explicitly a particular controller that further minimizes an entropy function for the SD system. The characterization lays the groundwork for synthesizing multirate control systems with multiple/mixed control specifications.

Keywords: multirate systems, $\mathcal{H}_\infty$ optimization, digital control, sampled-data systems, matrix factorization, nest operators.

1 Introduction

Multirate systems are abundant in industry [17]; there are several reasons for this:

- In *multivariable* digital control systems, often it is unrealistic, or sometimes impossible, to sample all physical signals uniformly at one single rate. In such situations, one is forced to use multirate sampling.
- In general one gets better performance if one can use faster A/D and D/A conversions; but this means higher cost in implementation. For signals with different bandwidths, better trade-offs between performance and implementation cost can be obtained using A/D and D/A converters at different rates.

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- Multirate controllers are in general time-varying. Thus multirate control systems can outperform single-rate systems; for example, gain margin improvement [26, 16], simultaneous stabilization [26], and decentralized control [2, 42].

The study of multirate systems started in late 1950's [28, 24, 25]. Early studies were focused on analysis and were solely for purely discrete-time systems, see also [31]. A renaissance of research on multirate systems has occurred since late 1980 with an increased interest in multirate controller design, e.g., stabilizing controller design and parametrization of all stabilizing controllers [11, 29, 34], LQG/LQR control [8, 1, 30], $\mathcal{H}_2$ optimal control [40, 41, 33], $\mathcal{H}_\infty$ control [40, 41, 10], ℓ_1 optimal control [15] and the work in [3, 20, 36]. With the recognition that many industrial control systems consist of an analog plant and a digital controller interconnected via A/D and D/A converters, direct optimal control of multirate systems has been studied in this sampled-data setting [40, 10, 33]. The existing techniques for multirate $\mathcal{H}_\infty$ control allow for computation of one $\mathcal{H}_\infty$ controller via a numerical convex optimization [41] or more easily via an explicit design [10]. The purpose of this paper is to characterize in an explicit way the set of all $\mathcal{H}_\infty$ suboptimal controllers and to find a particular $\mathcal{H}_\infty$ suboptimal controller which minimizes an entropy function.

In this paper we shall treat a general multirate setup. For this, we define the periodic sampler S_τ and the (zero-order) hold H_τ (the subscript denotes the period) as follows: S_τ maps a continuous signal to a discrete signal and is defined via

$$\psi = S_\tau y \Longleftrightarrow \psi(k) = y(k\tau).$$

H_τ maps discrete to continuous via

$$u = H_\tau v \Longleftrightarrow u(t) = v(k), \qquad k\tau \le t < (k+1)\tau.$$

(The signals may be vector-valued.) Note that the sampler and hold are synchronized at $t = 0$.

The general multirate system is shown in Figure 1. We have used continuous arrows for continuous signals and dotted arrows for discrete signals. Here, G_a is the continuous-time

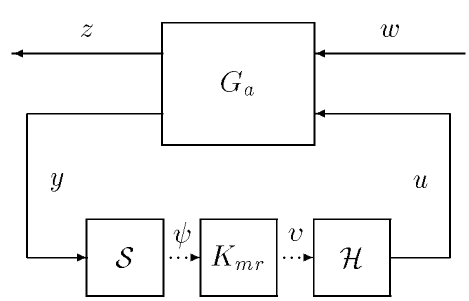

Figure 1: The general multirate sampled-data setup

generalized plant with two inputs, the exogenous input w and the control input u, and two outputs, the signal z to be controlled and the measured signal y. $\mathcal{S}$ and $\mathcal{H}$ are multirate sampling and hold operators and are defined as follows:

$$\mathcal{S} = \begin{bmatrix} S_{m_1 h} & & \\ & \ddots & \\ & & S_{m_p h} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H_{n_1 h} & & \\ & \ddots & \\ & & H_{n_q h} \end{bmatrix}.$$

These correspond to sampling p channels of y periodically with periods $m_i h$, $i = 1, \cdots, p$, respectively, and holding q channels of v with periods $n_j h$, $j = 1, \cdots, q$, respectively. Here m_i and n_j are different integers and h is a real number referred to as the *base period*. If we partition the signals accordingly

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_p \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_q \end{bmatrix},$$

then

$$\begin{aligned} \psi_i(k) &= y_i(k m_i h), \quad i = 1, \cdots, p, \\ u_j(t) &= v_j(k), \quad k n_j h \le t < (k+1) n_j h, \quad j = 1, \cdots, q. \end{aligned}$$

K_{mr} is a discrete-time multirate controller, implemented via a microprocessor; it is synchronized with $\mathcal{S}$ and $\mathcal{H}$ in the sense that it inputs a value from the i-th channel at times $k m_i h$ and outputs a value to the j-th channel at $k n_j h$.

In the general multirate setup of Figure 1, we assume throughout that G_a and K_{mr} are causal and linear. Furthermore, G_a is assume to be time-invariant and finite-dimensional, and K_{mr} is assumed to satisfy certain periodic property and to be finite-dimensional.

For periodicity of K_{mr}, let l be the least common multiple of the sampling and hold indices, $\{m_1, \cdots, m_p, n_1, \cdots, n_q\}$. Thus $\sigma := lh$ is the least common period for all sampling and hold channels. The multirate controller K_{mr} can be chosen so that $\mathcal{H} K_{mr} \mathcal{S}$ is σ-periodic in continuous time. For this, we need a few definitions.

Let ℓ be the space of sequences, perhaps vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. Let U be the unit time delay on ℓ and U^* the unit time advance. Define the integers

$$\begin{aligned} \bar{m}_i &= l/m_i, \quad i = 1, 2, \cdots, p \\ \bar{n}_j &= l/n_j, \quad j = 1, 2, \cdots, q. \end{aligned}$$

We say K_{mr} is *σ-periodic in real time* if

$$\begin{bmatrix} (U^*)^{\bar{n}_1} & & \\ & \ddots & \\ & & (U^*)^{\bar{n}_q} \end{bmatrix} K_{mr} \begin{bmatrix} U^{\bar{m}_1} & & \\ & \ddots & \\ & & U^{\bar{m}_p} \end{bmatrix} = K_{mr}.$$

This means shifting ψ_i by $\bar{m}_i$ time units ($\bar{m}_i m_i h = \sigma$) corresponds to shifting v_j by $\bar{n}_j$ units ($\bar{n}_j n_j h = \sigma$). Thus $\mathcal{H} K_{mr} \mathcal{S}$ is σ-periodic in continuous time iff K_{mr} is σ-periodic in real time.

Since G_a is LTI, it follows that the sampled-data system in Figure 1 is σ-periodic if K_{mr} is σ-periodic in real time. We shall refer to σ as the *system period.* We shall assume throughout the paper that K_{mr} is σ-periodic in real time. With all these assumptions, the controller K_{mr} can be implemented via difference equations [10]

$$\begin{aligned} \eta(k+1) &= A\eta(k) + \sum_{i=1}^{p} \sum_{s=0}^{\bar{m}_i - 1} (B_i)_s \psi_i(k\bar{m}_i + s), \\ v_j(k\bar{n}_j + r) &= (C_j)_r \eta(k) + \sum_{i=1}^{p} \sum_{s=0}^{\bar{m}_i - 1} (D_{ji})_{rs} \psi_i(k\bar{m}_i + s), \quad j = 1, 2, \ldots, q, \end{aligned}$$

where causality requires $(D_{ji})_{rs} = 0$ if $rn_j < sm_i$.

Our goal in this paper is two-fold: (1) characterize all feasible multirate controllers which internally stabilize the feedback system shown in Figure 1 and make the $\mathcal{L}_2$ induced norm less than a prespecified value, such controllers are called $\mathcal{H}_\infty$ suboptimal controllers; (2) among all $\mathcal{H}_\infty$ suboptimal controllers, find one which further minimizes an entropy function. Used with other optimization techniques, such a characterization, like its LTI counterpart [14, 21], is essential in designing control systems with simultaneous $\mathcal{H}_\infty$ and other performance requirements. The minimum entropy control, also like its LTI counterpart [32, 22, 23], gives a particular example of such multi-objective control problem in which an analytic solution exists.

Although the overall system shown in Figure 1 is hybrid (involving both continuous-time and discrete-time signals) and time-varying, the recently developed lifting technique enables us to convert the problem into an equivalent LTI discrete-time problem. However, the resulting control problem will have an undesirable and unconventional constraint on the LTI controller due to the causality requirement. This constraint is the main difficulty in designing optimal multirate systems. The recent introduction of the nest operators has proven to be effective in handling causality constraints in multirate design [10]. The results of this paper will be built on the nest operator technique.

We would like to remark here that the results in this paper extend directly to periodic discrete-time systems, i.e., direct application yields a characterization of all $\mathcal{H}_\infty$ suboptimal solutions which are periodic and causal; this result has not obtained before.

The paper is organized as follows. The next section reviews some basic facts about continuous-time periodic systems, introduces the concept of entropy for such systems, and establishes the connection between the entropy and a linear, exponential, quadratic, Gaussian cost function. Section 3 addresses topics on nest operators and nest algebra, which are the main tools to handle causality in this paper. Section 4 briefly discusses the procedure of converting our hybrid problem into an equivalent LTI problem with a causality constraint. Section 5 gives a characterization of all $\mathcal{H}_\infty$ suboptimal controllers and the minimum entropy controller. The appendices contain two long and involved proofs.

Preliminary results in this paper have been presented at several conferences: the Asian Control Conference (Tokyo, 1994), the IEEE Conference on Decision and Control (Florida, 1994), and the International Conference on Operator Theory and its Applications (Manitoba, 1994).

Finally, we introduce some notation. Given an operator K and two operator matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

the linear fractional transformation associated with P and K is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

and the star product of P and Q is

$$P \star Q = \begin{bmatrix} P_{11} + P_{12}Q_{11}(I - P_{22}Q_{11})^{-1}P_{21} & P_{12}(I - Q_{11}P_{22})^{-1}Q_{12} \\ Q_{21}(I - P_{22}Q_{11})^{-1}P_{21} & Q_{21}(I - P_{22}Q_{11})^{-1}P_{22}Q_{12} + Q_{22} \end{bmatrix}.$$

Here, we assume that the domains and co-domains of the operators are compatible and the inverses exist. With these definitions, we have

$$\mathcal{F}(P, \mathcal{F}(Q, K)) = \mathcal{F}(P \star Q, K).$$

2 Entropy of periodic systems

A multirate system as depicted in Section 1 is a continuous-time σ-periodic system. In this section, we review some basic concepts of periodic systems and introduce the concept of entropy.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and $f = \{f(k) : k = 1, 2, \ldots\}$ be a sequence of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. Then

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

is an operator-valued function on some subset of $\mathcal{C}$. We say that $\hat{F}$ belongs to $\mathcal{H}_\infty(\mathcal{X}, \mathcal{Y})$ if $\hat{F}$ is analytic in $\mathcal{D}$, the open unit disk, and

$$\sup_{\lambda \in \mathcal{D}} \|\hat{F}(\lambda)\| < \infty.$$

In this case, the left-hand side above is defined to be the $\mathcal{H}_\infty$ norm of $\hat{F}$, denoted by $\|\hat{F}\|_\infty$, the operator $\hat{F}(\mathrm{e}^{\mathrm{j}\omega})$ is bounded for almost every $\omega \in [-\pi, \pi)$, and

$$\operatorname*{ess\ sup}_{\omega \in [-\pi, \pi)} \|\hat{F}(\mathrm{e}^{\mathrm{j}\omega})\| = \|\hat{F}\|_\infty.$$

Now let $f = \{f(k) : k = 1, 2, \ldots\}$ be a sequence of Hilbert-Schmidt operators from $\mathcal{X}$ to $\mathcal{Y}$. The set of Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, $\| \cdot \|_{\mathrm{HS}}$, is a Hilbert space [19]. Then

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

is a Hilbert-space vector-valued function on some subset of $\mathcal{C}$. We say that $\hat{F}$ belongs to $\mathcal{H}_2(\mathcal{X},\mathcal{Y})$ if

$$\left(\sum_{k=0}^{\infty} \|f(k)\|_{\mathrm{HS}}^2\right)^{1/2} < \infty.$$

In this case, the left-hand side above is defined to be the $\mathcal{H}_2$ norm of $\hat{F}$, denoted by $\|\hat{F}\|_2$, the operator $\hat{F}(\mathrm{e}^{\mathrm{j}\omega})$ is Hilbert-Schmidt for almost every $\omega \in [-\pi, \pi)$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\hat{F}(\mathrm{e}^{\mathrm{j}\omega})\|_{\mathrm{HS}}^2 d\omega = \|\hat{F}\|_2^2.$$

Assume $\hat{F} \in \mathcal{H}_\infty(\mathcal{X},\mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X},\mathcal{Y})$ and $\|\hat{F}\|_\infty < 1$. Extending the entropy definition for matrix valued analytic functions [22, 23], we define the entropy of $\hat{F}$ as

$$\mathcal{I}(\hat{F}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det[I - \hat{F}^*(\mathrm{e}^{\mathrm{j}\omega})\hat{F}(\mathrm{e}^{\mathrm{j}\omega})]d\omega.$$

This entropy is well defined. Since $\hat{F}(\mathrm{e}^{\mathrm{j}\omega})$ is a Hilbert-Schmidt operator at almost every $\omega \in [-\pi, \pi)$, its singular values form a square-summable sequence $\{\sigma_k(\mathrm{e}^{\mathrm{j}\omega})\}$. Hence

$$\det[I - \hat{F}^*(\mathrm{e}^{\mathrm{j}\omega})\hat{F}(\mathrm{e}^{\mathrm{j}\omega})] = \prod_{k=1}^{\infty}[1 - \sigma_k^2(\mathrm{e}^{\mathrm{j}\omega})],$$

which converges to some number in $(0, 1)$ due to square-summability of $\{\sigma_k(\mathrm{e}^{\mathrm{j}\omega})\}$ and the fact that $\|\hat{F}\|_\infty < 1$. This also shows that $\mathcal{I}(\hat{F})$ is nonnegative.

Lemma 1 *Assume $\hat{F} \in \mathcal{H}_\infty(\mathcal{X},\mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X},\mathcal{Y})$ and $\|\hat{F}\|_\infty < 1$. Then*

(a) $\|\hat{F}\|_2^2 \le \mathcal{I}(\hat{F})$;

(b) for $\hat{U} = \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix} \in \mathcal{H}_\infty(\mathcal{X}\oplus\mathcal{Y}, \mathcal{Y}\oplus\mathcal{X})$ with $\hat{U}^\sim\hat{U} = I$, $U_{11} \in \mathcal{H}_2(\mathcal{X},\mathcal{Y})$, and $\hat{U}_{21}^{-1} \in \mathcal{H}_\infty(\mathcal{X},\mathcal{X})$,

$$\mathcal{I}[\mathcal{F}(\hat{U},\hat{F})] = \mathcal{I}(\hat{F}) + \mathcal{I}(\hat{U}_{11}) + 2\ln|\det[I - \hat{U}_{22}(0)\hat{F}(0)]|.$$

The proof of Lemma 1 is similar to that for the finite-dimensional, continuous-time case [32].

Now let us return to periodic systems. Let F_a be a continuous-time, σ-periodic, causal system described by the following integral operator

$$(F_a w)(t) = \int_0^t f_a(t,\tau)w(\tau)\,d\tau.$$

We assume that f_a, the matrix-valued impulse response of F_a, is locally square-integrable, i.e., every element is square-integrable on any compact subset of $\mathcal{R}^2$. The periodicity of F_a implies $f_a(t+T,\tau+T) = f_a(t,\tau)$, and the causality implies that $f_a(t,\tau) = 0$ if $\tau > t$.

The local square-integrability of f_a guarantees that F_a is a linear map from $\mathcal{L}_{2e}$ to $\mathcal{L}_{2e}$, the space of locally square-integrable functions of t. Given an arbitrary, positive integer l, let

$$\mathcal{K} = \mathcal{L}_2^l[0, \frac{\sigma}{l}).$$

Denote the space of $\mathcal{K}$-valued sequences by $\ell(\mathcal{K})$. Define the lifting operator $L_{\sigma,l} : \mathcal{L}_{2e} \to \ell(\mathcal{K})$ via

$$\omega = L_{\sigma,l} w \Longleftrightarrow$$
$$\{\omega(0), \omega(1), \ldots\} = \left\{ \begin{bmatrix} w(t) \\ \vdots \\ w(t+(l-1)\frac{\sigma}{l}) \end{bmatrix}, \begin{bmatrix} w(t+\sigma) \\ \vdots \\ w(t+\sigma+(l-1)\frac{\sigma}{l}) \end{bmatrix}, \ldots \right\}, \quad t \in [0, \frac{\sigma}{l}).$$

This lifting $L_{\sigma,l}$ gives an algebraic isomorphism between $\mathcal{L}_{2e}$ and $\ell(\mathcal{K})$ [40]. We use the obvious norm in $\mathcal{K}$:

$$\kappa = \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_l \end{bmatrix} \in \mathcal{K} \Longrightarrow \|\kappa\| = \left(\sum_{i=1}^{l} \|\kappa_i\|^2 \right)^{1/2},$$

where $\|\kappa_i\|$ is the norm on $\mathcal{L}_2[0, \frac{\sigma}{l})$. Denote by $\ell_2(\mathcal{K})$ the subset of $\ell(\mathcal{K})$ consisting of all sequences ω with

$$\left(\sum_{k=0}^{\infty} \|\omega(k)\|^2 \right)^{1/2} < \infty,$$

and define the norm on $\ell_2(\mathcal{K})$ to be the left-hand side of the above inequality. It is clear that $\omega \in \ell_2(\mathcal{K})$ if and only if $w \in \mathcal{L}_2$ and $L_{\sigma,l}$ is a Hilbert-space isometric isomorphism from $\mathcal{L}_2$ to $\ell_2(\mathcal{K})$.

Now we lift F_a to get $F := L_{\sigma,l} F_a L_{\sigma,l}^{-1}$. The lifted system $F : \ell(\mathcal{K}) \to \ell(\mathcal{K})$ can be described by

$$\zeta = F\omega \Longleftrightarrow \zeta(k) = \sum_{i=0}^{k} f(k-i)\omega(i), \quad k \geq 0,$$

where $f(k)$, $k = 0, 1, \ldots$, map $\mathcal{K}$ to $\mathcal{K}$ via

$$[f(k)\kappa](t)$$
$$= \int_0^{\frac{\sigma}{l}} \begin{bmatrix} f_a(t+k\sigma, \tau) & \cdots & f_a(t+k\sigma, \tau+(l-1)\frac{\sigma}{l}) \\ \vdots & & \vdots \\ f_a(t+k\sigma+(l-1)\frac{\sigma}{l}, \tau) & \cdots & f_a(t+k\sigma+(l-1)\frac{\sigma}{l}, \tau+(l-1)\frac{\sigma}{l}) \end{bmatrix} \begin{bmatrix} \kappa_1(\tau) \\ \vdots \\ \kappa_l(\tau) \end{bmatrix} d\tau,$$
$$t \in [0, \frac{\sigma}{l}).$$

The local square-integrability of $f_a(t,\tau)$ ensures that $f(k)$, $k \geq 0$, are Hilbert-Schmidt operators [44].

For σ-periodic F_a, the lifted system F is LTI in discrete time; its transfer function is defined as

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k.$$

So if $\hat{F} \in \mathcal{H}_\infty(\mathcal{K},\mathcal{K}) \cap \mathcal{H}_2(\mathcal{K},\mathcal{K})$ and $\|\hat{F}\|_\infty < 1$, its entropy can be defined.

We will define the $\mathcal{H}_\infty$ norm, $\mathcal{H}_2$ norm, and entropy of F_a to be those of $\hat{F}$ respectively. Actually, the $\mathcal{H}_\infty$ norm defined this way is indeed the $\mathcal{L}_2$-induced norm of F_a [7, 5, 38]; the $\mathcal{H}_2$ norm has natural interpretations in terms of impulse responses and white noise responses [6, 27]; the entropy not only provides an upper bound for the $\mathcal{H}_2$ norm as stated in Lemma 1, but also has a stochastic interpretation in terms of a linear, exponential, quadratic, Gaussian (LEQG) cost function, similar to the case of matrix-valued transfer functions [18].

To avoid unnecessary technicality, we will concentrate on finite-dimensional periodic systems, i.e., those F_a with finite-dimensional realizations, or equivalently, those F_a whose lifted transfer functions $\hat{F}$ have only a finite number of poles. (The multirate systems to be studied in Figure 1 fall in this class if both G_a and K_{mr} are finite-dimensional.) Let w be a Gaussian white noise with zero mean and unit covariance on the time interval $[0,\infty)$ and z the corresponding response: $z = F_a w$. Define an LEQG cost function for F_a as

$$\Omega_T = \frac{2}{T} \ln \boldsymbol{E} \left\{ \exp \left[\frac{1}{2} \int_0^T z'(t) z(t) dt \right] \right\}$$

where $\boldsymbol{E}(\cdot)$ means the expectation. The proof of the following theorem is given in Appendix A.

Theorem 1 *Given a finite-dimensional, σ-periodic system F_a, assume its lifted transfer function $\hat{F}$ satisfies $\hat{F} \in \mathcal{H}_\infty(\mathcal{K},\mathcal{K}) \cap \mathcal{H}_2(\mathcal{K},\mathcal{K})$ and $\|\hat{F}\|_\infty < 1$. Then* $\lim_{T\to\infty} \Omega_T = \mathcal{I}(\hat{F})/\sigma$.

Now we are ready to state our control problems associated with Figure 1 precisely:

> Given a continuous-time, finite-dimensional, LTI plant G_a and sampling and hold schemes $\mathcal{S}$ and $\mathcal{H}$,
>
> (1) characterize all feasible multirate controllers K_{mr} such that the feedback system is internally stable and
> $$\|\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})\|_\infty < 1;$$
>
> (2) find a particular controller from those obtained in (1) such that the entropy
> $$\mathcal{I}[\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})]$$
> is minimized.

These problems will be solved explicitly in Sections 5 and 6. Next, we present the required mathematical tool based on nest operators.

3 Nest operators

In this section, we address some issues on nest operators and nest algebra [4, 12], which are useful in the sequel. Our main purpose is to probe further the Arveson's distance problem, that is, we characterize explicitly all nest operators which are within a fixed distance from a given operator; we also give one such nest operator which minimizes an auxiliary entropy function. The same problems were also studied in the mathematical literature [43], but the solutions are different. Our results, based on the unitary dilation, provide further insight as well as certain numerical advantages; they take forms which are easily applicable to our control problems at hand.

Let $\mathcal{X}$ be a vector space. A *nest* in $\mathcal{X}$, denoted $\{\mathcal{X}_i\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the nonincreasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{n-1} \supseteq \mathcal{X}_n = \{0\}.$$

(A nest may be defined to contain an infinite number of spaces, but this generalization is not necessary in the sequel.)

Let $\mathcal{X}$ and $\mathcal{Y}$ be both Hilbert spaces. Denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of bounded linear operators $\mathcal{X} \to \mathcal{Y}$ and abbreviate it as $\mathcal{L}(\mathcal{X})$ if $\mathcal{X} = \mathcal{Y}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are equipped, respectively, with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ which have the same number of subspaces, say, $n+1$ as above. An operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be a *nest operator* if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_i, \qquad i = 0, 1, \cdots, n. \tag{1}$$

It is said to be a *strict nest operator* if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_{i+1}, \qquad i = 0, 1, 2, \cdots, n-1. \tag{2}$$

Let $\Pi_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i$ and $\Pi_{\mathcal{Y}_i} : \mathcal{Y} \to \mathcal{Y}_i$ be orthogonal projections. Then the condition in (1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_i}) T \Pi_{\mathcal{X}_i} = 0, \qquad i = 0, 1, \cdots, n,$$

and the condition in (2) is equivalent to

$$(I - \Pi_{\mathcal{Y}_{i+1}}) T \Pi_{\mathcal{X}_i} = 0, \qquad i = 0, 1, 2, \cdots, n-1.$$

Given the nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$, the set of all nest operators is denoted $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and abbreviated $\mathcal{N}(\{\mathcal{X}_i\})$ if $\{\mathcal{X}_i\} = \{\mathcal{Y}_i\}$; the set of all strict nest operators is denoted $\mathcal{N}_s(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and abbreviated $\mathcal{N}_s(\{\mathcal{X}_i\})$ if $\{\mathcal{X}_i\} = \{\mathcal{Y}_i\}$.

If we decompose the spaces $\mathcal{X}$ and $\mathcal{Y}$ in the following way

$$\mathcal{X} = (\mathcal{X}_0 \ominus \mathcal{X}_1) \oplus (\mathcal{X}_1 \ominus \mathcal{X}_2) \oplus \cdots \oplus (\mathcal{X}_{n-1} \ominus \mathcal{X}_n), \tag{3}$$

$$\mathcal{Y} = (\mathcal{Y}_0 \ominus \mathcal{Y}_1) \oplus (\mathcal{Y}_1 \ominus \mathcal{Y}_2) \oplus \cdots \oplus (\mathcal{Y}_{n-1} \ominus \mathcal{Y}_n), \tag{4}$$

then the associated matrix representation of T is

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$$

and $T \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ means that this matrix representation is (block) lower triangular: $T_{ij} = 0$ if $i > j$. The following useful lemmas can be proven readily by using the above matrix representation.

Lemma 2

(a) If $T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, then $T_2T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$.

(b) If $T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}_s(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, or if $T_1 \in \mathcal{N}_s(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, then $T_2T_1 \in \mathcal{N}_s(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$.

(c) $\mathcal{N}(\{\mathcal{X}_i\})$ forms an algebra, called a nest algebra*.*

In the rest of this section, we restrict our discussion to finite-dimensional spaces.

Lemma 3

(a) If $T \in \mathcal{N}_s(\{\mathcal{X}_i\})$, then $I - T$ is always invertible.

(b) If $T \in \mathcal{N}(\{\mathcal{X}_i\})$ and T is invertible, then $T^{-1} \in \mathcal{N}(\{\mathcal{X}_i\})$.

Lemma 4 (Generalized QR factorization) *Let $T \in \mathcal{L}(\mathcal{X})$.*

(a) There exist a unitary operator Q_1 on $\mathcal{X}$ and $R_1 \in \mathcal{N}(\{\mathcal{X}_i\})$ such that $T = Q_1R_1$.

(b) There exist $R_2 \in \mathcal{N}(\{\mathcal{X}_i\})$ and a unitary operator Q_2 on $\mathcal{X}$ such that $T = R_2Q_2$.

Lemma 5 (Generalized Cholesky factorization) *Let $T \in \mathcal{L}(\mathcal{X})$ and assume T is selfadjoint and nonnegative.*

(a) There exists $C_1 \in \mathcal{N}(\{\mathcal{X}_i\})$ such that $T = C_1^*C_1$.

(b) There exists $C_2 \in \mathcal{N}(\{\mathcal{X}_i\})$ such that $T = C_2C_2^*$.

The purpose of the rest of this section is to address the following two matrix problems: Given $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, (1) characterize all $N \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ such that $\|T + N\| < 1$; (2) find, among all N characterized in (1), the one which minimizes $\mathcal{I}(T + N)$. Here the entropy of a contractive matrix T is obtained as a special case from the entropy definition of a contractive Hilbert-Schmidt operator-valued function:

$$\mathcal{I}(T) = -\ln\det(I - T^*T).$$

These two matrix problems are closely related to and are actually simple special cases of the main problems of this paper: Characterize all $\mathcal{H}_\infty$ suboptimal controllers and find the minimum entropy controller.

We shall need some more notation. With $\mathcal{X}$ and $\mathcal{Y}$ as before, introduce two more finite-dimensional inner-product spaces $\mathcal{Z}$ and $\mathcal{W}$. A linear operator $T \in \mathcal{L}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{Z} \oplus \mathcal{W})$ is partitioned as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

with $T_{11} \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $T_{21} \in \mathcal{L}(\mathcal{X}, \mathcal{W})$, etc. For nests $\{\mathcal{X}_i\}$, $\{\mathcal{Y}_i\}$, $\{\mathcal{Z}_i\}$, $\{\mathcal{W}_i\}$ in $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{W}$, respectively, all with $n+1$ subspaces, the nests $\{X_i \oplus \mathcal{Y}_i\}$ and $\{\mathcal{Z}_i \oplus \mathcal{W}_i\}$ are defined in the obvious way. Hence writing

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{N}(\{X_i \oplus \mathcal{Y}_i\}, \{\mathcal{Z}_i \oplus \mathcal{W}_i\}),$$

means $T_{11} \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$, $T_{21} \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{W}_i\})$, etc.

Theorem 2 *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The following statements are equivalent:*

(a) There exists $N \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ such that $\|T + N\| < 1$.

(b) $\max_i \|(I - \Pi_{\mathcal{Y}_i})T|_{\mathcal{X}_i}\| < 1$.

(c) There exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{N}(\{\mathcal{X}_i \oplus \mathcal{Y}_i\}, \{\mathcal{Y}_i \oplus \mathcal{X}_i\})$$

with P_{12} and P_{21} both invertible and $P_{22} \in \mathcal{N}_s(\{\mathcal{Y}_i\}, \{\mathcal{X}_i\})$ such that

$$\begin{bmatrix} T + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary.

The proof of Theorem 2 is given in Appendix B. This theorem can be used to solve our first matrix problem.

Theorem 3 *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and assume condition (c) in Theorem 2 is satisfied. Then the set of all $N \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ such that $\|T + N\| < 1$ is given by*

$$\{N = \mathcal{F}(P, U) : \ U \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\}) \text{ and } \|U\| < 1\}. \tag{5}$$

Proof: Since

$$\begin{bmatrix} T+P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary and P_{12}, P_{21} are invertible, it follows from [35] that the map

$$U \mapsto \mathcal{F}\left(\begin{bmatrix} T+P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, U\right) = T + \mathcal{F}(P,U)$$

is a bijection from the open unit ball of $\mathcal{L}(\{\mathcal{X}\},\{\mathcal{Y}\})$ onto itself. What is left to show is that $\mathcal{F}(P,U) \in \mathcal{N}(\{\mathcal{X}_i\},\{\mathcal{Y}_i\})$ iff $U \in \mathcal{N}(\{\mathcal{X}_i\},\{\mathcal{Y}_i\})$. The "if" part follows from Lemma 2 by noting $P \in \mathcal{N}(\{\mathcal{X}_i \oplus \mathcal{Y}_i\},\{\mathcal{Y}_i \oplus \mathcal{X}_i\})$. For the "only if" part, assume $N := \mathcal{F}(P,U) \in \mathcal{N}(\{\mathcal{X}_i\},\{\mathcal{Y}_i\})$ for some $U \in \mathcal{L}(\{\mathcal{X}\},\{\mathcal{Y}\})$; we need to show that U too belongs to $\mathcal{N}(\{\mathcal{X}_i\},\{\mathcal{Y}_i\})$. From

$$N = P_{11} + P_{12}U(I - P_{22}U)^{-1}P_{21},$$

we obtain after some algebra

$$P_{12}^{-1}(N - P_{11})P_{21}^{-1} = [I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22}]U. \tag{6}$$

Since

$$\begin{aligned} I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22} &= I + P_{12}^{-1}P_{12}U(I - P_{22}U)^{-1}P_{21}P_{21}^{-1}P_{22}] \\ &= I + U(I - P_{22}U)^{-1}P_{22} \\ &= (I - UP_{22})^{-1}, \end{aligned}$$

it follows that $I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22}$ is invertible. Hence from (6)

$$U = [I + P_{12}^{-1}(N - P_{11})P_{21}^{-1}P_{22}]^{-1}P_{12}^{-1}(N - P_{11})P_{21}^{-1}$$

Therefore U belongs to $\mathcal{N}(\{\mathcal{X}_i\},\{\mathcal{Y}_i\})$ by Lemma 2. □

The characterization in Theorem 3 also renders an easy solution to the second matrix problem.

Theorem 4 *Let $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and assume condition (c) in Theorem 2 is satisfied. Then the unique N which satisfies $\|T+N\| < 1$ and minimizes $\mathcal{I}(T+N)$ is given by $N = P_{11}$.*

Proof: According to Theorem 3, all N satisfying $\|T+N\| < 1$ are characterized by (5). Consequently, all resulting $T+N$ are given by

$$\left\{\mathcal{F}\left(\begin{bmatrix} T+P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, U\right) : \ U \in \mathcal{N}(\{\mathcal{X}_i\},\{\mathcal{Y}_i\}) \text{ and } \|U\| < 1\right\}.$$

By Lemma 1, we obtain

$$\mathcal{I}(T+N) = \mathcal{I}(U) + \mathcal{I}(T+P_{11}) + 2\ln|\det(I - P_{22}U)|.$$

Notice that the second term is independent of U and $P_{22}U \in \mathcal{N}_s(\{\mathcal{Y}_i\},\{\mathcal{Y}_i\}$, which implies that the third term is zero. Therefore the minimizing U is 0 and hence $N = P_{11}$. □

One implication of Theorem 4 is that although P in condition (c) of Theorem 2 is not unique, P_{11} is uniquely determined.

4 Equivalent LTI systems

Our main problems deal with hybrid time-varying systems. Following [10] and [40], we can reduce the control problem to an equivalent one involving only finite-dimensional LTI systems. In this section we briefly review the reduction process. The detailed justification is referred to [10], [40], and [5]. Our emphasis here is on the relationship between the entropy of the original system and the equivalent LTI system.

We start with a state model of G_a:

$$\hat{G}_a(s) = \left[\begin{array}{c|cc} A_a & B_{a1} & B_{a2} \\ \hline C_{a1} & 0 & D_{a12} \\ C_{a2} & 0 & 0 \end{array}\right].$$

For an integer $m > 0$, define the discrete lifting operator L_m via

$$L_m\{\psi(0), \psi(1), \cdots\} = \left\{ \begin{bmatrix} \psi(0) \\ \vdots \\ \psi(m-1) \end{bmatrix}, \begin{bmatrix} \psi(m) \\ \vdots \\ \psi(2m-1) \end{bmatrix}, \cdots \right\}.$$

Denote

$$L_M = \begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix}, L_N = \begin{bmatrix} L_{\bar{n}_1} & & \\ & \ddots & \\ & & L_{\bar{n}_q} \end{bmatrix}.$$

and recall the continuous lifting operator $L_{\sigma,l}$ in Section 2: Here we take $\sigma = lh$. We lift G_a and K_{mr} by defining

$$\tilde{G} = \begin{bmatrix} L_{\sigma,l} & \\ & L_M \mathcal{S} \end{bmatrix} G_a \begin{bmatrix} L_{\sigma,l}^{-1} & \\ & \mathcal{H} L_N^{-1} \end{bmatrix}$$

and

$$K = L_N K_{mr} L_M^{-1}.$$

It is easy to check that $\tilde{G}$ and K are LTI systems, so they have transfer functions $\hat{\tilde{G}}(\lambda)$ and $\hat{K}(\lambda)$. By definitions,

$$\begin{aligned} \|\mathcal{F}(G_a, \mathcal{H} K_{mr} \mathcal{S})\|_\infty &= \|\mathcal{F}(\hat{\tilde{G}}, \hat{K})\|_\infty \\ \mathcal{I}[\mathcal{F}(G_a, \mathcal{H} K_{mr} \mathcal{S})] &= \mathcal{I}[\mathcal{F}(\hat{\tilde{G}}, \hat{K})]. \end{aligned}$$

A state-space realization of $\tilde{G}$ can be computed:

$$\hat{\tilde{G}}(\lambda) = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right].$$

Due to the causality of G_a and K_{mr}, the lifted systems $\tilde{G}$ and K have some special structures which can be easily characterized using nest operators.

Write

$$\tilde{\omega} = L_{\sigma,l} w, \quad \tilde{\zeta} = L_{\sigma,l} z, \quad \epsilon = L_N v, \quad \phi = L_M \psi.$$

Then

$$\phi(0) = \begin{bmatrix} \psi_1(0)' & \cdots & \psi_1(\bar{m}_1 - 1)' & \cdots & \psi_p(0)' & \cdots & \psi_p(\bar{m}_p - 1)' \end{bmatrix}'.$$

Note that $\psi_i(k)$ is sampled at $t = km_i h$. Similarly,

$$\epsilon(0) = \begin{bmatrix} v_1(0)' & \cdots & v_1(\bar{n}_1 - 1)' & \cdots & v_q(0)' & \cdots & v_q(\bar{n}_q - 1)' \end{bmatrix}'$$

and $v_j(k)$ occurs at $t = kn_j h$. For $r = 0, 1, \ldots, l$, define

$$\begin{aligned} \tilde{\mathcal{W}}_r &= \{\tilde{\omega}(0) : \tilde{\omega}_1(0) = \tilde{\omega}_2(0) = \cdots = \tilde{\omega}_r(0) = 0\}, \\ \tilde{\mathcal{Z}}_r &= \{\tilde{\zeta}(0) : \tilde{\zeta}_1(0) = \tilde{\zeta}_2(0) = \cdots = \tilde{\zeta}_r(0) = 0\}, \\ \mathcal{U}_r &= \{\epsilon(0) : \; v_j(k) = 0 \text{ if } kn_j < r\}, \\ \mathcal{Y}_r &= \{\phi(0) : \; \psi_i(k) = 0 \text{ if } km_i < r\}. \end{aligned}$$

Then the D-blocks in the lifted plant satisfy

$$\tilde{D}_{11} \in \mathcal{N}(\{\tilde{\mathcal{W}}_r\}, \{\tilde{\mathcal{Z}}_r\}), \quad (7)$$
$$\tilde{D}_{12} \in \mathcal{N}(\{\mathcal{U}_r\}, \{\tilde{\mathcal{Z}}_r\}), \quad (8)$$
$$\tilde{D}_{21} \in \mathcal{N}_s(\{\tilde{\mathcal{W}}_r\}, \{\mathcal{Y}_r\}), \quad (9)$$
$$\tilde{D}_{22} \in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}), \quad (10)$$

and for K_{rm} to be causal,

$$\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\}). \quad (11)$$

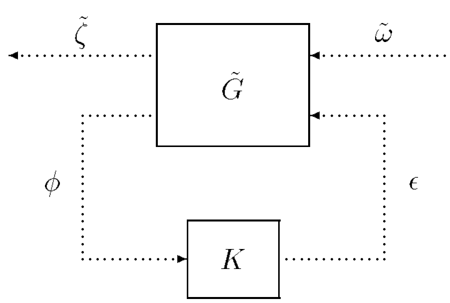

Figure 2: The lifted system

Hence we have arrived at an equivalent LTI problem, shown in Figure 2, with plant $\tilde{G}$ and controller K. Note that equations (7)–(10) give special structures of $\tilde{G}$ that can be exploited, whereas (11) is a design constraint on K that has to be respected in order for K to correspond to a causal K_{mr}.

The signals $\tilde{\omega}$ and $\tilde{\zeta}$ in Figure 2 take values in infinite-dimensional spaces. In other words, $\tilde{B}_1, \tilde{C}_1, \tilde{D}_{11}, \tilde{D}_{12}, \tilde{D}_{21}$ are operators with either domain or co-domain being infinite-dimensional spaces. To overcome this difficulty, we observe that all these operators except $\tilde{D}_{11}$ have finite rank.

Due to the particular choice of decomposition of $\mathcal{W}$ and $\mathcal{Z}$, the operator D_{11} takes a lower-triangular Toeplitz form:

$$\tilde{D}_{11} = \begin{bmatrix} (\tilde{D}_{11})_0 & & 0 \\ \vdots & \ddots & \\ (\tilde{D}_{11})_{l-1} & \cdots & (D_{11})_0 \end{bmatrix}.$$

The only block with infinite rank is $(\tilde{D}_{11})_0$. Our next step is to get rid of this by a linear fractional transformation. Since $\tilde{D}_{21} \in \mathcal{N}_s(\{\tilde{\mathcal{W}}_r\}, \{\mathcal{Y}_r\})$, the diagonal blocks of

$$\mathcal{F}(\hat{\tilde{G}}, \hat{K})(0) = \mathcal{F}[\hat{\tilde{G}}(0), \hat{K}(0)] = \tilde{D}_{11} + \tilde{D}_{12}\hat{K}(0)[I + \tilde{D}_{22}\hat{K}(0)]^{-1}\tilde{D}_{21}$$

are invariant for any K satisfying (11). Therefore $\|(\tilde{D}_{11})_0\| < 1$ is a necessary condition for the solvability of our $\mathcal{H}_\infty$ control problem. From now on we assume this condition is satisfied.

Define a diagonal operator matrix

$$U_{11} = \begin{bmatrix} -(\tilde{D}_{11})_0 & & \\ & \ddots & \\ & & -(\tilde{D}_{11})_0 \end{bmatrix}$$

and a Julian operator matrix

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & (I - U_{11}U_{11}^*)^{\frac{1}{2}} \\ (I - U_{11}^*U_{11})^{\frac{1}{2}} & -U_{11}^* \end{bmatrix}.$$

Let

$$\bar{G} = U \star \tilde{G}.$$

Then it is well-known [35] that $\|\mathcal{F}(\hat{\tilde{G}}, \hat{K})\|_\infty < 1$ iff $\|\mathcal{F}(\hat{\bar{G}}, \hat{K})\|_\infty < 1$. The relationship between the entropies is given in the following lemma.

Lemma 6

$$\mathcal{I}[\mathcal{F}(\hat{\bar{G}}, \hat{K})] = \mathcal{I}[\mathcal{F}(\hat{\tilde{G}}, \hat{K})] + l \ln\det[I - (D_{11})_0^*(D_{11})_0].$$

Proof: By Lemma 1,

$$\mathcal{I}[\mathcal{F}(\hat{\bar{G}}, \hat{K})] = \mathcal{I}[\mathcal{F}(\hat{\tilde{G}}, \hat{K})] + \mathcal{I}(U_{11}) + 2\ln|\det\{I - U_{22}\mathcal{F}[\hat{\tilde{G}}(0), \hat{K}(0)]\}|.$$

Since U_{11} is a constant operator function,

$$\mathcal{I}(U_{11}) = -\ln\det[I - U_{11}^*U_{11}] = -\ln\det[I - (\tilde{D}_{11})_0^*(\tilde{D}_{11})_0]^l = -l\ln\det[I - (\tilde{D}_{11})_0^*(\tilde{D}_{11})_0].$$

Note that $U_{22} \in \mathcal{N}(\{\tilde{\mathcal{Z}}_r\}, \{\tilde{\mathcal{W}}_r\})$ and

$$\mathcal{F}[\hat{\tilde{G}}(0), \hat{K}(0)] = \tilde{D}_{11} + \tilde{D}_{12}\hat{K}(0)[I - \tilde{D}_{22}\hat{K}(0)]^{-1}\tilde{D}_{21},$$

whose first term is in $\mathcal{N}(\{\tilde{\mathcal{W}}_r\}, \{\tilde{\mathcal{Z}}_r\})$ and second term in $\mathcal{N}_s(\tilde{\mathcal{W}}_r\}, \{\tilde{\mathcal{Z}}_r\})$. Hence

$$\begin{aligned}
\ln|\det\{I - U_{22}\mathcal{F}[\hat{\tilde{G}}(0), \hat{K}(0)]\}| &= \ln|\det(I - U_{22}\tilde{D}_{11})| \\
&= \ln\det[I - (\tilde{D}_{11})_0^*(\tilde{D}_{11})_0]^l \\
&= l\ln\det[I - (\tilde{D}_{11})_0^*(\tilde{D}_{11})_0].
\end{aligned}$$

The result then follows. □

A state-space model of $\bar{G}$ can again be computed:

$$\hat{\bar{G}}(\lambda) = \begin{bmatrix} \hat{\bar{G}}_{11}(\lambda) & \hat{\bar{G}}_{12}(\lambda) \\ \hat{\bar{G}}_{21}(\lambda) & \hat{\bar{G}}_{22}(\lambda) \end{bmatrix} = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right].$$

Since U_{11} is diagonal, i.e., $U_{11} \in \mathcal{N}(\{\tilde{\mathcal{W}}_r\}, \{\tilde{\mathcal{Z}}_r\})$ and $U_{11}^* \in \mathcal{N}(\{\tilde{\mathcal{Z}}_r\}, \{\tilde{\mathcal{W}}_r\})$, it follows

$$\begin{aligned}
\bar{D}_{11} &= U_{11} + U_{12}\tilde{D}_{11}(I - U_{22}\tilde{D}_{11})^{-1}U_{21} \in \mathcal{N}_s(\{\tilde{\mathcal{W}}_r\}, \{\tilde{\mathcal{Z}}_r\}) \\
\bar{D}_{12} &= U_{12}(I - \tilde{D}_{11}U_{22})^{-1}\tilde{D}_{12} \in \mathcal{N}(\{\mathcal{U}_r\}, \{\tilde{\mathcal{Z}}_r\}) \\
\bar{D}_{21} &= \tilde{D}_{21}(I - U_{22}\tilde{D}_{11})^{-1}U_{21} \in \mathcal{N}_s(\{\tilde{\mathcal{W}}_r\}, \{\mathcal{Y}_r\}) \\
\bar{D}_{22} &= \tilde{D}_{21}(I - U_{22}\tilde{D}_{11})^{-1}U_{22}\tilde{D}_{12} + \tilde{D}_{22} \in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}).
\end{aligned}$$

Note that the diagonal blocks of $\tilde{D}_{11}$ has been cancelled by the linear fractional transformation, resulting in a strictly (block) lower-triangular $\bar{D}_{11}$. Then the advantage of $\bar{G}$ over $\tilde{G}$ is that all operators $\bar{B}_1$, $\bar{C}_1$, $\bar{D}_{11}$, $\bar{D}_{12}$, and $\bar{D}_{21}$ are of finite rank. Therefore, if we define

$$\mathcal{Z} = \operatorname{Im}[\bar{C}_1 \ \bar{D}_{11} \ \bar{D}_{12}], \quad \mathcal{W} = \left(\operatorname{Ker}\begin{bmatrix} \bar{B}_1 \\ \bar{D}_{11} \\ \bar{D}_{21} \end{bmatrix}\right)^-$$

and

$$G = \begin{bmatrix} \Pi_{\mathcal{Z}}\bar{G}_{11}|_{\mathcal{W}} & \Pi_{\mathcal{Z}}\bar{G}_{12} \\ \bar{G}_{21}|_{\mathcal{W}} & \bar{G}_{22} \end{bmatrix},$$

then G has finite-dimensional input and output spaces and

$$\begin{aligned}
\|\mathcal{F}(\hat{G}, \hat{K})\|_\infty &= \|\mathcal{F}(\hat{\bar{G}}, \hat{K})\|_\infty \\
\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})] &= \mathcal{I}[\mathcal{F}(\hat{\bar{G}}, \hat{K})].
\end{aligned}$$

The nests $\{\tilde{\mathcal{W}}_r\}$ and $\{\tilde{\mathcal{Z}}_r\}$ induce nests in $\mathcal{W}$ and $\mathcal{Z}$ in a natural way:

$$\mathcal{W}_r = \mathcal{W} \cap \tilde{\mathcal{W}}_r, \quad \mathcal{Z}_r = \mathcal{W} \cap \tilde{\mathcal{Z}}_r.$$

Assume that a state-space model of G is:

$$\hat{G}(\lambda) = \begin{bmatrix} \hat{G}_{11}(\lambda) & \hat{G}_{12}(\lambda) \\ \hat{G}_{21}(\lambda) & \hat{G}_{22}(\lambda) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

The following structure of G is inherited from that of $\bar{G}$:

$$\begin{aligned} D_{11} &\in \mathcal{N}_s(\mathcal{W}_r\}, \{\mathcal{Z}_r\}) & (12) \\ D_{12} &\in \mathcal{N}(\{\mathcal{U}_r\}, \{\mathcal{Z}_r\}) & (13) \\ D_{21} &\in \mathcal{N}_s(\{\mathcal{W}_r\}, \{\mathcal{Y}_r\}) & (14) \\ D_{22} &\in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}). & (15) \end{aligned}$$

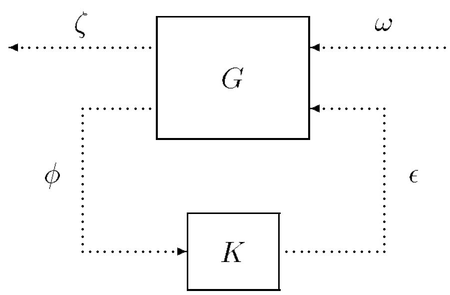

Figure 3: The equivalent finite-dimensional LTI system

In summary, our original hybrid time-varying control problem with plant G_a and controller K_{mr} can be converted into a finite-dimensional LTI control problem with plant G and controller K, as shown in Figure 3, in the sense that the system in Figure 3 is internally stable iff the system in Figure 1 is internally stable,

$$\|\mathcal{F}(\hat{G}, \hat{K})\|_\infty < 1 \iff \|\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})\|_\infty < 1,$$

and

$$\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})] = \mathcal{I}[\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})] + l \ln \det[I - (D_{11})_0^*(D_{11})_0].$$

A state-space model of G can be computed from that of G_a using the techniques developed in [5]. Any K satisfying (11) resulted from the design can be converted into a feasible multirate controller K_{mr}. We would like to emphasize, however, that the finite-dimensional LTI problem has a nonconventional constraint on the controller K given by (11). This constraint is the *causality constraint*. Also, the LTI plant G obtained from G_a will automatically satisfy (12)–(15).

5 All $\mathcal{H}_\infty$ suboptimal controllers and the minimum entropy controller.

In this section, we first characterize all $\hat{K}$ satisfying the causality constraint (11) such that the system shown in Figure 3 is internally stable and $\|\mathcal{F}(\hat{G},\hat{K})\|_\infty < 1$. This problem differs from the standard $\mathcal{H}_\infty$ problem only in the causality constraint on $\hat{K}$ and is hence called a constrained $\mathcal{H}_\infty$ problem. Our strategy in solving this problem is first to characterize all $\hat{K}$ such that the system in Figure 4 is internally stable and $\|\mathcal{F}(\hat{G},\hat{K})\|_\infty < 1$ without considering the causality constraint (this is a standard $\mathcal{H}_\infty$ problem) and then choose, if possible, from this characterization all those satisfying the causality constraint.

Several solutions to the standard $\mathcal{H}_\infty$ problem exist in the literature. Here we adopt the solution in [21]. Note that it is assumed in [21] that $D_{12}^*D_{21} > 0$ and $D_{21}D_{21}^* > 0$; these assumptions are not satisfied for the equivalent LTI system G. However, they are not essential and the solution in [21] can be modified accordingly by following, e.g., the idea in [37]. Assume the solvability conditions are satisfied, then all stabilizing controllers K satisfying $\|\mathcal{F}(\hat{G},\hat{K})\|_\infty < 1$ are characterized by

$$\left\{\hat{K} = \mathcal{F}\left(\begin{bmatrix} 0 & I \\ I & -D_{22}\end{bmatrix} \star \hat{M}, \hat{\Phi}\right) : \hat{\Phi} \in \mathcal{RH}_\infty, \|\hat{\Phi}\|_\infty < 1,\ I + D_{22}\mathcal{F}[\hat{M}(0),\hat{\Phi}(0)] \text{ is invertible}\right\}, \tag{16}$$

where $\hat{M} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22}\end{bmatrix}$ is not uniquely given in [21] and by using Lemma 5 we can always choose $\hat{M}$ so that

$$\begin{aligned} \hat{M}_{12}(0) &\in \mathcal{N}(\{\mathcal{U}_r\}), \\ \hat{M}_{21}(0) &\in \mathcal{N}(\{\mathcal{Y}_r\}), \\ \hat{M}_{22}(0) &= 0, \end{aligned}$$

and furthermore, $\hat{M}_{12}(0)$ and $\hat{M}_{21}(0)$ are invertible.

Theorem 5 *The constrained $\mathcal{H}_\infty$ problem is solvable iff the corresponding unconstrained problem is solvable and*

$$\max_r \|(I - \Pi_{\mathcal{U}_r})\hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1}|_{\mathcal{Y}_r}\| < 1. \tag{17}$$

Proof: Obviously, the corresponding unconstrained problem has to be solvable in order for the constrained problem to be solvable. Assume that the unconstrained problem is solvable. Since $D_{22} \in \mathcal{N}_s(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$, it follows that $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$ iff

$$\mathcal{F}[\hat{M}(0),\hat{\Phi}(0)] = \hat{M}_{11}(0) + \hat{M}_{12}(0)\hat{\Phi}(0)\hat{M}_{21}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\}).$$

Pre- and post-multiply this by $\hat{M}_{12}(0)^{-1}$ and $\hat{M}_{21}(0)^{-1}$ respectively to get

$$\hat{M}_{12}(0)^{-1}\mathcal{F}[\hat{M}(0),\hat{\Phi}(0)]\hat{M}_{21}(0)^{-1} = \hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \hat{\Phi}(0).$$

It follows from Theorem 1 that in order to have $\mathcal{F}[\hat{M}(0),\hat{\Phi}(0)] \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$ and $\|\hat{\Phi}(0)\| < 1$, we must have (17). Conversely, if (17) is true, then there exists a constant matrix Φ with $\|\Phi\| < 1$ such that

$$\hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \Phi \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\}).$$

Hence

$$\hat{K} = \mathcal{F}\left(\begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \star \hat{M}, \Phi\right)$$

achieves $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$. □

If the conditions in Theorem 5 are satisfied, then there exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\},\{\mathcal{U}_r \oplus \mathcal{Y}_r\})$$

with $P_{22} \in \mathcal{N}_s(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$ and P_{12} and P_{21} invertible such that

$$U = \begin{bmatrix} -\hat{M}_{12}^{-1}(0)\hat{M}_{11}(0)\hat{M}_{21}^{-1}(0) + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary. Define

$$\hat{N} = \begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \star \hat{M} \star U.$$

It is easy to check that $\hat{N}(0) \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\},\{\mathcal{U}_r \oplus \mathcal{Y}_r\})$, $\hat{N}_{12}(0)$ and $\hat{N}_{21}(0)$ are invertible, and $\hat{N}_{22}(0) \in \mathcal{N}_s(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$. By setting $\hat{\Phi} = \mathcal{F}(U,\hat{\Psi})$, the set (16) can be rewritten as

$$\{\hat{K} = \mathcal{F}(\hat{N},\hat{\Psi}) : \hat{\Psi} \in \mathcal{RH}_\infty, \|\hat{\Psi}\|_\infty < 1, I - \hat{N}_{22}(0)\hat{\Psi}(0) \text{ is invertible}\}.$$

Now we can state the main result of this paper.

Theorem 6 *Assume the solvability of the constrained $\mathcal{H}_\infty$ problem. Then the set of all controllers solving the problem is given by*

$$\{\hat{K} = \mathcal{F}(\hat{N},\hat{\Psi}) : \hat{\Psi} \in \mathcal{RH}_\infty, \|\hat{\Psi}\|_\infty < 1, \hat{\Psi}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})\ \}. \tag{18}$$

Proof: First notice that $I - \hat{N}_{22}(0)\hat{\Psi}(0)$ is always invertible if $\hat{\Psi}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$. Since $\hat{N}(0) \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\},\{\mathcal{U}_r \oplus \mathcal{Y}_r\})$ and $\hat{N}_{12}(0)$ and $\hat{N}_{21}(0)$ are invertible, it follows that $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$ iff $\hat{\Psi}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$. Then the result follows immediately. □

In the rest of this section, we show that the central controller obtained by setting $\hat{\Psi} = 0$ in (18) is the controller which minimizes $\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})]$.

Now let us go back to the characterization given in [21]. It is known (see [32] for the continuous-time case) that if all $\mathcal{H}_\infty$ suboptimal controllers are characterized by (16), then all $\mathcal{H}_\infty$ suboptimal closed loop transfer function is characterized by

$$\mathcal{F}(\hat{G}, \hat{K}) = \left\{ \mathcal{F}\left(\hat{R}, \begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix} \right) : \hat{\Phi} \in \mathcal{RH}_\infty, \|\hat{\Phi}\|_\infty < 1,\ I + D_{22}\mathcal{F}[\hat{M}(0), \hat{\Phi}(0)] \text{ is invertible} \right\}$$

where

$$\hat{R} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & \hat{R}_{13} \\ \hat{R}_{21} & \hat{R}_{22} & \hat{R}_{23} \\ \hat{R}_{31} & \hat{R}_{32} & \hat{R}_{33} \end{bmatrix} \in \mathcal{RH}_\infty$$

is para-unitary satisfying $\begin{bmatrix} \hat{R}_{21} \\ \hat{R}_{31} \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$. Clearly we have

$$\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & -D_{22} \end{bmatrix} \star \hat{M}$$

and $\hat{R}_{22}(0) = 0$. Because of this, the $\mathcal{H}_\infty$ controller without the causality constraint which minimizes the entropy $\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})]$ is conveniently given by $\hat{K} = \mathcal{F}(\hat{M}, 0) = \hat{M}_{11}$.

Notice that $\hat{\Phi} = \mathcal{F}(U, \hat{\Psi})$ gives

$$\begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{F}\left(V, \begin{bmatrix} \hat{\Psi} & 0 \\ 0 & 0 \end{bmatrix} \right)$$

where

$$V = \begin{bmatrix} -\hat{M}_{12}^{-1}(0)\hat{M}_{11}(0)\hat{M}_{21}^{-1}(0) + P_{11} & 0 & P_{12} & 0 \\ 0 & 0 & 0 & I \\ P_{21} & 0 & P_{22} & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Consequently, if we characterize the controller using (18), then all $\mathcal{H}_\infty$ suboptimal closed-loop transfer functions are

$$\mathcal{F}(\hat{G}, \hat{K}) = \left\{ \mathcal{F}\left(\hat{S}, \begin{bmatrix} \hat{\Psi} & 0 \\ 0 & 0 \end{bmatrix} \right) : \hat{\Psi} \in \mathcal{RH}_\infty, \|\hat{\Psi}\|_\infty < 1, \hat{\Psi}(0) \in \mathcal{N}(\{\mathcal{Y}_r)\}, \{\mathcal{U}_r\}) \right\}$$

where

$$\hat{S} = \begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} & \hat{S}_{13} \\ \hat{S}_{21} & \hat{S}_{22} & \hat{S}_{23} \\ \hat{S}_{31} & \hat{S}_{32} & \hat{S}_{33} \end{bmatrix} = \hat{R} \star V \in \mathcal{RH}_\infty.$$

Since $\hat{R}$ is para-unitary and V is unitary, it follows that $\hat{S}$ is para-unitary. It can be checked that $\begin{bmatrix} \hat{S}_{21} \\ \hat{S}_{31} \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$ and $S_{22}(0) \in \mathcal{N}_s(\{\mathcal{Y}_r\})$. By Lemma 1,

$$\begin{aligned}
\mathcal{I}[\mathcal{F}(\hat{G},\hat{K})] &= \mathcal{I}\left(\begin{bmatrix} \hat{\Psi} & 0 \\ 0 & 0 \end{bmatrix}\right) + \mathcal{I}(\hat{S}_{11}) + 2\ln\det\left(I - \begin{bmatrix} \hat{S}_{22}(0) & \hat{S}_{23}(0) \\ \hat{S}_{32}(0) & \hat{S}_{33}(0) \end{bmatrix}\begin{bmatrix} \hat{\Psi}(0) & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \mathcal{I}(\hat{\Psi}) + \mathcal{I}(\hat{S}_{11}) + 2\ln|\det[I - \hat{S}_{22}(0)\hat{\Psi}(0)]| \\
&= \mathcal{I}(\hat{\Psi}) + \mathcal{I}(\hat{S}_{11}).
\end{aligned}$$

The last equality is due to the fact $\hat{S}_{22}(0)\hat{\Psi}(0) \in \mathcal{N}_s(\{\mathcal{Y}_r\})$. Therefore, the minimum of $\mathcal{I}[\mathcal{F}(\hat{G},\hat{K})]$ is achieved at $\hat{\Psi} = 0$. The following theorem is thus obtained.

Theorem 7 *The minimum entropy controller is given by* $\hat{K} = \hat{N}_{11}$.

Appendix A: Proof of Theorem 1

The proof of Theorem 1 follows from the idea in [18] but has two complications: (1) operator-valued transfer functions are treated, which requires dealing with random variables in Hilbert spaces [39]; (2) signals are defined on time $[0,\infty)$ instead of $(-\infty,\infty)$, which requires treating nonstationary stochastic processes. Since F_a is linear, it follows that z is a Gaussian process. Define z_T as the stochastic process on $[0,T]$ such that $z_T(t) = z(t)$ for $t \in [0,T]$. Then z_T can be considered as a Gaussian random variable in the Hilbert space $\mathcal{L}_2[0,T]$. The covariance operator $V_T : \mathcal{L}_2[0,T] \to \mathcal{L}_2[0,T]$ is then given by $(t \in [0,T])$

$$\begin{aligned}
(V_T x)(t) &= \boldsymbol{E}\left[z_T(t)\int_0^T z_T'(\tilde{t})x(\tilde{t})d\tilde{t}\right] \\
&= \int_0^T \boldsymbol{E}[z_T(t)z_T'(\tilde{t})]x(\tilde{t})d\tilde{t} \\
&= \int_0^T \boldsymbol{E}\left[\int_0^T f_a(t,\tau)w(\tau)d\tau \int_0^T w'(\tilde{\tau})f_a'(\tilde{t},\tilde{\tau})d\tilde{\tau}\right]x(\tilde{t})d\tilde{t} \\
&= \int_0^T\int_0^T\int_0^T f_a(t,\tau)\boldsymbol{E}[w(\tau)w'(\tilde{\tau})]f_a'(\tilde{t},\tilde{\tau})x(\tilde{t})d\tau d\tilde{\tau} d\tilde{t} \\
&= \int_0^T\int_0^T\int_0^T f_a(t,\tau)\delta(\tau-\tilde{\tau})f_a'(\tilde{t},\tilde{\tau})x(\tilde{t})d\tau d\tilde{\tau} d\tilde{t} \\
&= \int_0^T\int_0^T f_a(t,\tau)f_a'(\tilde{t},\tilde{\tau})x(\tilde{t})d\tau d\tilde{t} \\
&= (F_a F_a^* x)(t).
\end{aligned}$$

This shows that $V_T = \Pi_{\mathcal{L}_2[0,T]}F_aF_a^*|_{\mathcal{L}_2[0,T]}$. Since $\Pi_{\mathcal{L}_2[0,T]}F_a|_{\mathcal{L}_2[0,T]}$ is a contractive Hilbert-Schmidt operator and F_a is causal, it follows that V_T is a selfadjoint contractive nuclear operator. Let the Schmidt expansion of V_T be

$$V_T = \sum_{i=1}^{\infty} \sigma_i \langle \cdot, v_i \rangle v_i.$$

Then z_T can be expressed as

$$z_T = \sum_{i=1}^{\infty} \alpha_i v_i$$

and α_i, $i = 1, 2, \ldots$, are independent scalar Gaussian random variables with covariance σ_i. Hence

$$\begin{aligned} \boldsymbol{E}\left\{\exp\left[\frac{1}{2}\int_0^T z'(t)z(t)dt\right]\right\} &= \boldsymbol{E}\left\{\exp\left[\frac{1}{2}\langle z_T, z_T\rangle\right]\right\} = \boldsymbol{E}\left\{\exp\left[\frac{1}{2}\sum_{i=1}^{\infty}\alpha_i^2\right]\right\} \\ &= \prod_{i=1}^{\infty} \boldsymbol{E}\{\exp \alpha_i^2/2\} = \prod_{i=1}^{\infty}(1-\sigma_i)^{-1/2} = [\det(I - V_T)]^{-1/2}. \end{aligned}$$

Now lift w to get ω and lift z to get ζ. Then $z = F_a w$ is equivalent to $\zeta = F\omega$ and F has a matrix representation

$$F = \begin{bmatrix} f(0) & 0 & & \\ f(1) & f(0) & 0 & \\ f(2) & f(1) & f(0) & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

Let F_K be the leading $K \times K$ submatrix of F. Then

$$\boldsymbol{E}\left\{\exp\left[\frac{1}{2}\int_0^{K\sigma} z'(t)z(t)dt\right]\right\} = \det(I - F_K F_K^*)^{-1/2}.$$

Since $\hat{F}$ has only finite number of poles, the infinite Hankel matrix

$$H = \begin{bmatrix} f(1) & f(2) & f(3) & \cdots \\ f(2) & f(3) & f(4) & \cdots \\ f(3) & f(4) & f(5) & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

has finite rank. Let H_K be the first K block rows of H and define

$$W_K = F_K F_K^* + H_K H_K^*.$$

Notice that W_K is a selfadjoint Toeplitz matrix

$$W_K = \begin{bmatrix} w(0) & w(-1) & w(-2) & \cdots & w(-K+1) \\ w(1) & w(0) & w(-1) & \cdots & w(-K+2) \\ w(2) & w(1) & w(0) & \cdots & w(-K+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w(K-1) & w(K-2) & w(K-3) & \cdots & w(0) \end{bmatrix}$$

and $w(i)$ is the i-th Fourier coefficient of $\hat{F}\hat{F}^{\sim}$, where $\hat{F}^{\sim}(\lambda) = \hat{F}(\bar{\lambda}^{-1})^*$. Denote by $\sigma_i(W_K)$ and $\sigma_i(F_K F_K^*)$, $i = 1, 2, \ldots$, the singular values of W_K and $F_K F_K^*$ respectively assuming ordered nondecreasingly. Then

$$\sum_{i=1}^{\infty} |\sigma_i(W_K) - \sigma_i(F_K F_K^*)| \leq \mathrm{tr} H_K H_K^* \leq \mathrm{tr} H H^* < \infty.$$

Since $\sigma_i(W_K)$ and $\sigma_i(F_K F_K^*)$ are all contained in $[-\|\hat{F}\|_\infty^2, \|\hat{F}\|_\infty^2]$, it follows that

$$\begin{aligned} |\ln\det(I - F_K F_K^*) - \ln\det(I - W_K)| &= \left| \sum_{i=1}^{\infty} \ln[1 - \sigma_i(F_K F_K^*)] - \sum_{i=1}^{\infty} \ln[1 - \sigma_i(W_K)] \right| \\ &= \left| \sum_{i=1}^{\infty} \frac{-1}{1 - \xi_i} [\sigma_i(F_K F_K^*) - \sigma_i(W_K)] \right| \end{aligned}$$

for some $\xi_i \in [-\|\hat{F}\|_\infty^2, \|\hat{F}\|_\infty^2]$. This shows that

$$|\ln\det(I - F_K F_K^*) - \ln\det(I - W_K)| \leq \frac{1}{1 - \|\hat{F}\|_\infty^2} \sum_{i=1}^{\infty} |\sigma_i(W_K) - \sigma(F_K F_K^*)| \leq \frac{1}{1 - \|\hat{F}\|_\infty^2} \mathrm{tr} H H^*.$$

Hence by using the operator-valued strong Szego-Widom limit theorem [9, Theorem 6.4],

$$\begin{aligned} \lim_{K\to\infty} \Omega_{K\sigma} &= -\lim_{K\to\infty} \frac{1}{K\sigma} \ln\det(I - F_K F_K^*) = -\lim_{K\to\infty} \frac{1}{K\sigma} \ln\det(I - W_K) \\ &= -\frac{1}{2\pi\sigma} \int_{-\pi}^{\pi} \ln\det[I - \hat{F}(\mathrm{e}^{\mathrm{j}\omega}) \hat{F}^*(\mathrm{e}^{\mathrm{j}\omega})] d\omega = \frac{1}{\sigma} \mathcal{I}(\hat{F}). \end{aligned}$$

Notice that for $K\sigma < T < (K+1)\sigma$,

$$\frac{K}{K+1} \Omega_{K\sigma} \leq \Omega_T \leq \frac{K+1}{K} \Omega_{(K+1)\sigma}.$$

Therefore, $\lim_{T\to\infty} \Omega_T = \mathcal{I}(\hat{F})/\sigma$.

Appendix B: Proof of Theorem 2

The equivalence of (a) and (b) follows from the Arveson's distance formula [12]. That (c) implies (a) is obvious. It remains to show that (b) implies (c). For this, we need a technical lemma.

Lemma 7 *Assume the matrices E, F, and H, of appropriate dimensions, satisfy the conditions:*

$$\begin{bmatrix} E & F \end{bmatrix} \begin{bmatrix} E^* \\ F^* \end{bmatrix} = I, \quad \left\| \begin{bmatrix} F \\ H \end{bmatrix} \right\| < 1.$$

Then there exists a matrix G satisfying

$$\| \begin{bmatrix} E & F \\ G & H \end{bmatrix} \| \leq 1, \qquad \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} E^* \\ F^* \end{bmatrix} = 0, \quad \| \begin{bmatrix} G & H \end{bmatrix} \| < 1.$$

An explicit formula for such a matrix is: $G = -HF^*(EE^*)^{-1}E$.

Proof: It follows from [13] that there exists a matrix G such that

$$\| \begin{bmatrix} E & F \\ G & H \end{bmatrix} \| \leq 1.$$

Among all such G characterized in [13] in terms of a free contractive matrix, the "central" one obtained by setting the free contractive matrix to zero is

$$G = -HF^*(I - FF^*)^{-1}E = -HF^*(EE^*)^{-1}E.$$

Using this G, we have

$$\begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} E^* \\ F^* \end{bmatrix} = -HF^*(EE^*)^{-1}EE^* + HF^* = 0.$$

and

$$\begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} G^* \\ H^* \end{bmatrix} = HF^*(EE^*)^{-1}FH^* + HH^* = H(I - F^*F)^{-1}H^* < I.$$

The last inequality follows from $\| \begin{bmatrix} F \\ H \end{bmatrix} \| < 1$. □

To avoid awkward notation in the proof of Theorem 2, we redefine

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

Under the decompositions of $\mathcal{X}$ and $\mathcal{Y}$ in (3)-(4), we get the matrix representation

$$\begin{bmatrix} T+A & B \\ C & D \end{bmatrix} = \begin{bmatrix} T_{11}+A_{11} & T_{12} & \cdots & T_{1n} & B_{11} & 0 & \cdots & 0 \\ T_{21}+A_{21} & T_{22}+A_{22} & \cdot & T_{2n} & B_{21} & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n1}+A_{n1} & T_{n2}+A_{n2} & \cdots & T_{nn}+A_{nn} & B_{n1} & B_{n2} & \cdots & B_{nn} \\ C_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ C_{21} & C_{22} & \cdots & 0 & D_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} & D_{n1} & D_{n2} & \cdots & 0 \end{bmatrix}.$$

Statement (b) becomes

$$\max_i \| \begin{bmatrix} T_{1(i+1)} & \cdots & T_{1n} \\ \vdots & & \vdots \\ T_{i(i+1)} & \cdots & T_{in} \end{bmatrix} \| < 1.$$

We need to decide A_{ij}, B_{ij}, C_{ij}, for $i \geq j$, and D_{ij} for $i > j$. This will be done in the following order: In the i-th step, determine those blocks in the $(n+i)$-th row and the i-th row.

Step 1:

Set $C_{11} = I$, $T_{11} + A_{11} = 0$, and choose B_{11} so that

$$\begin{bmatrix} T_{12} & \cdots & T_{1n} & B_{11} \end{bmatrix}$$

is a co-isometry. Statement (b) implies that any B_{11} chosen in this way is nonsingular.

Step i, $i = 2, \ldots, n-1$:

Set $C_{i1} = 0$ and choose the rest of the $(n+i)$-th row so that it is a co-isometry and is orthogonal to all of the previously determined rows. This requires

$$\begin{bmatrix} C_{i2} & \cdots & C_{ii} & D_{i1} & \cdots & D_{i(i-1)} \end{bmatrix}^*$$

to be an isometry onto the kernel of

$$\begin{bmatrix} T_{12} & \cdots & T_{1i} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ T_{(i-1)2} + A_{(i-1)2} & \cdots & T_{(i-1)i} & B_{(i-1)1} & \cdots & B_{(i-1)(i-1)} \\ C_{22} & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{(i-1)2} & \cdots & 0 & D_{(i-1)1} & \cdots & 0 \end{bmatrix}.$$

Then set $T_{i1} + A_{i1} = 0$ and choose

$$\begin{bmatrix} T_{i2} + A_{i2} & \cdots & T_{ii} + A_{ii} & B_{i1} & \cdots & B_{i(i-1)} \end{bmatrix}$$

in such a way so that

$$\begin{bmatrix} T_{12} & \cdots & T_{1i} & T_{1(i+1)} & \cdots & T_{1n} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ T_{(i-1)2} + A_{(i-1)2} & \cdots & T_{(i-1)i} & T_{(i-1)(i+1)} & \cdots & T_{(i-1)n} & B_{(i-1)1} & \cdots & B_{(i-1)(i-1)} \\ T_{i2} + A_{i2} & \cdots & T_{ii} + A_{ii} & T_{i(i+1)} & \cdots & T_{in} & B_{i1} & \cdots & B_{i(i-1)} \\ C_{22} & \cdots & 0 & 0 & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ C_{i2} & \cdots & C_{ii} & 0 & \cdots & 0 & D_{i1} & \cdots & D_{i(i-1)} \end{bmatrix}$$

is a contraction and it is orthogonal to all previously determined block rows. This is possible following Lemma 7, condition (c), and the fact that

$$\begin{bmatrix} T_{12} & \cdots & T_{1i} & T_{1(i+1)} & \cdots & T_{1n} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ T_{(i-1)2}+A_{(i-1)2} & \cdots & T_{(i-1)i} & T_{(i-1)(i+1)} & \cdots & T_{(i-1)n} & B_{(i-1)1} & \cdots & B_{(i-1)(i-1)} \\ C_{22} & \cdots & 0 & 0 & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ C_{i2} & \cdots & C_{ii} & 0 & \cdots & 0 & D_{i1} & \cdots & D_{i(i-1)} \end{bmatrix}$$

is a co-isometry. Finally determine B_{ii} so that

$$\begin{bmatrix} T_{i2}+A_{i2} & \cdots & T_{ii}+A_{ii} & T_{i(i+1)} & \cdots & T_{in} & B_{i1} & \cdots & B_{ii} \end{bmatrix}$$

is a co-isometry. By Lemma 7, any B_{ii} chosen in such a way is nonsingular.
Step n:

Set $C_{n1} = 0$ and choose the rest of the $2n$-th row so that it is orthogonal to all the previously determined rows. This requires

$$\begin{bmatrix} C_{n2} & \cdots & C_{nn} & D_{n1} & \cdots & D_{n(n-1)} \end{bmatrix}^*$$

to be an isometry onto the kernel of

$$\begin{bmatrix} T_{12} & \cdots & T_{1n} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ T_{(n-1)2}+A_{(n-1)2} & \cdots & T_{(n-1)n} & B_{(n-1)1} & \cdots & B_{(n-1)(n-1)} \\ C_{22} & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{(n-1)2} & \cdots & 0 & D_{(n-1)1} & \cdots & 0 \end{bmatrix}.$$

Finally set

$$\begin{bmatrix} T_{n1}+A_{n1} & \cdots & T_{nn}+A_{nn} & B_{n1} & \cdots & B_{n(n-1)} \end{bmatrix} = 0$$

and $B_{nn} = I$.

The above construction guarantees that the matrix

$$\begin{bmatrix} T+A & B \\ C & D \end{bmatrix} \tag{19}$$

is unitary, B is invertible, and $D \in \mathcal{N}_s(\{\mathcal{Y}_i\},\{\mathcal{X}_i\})$. The invertibility of C follows from that of B and the fact that the matrix in (19) is unitary.

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