Proceedings of the 33rd
Conference on Decision and Control
Lake Buena Vista, FL - December 1994

Multirate Sampled-Data Systems: All $\mathcal{H}_\infty$ Suboptimal Controllers and the Minimum Entropy Controller*

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Abstract

For a general multirate SD (sampled-data) system, we characterize explicitly the set of all causal, stabilizing controllers that achieve a certain $\mathcal{H}_\infty$ norm bound; moreover, we give explicitly a particular controller that further minimizes an entropy function for the SD system. The characterization lays the groundwork for synthesizing multirate control systems with multiple/mixed control specifications.

1. Introduction

Multirate systems are abundant in industry [10]; there are several reasons for this: (1) In *multivariable* digital control systems, often it is unrealistic to sample all physical signals uniformly at one single rate. (2) For signals with different bandwidths, better trade-offs between performance and implementation cost can be obtained using A/D and D/A converters at different rates. (3) Multirate control systems can outperform single-rate systems; for example, gain margin improvement and simultaneous stabilization [18].

Since late 1950's [19], multirate systems have been studied extensively. Recent interests are reflected in the LQG/LQR designs [1, 22], the parametrization of all stabilizing controllers [21, 25], the work in [2, 14, 8], and among others.

In this paper we shall treat a general multirate setup. Two basic elements in SD systems are S_τ, the periodic sampler, and H_τ, the (zero-order) hold, both with period τ and synchronized at $t = 0$. The general multirate system is shown in Figure 1. Here, G_a is a continuous-time, LTI

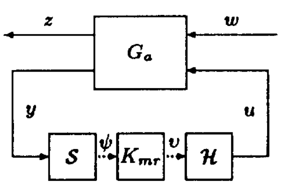

Figure 1: The general multirate setup

generalized plant; $\mathcal{S}$ and $\mathcal{H}$ are multirate sampling and hold operators and are defined as follows:

$$\mathcal{S} = \begin{bmatrix} S_{m_1 h} & & \\ & \ddots & \\ & & S_{m_p h} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H_{n_1 h} & & \\ & \ddots & \\ & & H_{n_q h} \end{bmatrix},$$

with m_i and n_j being integers and h the base period; and K_{mr} is a discrete-time multirate controller satisfying three properties: periodicity, causality, and finite-dimensionality [7]. (Such controllers are called admissible.)

Our goal here is two-fold: (1) to characterize all admissible K_{mr} which achieve a certain $\mathcal{H}_\infty$ norm bound; and (2) to find a particular K_{mr} which further minimizes an entropy function. This characterization, like the LTI result [11, 9], is essential in designing control systems with multiple specifications, possibly using convex optimization [6]. The minimum entropy control, also like its LTI counterpart [23, 17, 16], is such a multi-objective control problem in which an analytic solution exists.

The problem will be formulated in continuous time. Since the multirate system in Figure 1 is periodic, the well-known lifting technique is applicable to reduce the problem to an LTI one. However, the difficulty lies in that lifted controllers have a causality constraint [21, 25]. An effective framework based on nest operators is introduced recently in [7] to tackle this constraint. The results in this paper is based on further development in this framework.

The paper is organized as follows. The next section gives background material on nest operators and nest algebra. Section 3 introduces the concept of entropy for continuous-time periodic systems. Section 4 discusses converting our SD problem into an equivalent LTI problem with a causality constraint. The characterization of all $\mathcal{H}_\infty$ suboptimal controllers is given in Section 5 and the minimum entropy controller in Section 6. The details and proofs are contained in the full paper [24].

Now we introduce some notation. Given an operator K and two operator matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

the linear fractional transformation associated with P and K is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

*The research was supported by Hong Kong Research Grants Council and the Natural Sciences and Engineering Research Council of Canada.

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and the star product of P and Q is

$$P \star Q = \begin{bmatrix} P_{11} + P_{12}Q_{11}(I - P_{22}Q_{11})^{-1}P_{21} & P_{12}(I - Q_{11}P_{22})^{-1}Q_{12} \\ Q_{21}(I - P_{22}Q_{11})^{-1}P_{21} & Q_{21}(I - P_{22}Q_{11})^{-1}P_{22}Q_{12} + Q_{22} \end{bmatrix}.$$

Here, we assume that the domains and co-domains of the operators are compatible and the inverses exist. With these definitions, we have

$$\mathcal{F}(P, \mathcal{F}(Q, K)) = \mathcal{F}(P \star Q, K).$$

2. Preliminary

In this section we develop the necessary techniques to handle nest operators. The discussion in this section is intended to be general.

Let $\mathcal{X}$ be a vector space. A *nest* in $\mathcal{X}$, denoted $\{\mathcal{X}_i\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the nonincreasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{n-1} \supseteq \mathcal{X}_n = \{0\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be both Hilbert spaces. Denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of linear operators $\mathcal{X} \to \mathcal{Y}$ and abbreviate as $\mathcal{L}(\mathcal{X})$ if $\mathcal{X} = \mathcal{Y}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are equipped respectively with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ which have the same number of subspaces, say, $n+1$. A linear map $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be a *nest operator* if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_i, \quad i = 0, 1, \cdots, n, \tag{1}$$

and a *strict* nest operator if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_{i+1}, \quad i = 1, 2, \cdots, n. \tag{2}$$

Let $\Pi_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i$ and $\Pi_{\mathcal{Y}_i} : \mathcal{Y} \to \mathcal{Y}_i$ be orthogonal projections. Then the condition in (1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_i})T|_{\mathcal{X}_i} = 0, \quad i = 0, 1, \cdots, n,$$

and similarly for the condition in (2). Given nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$, the set of all nest operators is denoted $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and abbreviated $\mathcal{N}(\{\mathcal{X}_i\})$ if $\{\mathcal{X}_i\} = \{\mathcal{Y}_i\}$; similarly for strict nest operators and the symbol $\mathcal{N}_s$.

Lemma 1:

(a) If $T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, then $T_2T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$.

(b) If $T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}_s(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, or if $T_1 \in \mathcal{N}_s(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, then $T_2T_1 \in \mathcal{N}_s(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$.

(c) $\mathcal{N}(\{\mathcal{X}_i\})$ forms an algebra, called *nest algebra*.

(d) If $T \in \mathcal{N}_s(\{\mathcal{X}_i\})$, then $(I - T)^{-1}$ exists.

(e) If $T \in \mathcal{N}(\{\mathcal{X}_i\})$ and T is invertible, then $T^{-1} \in \mathcal{N}(\{\mathcal{X}_i\})$.

Lemma 2: Let $T \in \mathcal{L}(\mathcal{X})$.

(a) There exist a unitary operator U_1 on $\mathcal{X}$ and an operator R_1 in $\mathcal{N}(\{\mathcal{X}_i\})$ such that $T = U_1R_1$.

(b) There exist an operator R_2 in $\mathcal{N}(\{\mathcal{X}_i\})$ and a unitary operator U_2 on $\mathcal{X}$ such that $T = R_2U_2$.

If we decompose the spaces $\mathcal{X}$ and $\mathcal{Y}$ in the following way

$$\begin{aligned} \mathcal{X} &= (\mathcal{X}_0 \ominus \mathcal{X}_1) \oplus (\mathcal{X}_1 \ominus \mathcal{X}_2) \oplus \cdots \oplus (\mathcal{X}_{n-1} \ominus \mathcal{X}_n), \\ \mathcal{Y} &= (\mathcal{Y}_0 \ominus \mathcal{Y}_1) \oplus (\mathcal{Y}_1 \ominus \mathcal{Y}_2) \oplus \cdots \oplus (\mathcal{Y}_{n-1} \ominus \mathcal{Y}_n), \end{aligned}$$

then the associated matrix representation of $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}.$$

$T \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ means that this matrix representation is (block) lower triangular: $T_{ij} = 0$ if $i > j$. $T \in \mathcal{N}_s(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ means that this matrix representation is strictly (block) lower triangular: $T_{ij} = 0$ if $i \geq j$.

Now we restrict our discussion to finite-dimensional spaces. We shall give a result which strengthens Arveson's distance formula. For this, we need some more notation. With $\mathcal{X}$ and $\mathcal{Y}$ as before, introduce two more finite-dimensional Hilbert spaces $\mathcal{Z}$ and $\mathcal{W}$. A linear operator $T \in \mathcal{L}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{Z} \oplus \mathcal{W})$ is partitioned as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix},$$

with $T_{11} \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $T_{21} \in \mathcal{L}(\mathcal{X}, \mathcal{W})$, etc. For nests $\{\mathcal{X}_i\}$, $\{\mathcal{Y}_i\}$, $\{\mathcal{Z}_i\}$, $\{\mathcal{W}_i\}$ in $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{W}$ respectively, all with $n+1$ subspaces, the nests $\{\mathcal{X}_i \oplus \mathcal{Y}_i\}$ and $\{\mathcal{Z}_i \oplus \mathcal{W}_i\}$ are defined in the obvious way. Hence writing

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{N}(\{X_i \oplus \mathcal{Y}_i\}, \{\mathcal{Z}_i \oplus \mathcal{W}_i\})$$

means $T_{11} \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$, $T_{21} \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{W}_i\})$, etc.

Theorem 1: Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The following statements are equivalent:

(a) $\max_i \|(I - \Pi_{\mathcal{Y}_i})T|_{\mathcal{X}_i}\| < 1$.

(b) There exists $N \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ such that $\|T + N\| < 1$.

(c) There exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{N}(\{\mathcal{X}_i \oplus \mathcal{Y}_i\}, \{\mathcal{Y}_i \oplus \mathcal{X}_i\})$$

with P_{12} and P_{21} both invertible such that

$$\begin{bmatrix} T + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary.

The equivalence of (a) and (b) is actually given by Arveson's distance formula. The proof of the equivalence of (c) and (a) is rather involved and is left in [24]. The existence of P in (c) is proven constructively and hence P can be computed if condition (a) in Theorem 1 holds, which is easily verifiable. With Theorem 1, we can easily characterize all $N \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ such that $\|T + N\| < 1$. This is done in [24] but is not needed in the following.

3. Entropy for Periodic Systems

With reference to Figure 1, let l be the least common multiple of the integers, $\{m_1, \cdots, m_p, n_1, \cdots, n_q\}$ and $\sigma = lh$. If K_{mr} is admissible, the control path $\mathcal{H}K_{mr}\mathcal{S}$ is σ-periodic in continuous time [7] and hence the system mapping w to z in Figure 1, $\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})$, is also σ-periodic. This periodicity allows an entropy function to be defined for the SD system. For this, we need to lift the σ-periodic system $\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})$ as in [3, 5] to get an LTI discrete-time system, say, F_l. This lifted system has an operator-valued transfer function $\hat{F}_l(\lambda)$: For every λ in the region of convergence, $F_l(\lambda)$ is a bounded operator $\mathcal{K} \to \mathcal{K}$, where $\mathcal{K} = \mathcal{L}_2[0, \sigma)$. (Here we used the λ-transforms instead of the z-transforms, where $\lambda = z^{-1}$.)

For a general discussion, let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and $f = \{f(k) : k = 0, 1, 2, \cdots\}$ be a sequence of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. Then

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

defines an operator-valued function on some subset of $\mathcal{C}$, the complex plane. We say that $\hat{F}$ belongs to $\mathcal{H}_\infty(\mathcal{X}, \mathcal{Y})$ if $\hat{F}$ is analytic in $\mathcal{D}$, the open unit disk of $\mathcal{C}$, and

$$\sup_{\lambda \in \mathcal{D}} \|\hat{F}(\lambda)\| < \infty.$$

In this case, the left-hand side is defined to be the $\mathcal{H}_\infty$ norm of $\hat{F}$, denoted by $\|\hat{F}\|_\infty$, the operator $\hat{F}(e^{j\omega})$ is bounded for almost every $\omega \in [-\pi, \pi)$, and

$$\operatorname{ess} \sup_{\omega \in [-\pi, \pi)} \|\hat{F}(e^{j\omega})\| = \|\hat{F}\|_\infty.$$

Now let f be a sequence of Hilbert-Schmidt operators from $\mathcal{X}$ and $\mathcal{Y}$. The set of Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, $\|\cdot\|_{HS}$, is a Hilbert space [13]. Then

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

is a Hilbert-space vector-valued function on some subset of $\mathcal{C}$. We say that $\hat{F}$ belongs to $\mathcal{H}_2(\mathcal{X}, \mathcal{Y})$ if

$$\left(\sum_{k=0}^{\infty} \|f(k)\|_{HS}^2\right)^{1/2} < \infty.$$

In this case, the left-hand side is taken to be the $\mathcal{H}_2$ norm of $\hat{F}$, denoted by $\|\hat{F}\|_2$, the operator $\hat{F}(e^{j\omega})$ is Hilbert-Schmidt for almost every $\omega \in [-\pi, \pi)$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\hat{F}(e^{j\omega})\|_{HS}^2 \, d\omega = \|\hat{F}\|_2^2.$$

Assume $\hat{F} \in \mathcal{H}_\infty(\mathcal{X}, \mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X}, \mathcal{Y})$ and $\|\hat{F}\|_\infty < 1$. Define the entropy of $\hat{F}$ to be

$$\mathcal{I}(\hat{F}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det[I - \hat{F}^*(e^{j\omega})\hat{F}(e^{j\omega})] \, d\omega.$$

This entropy is well-defined: Since $\hat{F}(e^{j\omega})$ is a Hilbert-Schmidt operator at almost every $\omega \in [-\pi, \pi)$, its singular values form a square summable sequence $\{\sigma_k(e^{j\omega})\}$ with $\sigma_k(e^{j\omega}) \le \|\hat{F}\| < 1$; hence

$$\det[I - \hat{F}^*(e^{j\omega})\hat{F}(e^{j\omega})] = \prod_{k=1}^{\infty}[1 - \sigma_k^2(e^{j\omega})]$$

and it converges to some number in $(0, 1)$. This also shows that $\mathcal{I}(\hat{F})$ is nonnegative. The following lemma can be shown in a similar way as its continuous-time matrix-valued counterpart [23].

Lemma 3: Assume $\hat{F} \in \mathcal{H}_\infty(\mathcal{X}, \mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X}, \mathcal{Y})$ and $\|\hat{F}\|_\infty < 1$. Then

(a) $\|\hat{F}\|_2^2 \le \mathcal{I}(\hat{F})$;

(b) for $\hat{U} = \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix} \in \mathcal{H}_\infty(\mathcal{Y} \oplus \mathcal{X}, \mathcal{X} \oplus \mathcal{Y})$ with $\hat{U}^\sim \hat{U} = I$ and $\hat{U}_{21}^{-1} \in \mathcal{H}_\infty(\mathcal{Y}, \mathcal{Y})$,

$$\begin{aligned} \mathcal{I}[\mathcal{F}(\hat{U}, \hat{F})] &= \mathcal{I}(\hat{F}) + \mathcal{I}(\hat{U}_{11}) \\ &\quad + 2\ln\{\det[(I - \hat{U}_{22}(0)\hat{F}(0)]\}. \end{aligned}$$

Returning to the multirate SD system in Figure 1, the lifted map $w \mapsto z$ has an operator-valued transfer function $\hat{F}_l(\lambda)$. If the closed-loop SD system is internally stable [7], then under a mild condition, $\hat{F}_l \in \mathcal{H}_\infty(\mathcal{K}, \mathcal{K}) \cap \mathcal{H}_2(\mathcal{K}, \mathcal{K})$; moreover, $\|\hat{F}_l\|_\infty$ equals the $\mathcal{L}_2$-induced norm of the continuous-time system $\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})$; $\|\hat{F}_l\|_2$ equals the $\mathcal{H}_2$ norm of the same system, which is introduced in [4] in terms of impulse responses of SD systems; and the $\mathcal{H}_2$ norm is bounded above by the entropy $\mathcal{I}(\hat{F}_l)$, which will be referred to as the entropy for the system $\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})$.

4. Reduction to a Matrix-Valued Problem

Now we are ready to state our problems precisely: Given G_a, $\mathcal{S}$ and $\mathcal{H}$ in Figure 1, (1) characterize all admissible, stabilizing K_{mr} to achieve $\|\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})\|_\infty < 1$ (such controllers are called suboptimal ones); (2) find a particular K_{mr} from those in (1) to minimize the entropy $\mathcal{I}[\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})]$.

For motivation of the minimum entropy control, see [23]. If only the first problem is concerned, one can use the results in [27] or [7] to reduce the problem into an LTI discrete-time problem involving matrix-valued transfer functions only. But these results may not preserve the entropy. To tackle the two problems at the same time, a different reduction process is developed based on lifting, the details being given in [24]. This can be summarized below.

(A) Start with a state model of G_a:

$$\hat{G}_a(s) = \left[\begin{array}{c|cc} A_a & B_{a1} & B_{a2} \\ \hline C_{a1} & 0 & D_{a12} \\ C_{a2} & 0 & 0 \end{array}\right].$$

Lift G_a and perform certain mathematical operation to the lifted system. This will result in a discrete-time LTI system G with matrix-valued transfer function $\hat{G}$.

(B) Define K to be the lifted K_{mr} as in [7] and then K_{mr} is admissible iff K is finite-dimensional, LTI and causal with the direct feedthrough term being a nest operator, namely, $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$, where the nests $\{\mathcal{Y}_r\}$ and $\{\mathcal{U}_r\}$ are defined similarly as in [7].

This becomes our equivalent LTI system in the following sense. Let $\mathcal{F}(G, K)$ be the closed-loop system $\omega \mapsto \zeta$ in Figure 2; then internal stability of $\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})$ is

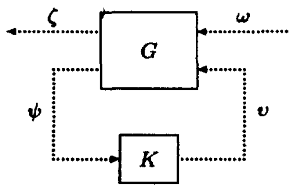

Figure 2: The equivalent LTI system

equivalent to that of $\mathcal{F}(G, K)$,

$$\|\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})\|_\infty < 1 \Leftrightarrow \|\mathcal{F}(\hat{G}, \hat{K})\|_\infty < 1,$$

and

$$\mathcal{I}[\mathcal{F}(G_a, \mathcal{H}K_{mr}\mathcal{S})] = \mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})] + \text{constant}.$$

Our equivalent problems for the setup in Figure 2 are now: (1) characterize all proper, stabilizing K with $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$ to achieve $\|\mathcal{F}(\hat{G}, \hat{K})\|_\infty < 1$; (2) find a particular K from those in (1) to minimize the entropy $\mathcal{I}[\mathcal{F}(\hat{G}, \hat{K})]$. These problems are standard except the causality condition on $\hat{K}(0)$, which is our main concern in the next two sections.

Note that a state space model of in G can be computed based on that of G_a, see [27, 7]; by some special consideration [24], G has the following causality structures:

$$\begin{aligned}
\hat{G}_{11}(0) &= \in \mathcal{N}_s(\{\mathcal{W}_r\}, \{\mathcal{Z}_r\}),\\
\hat{G}_{12}(0) &= \in \mathcal{N}(\{\mathcal{U}_r\}, \{\mathcal{Z}_r\}),\\
\hat{G}_{21}(0) &= \in \mathcal{N}_s(\{\mathcal{W}_r\}, \{\mathcal{Y}_r\}),\\
\hat{G}_{22}(0) &= \in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}),
\end{aligned}$$

where the nests $\{\mathcal{W}_r\}$ and $\{\mathcal{Z}_r\}$ are defined appropriately in [24]. Also note that under mild conditions, $\hat{G}$ is stabilizable and $\hat{G}_{12}$, $\hat{G}_{21}$ have no zeros on the unit circle.

5. All $\mathcal{H}_\infty$ Suboptimal Controllers

In this section, we tackle the constrained $\mathcal{H}_\infty$ problem: Find all $\mathcal{H}_\infty$ suboptimal controllers in Figure 2 which are proper, stabilizing and satisfy the causality condition. We remark here that one such controller can be computed by results in [28, 7].

First we characterize all stabilizing controllers. Instead of using the traditional Youla parameterization, we use a result in [20]: There exists

$$\hat{F} = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ \hat{F}_{21} & \hat{F}_{22} \end{bmatrix} \in \mathcal{RH}_\infty$$

with

$$\begin{aligned}
\hat{F}_{11}(0) &\in \mathcal{N}_s(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\}),\\
\hat{F}_{12}(0) &\in \mathcal{N}(\{\mathcal{U}_r\}, \{\mathcal{U}_r\}),\\
\hat{F}_{21}(0) &\in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{Y}_r\}),\\
\hat{F}_{22}(0) &\in \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\})
\end{aligned}$$

and $\hat{F}_{12}(0)$, $\hat{F}_{21}(0)$ being invertible such that all stabilizing controllers $\hat{K}$ satisfying $\hat{K}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$ are characterized by

$$\{\hat{K} = \mathcal{F}(\hat{F}, \hat{Q}) : \ \hat{Q} \in \mathcal{RH}_\infty, \ \hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})\}.$$

Note that the causality constraint is passed to $\hat{Q}$. Then the closed-loop system becomes

$$\mathcal{F}(\hat{G}, \hat{K}) = \mathcal{F}(G, \mathcal{F}(\hat{F}, \hat{Q})) = \mathcal{F}(\hat{T}, \hat{Q}),$$

where $\hat{T}$ is the star product $\hat{G} \star \hat{F}$:

$$\hat{T} = \begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} \\ \hat{T}_{21} & 0 \end{bmatrix} \in \mathcal{RH}_\infty.$$

Clearly

$$\hat{T}(0) \in \mathcal{N}(\{\mathcal{W}_r \oplus \mathcal{U}_r\}, \{\mathcal{Z}_r \oplus \mathcal{Y}_r\}).$$

Now let us put aside the causality constraint on $\hat{Q}$ temperarily and characterize all $\hat{Q} \in \mathcal{RH}_\infty$ such that $\|\mathcal{F}(\hat{T}, \hat{Q})\|_\infty < 1$. This is a standard $\mathcal{H}_\infty$ model-matching problem. If this unconstrained problem is solvable, its solution is given by [12, 15]

$$\{\hat{Q} = \mathcal{F}(\hat{L}, \hat{\Phi}) : \ \hat{\Phi} \in \mathcal{RH}_\infty, \ \|\hat{\Phi}\|_\infty < 1\},$$

where

$$\hat{L} = \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{bmatrix} \in \mathcal{RH}_\infty$$

with $\hat{L}_{12}$, $\hat{L}_{21}$ invertible over $\mathcal{RH}_\infty$ and $\|\hat{L}_{22}\|_\infty < 1$.

Define the Julian matrix

$$V = \begin{bmatrix} \hat{L}_{22}'(0) & \left[I - \hat{L}_{22}'(0)\hat{L}_{22}(0)\right]^{1/2} \\ \left[I - \hat{L}_{22}(0)\hat{L}_{22}'(0)\right]^{1/2} & -\hat{L}_{22}(0) \end{bmatrix}.$$

and perform the factorizations by Lemma 2

$$\begin{aligned}
\hat{L}_{12}(0)[I - V_{11}\hat{L}_{22}(0)]^{-1}V_{12} &= R_1U_1,\\
V_{21}[I - \hat{L}_{22}(0)V_{11}]^{-1}\hat{L}_{21}(0) &= U_2R_2
\end{aligned}$$

so that U_1 and U_2 are unitary matrices and $R_1 \in \mathcal{N}(\{\mathcal{U}_r\})$, $R_2 \in \mathcal{N}(\{\mathcal{Y}_r\})$. Let

$$U = \begin{bmatrix} 0 & U_1' \\ U_2' & 0 \end{bmatrix}.$$

Since V and U are unitary, so is $V \star U$. Hence the set of $\hat{Q}$ solving the unconstrained problem can be also characterized by

$$\{\hat{Q} = \mathcal{F}(\hat{M}, \hat{\Phi}) : \quad \hat{\Phi} \in \mathcal{RH}_\infty, \quad \|\hat{\Phi}\|_\infty < 1\},$$

where $M = L \star V \star U$. It is easy to check that $\hat{M}_{12}(0)$ and $\hat{M}_{21}(0)$ are invertible and

$$\begin{aligned} \hat{M}_{12}(0) &\in \mathcal{N}(\{\mathcal{U}_r\}), \\ \hat{M}_{21}(0) &\in \mathcal{N}(\{\mathcal{Y}_r\}), \\ \hat{M}_{22}(0) &= 0. \end{aligned}$$

Now let us return to the constrained $\mathcal{H}_\infty$ problem.

Theorem 2: The constrained $\mathcal{H}_\infty$ problem is solvable iff the corresponding unconstrained problem is solvable and

$$\max_r \|(I - \Pi_{\mathcal{U}_r})\hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1}|_{\mathcal{Y}_r}\| < 1. \quad (3)$$

Proof: The direct feedthrough term of $\hat{Q} = \mathcal{F}(\hat{M}, \hat{\Phi})$ gives

$$\hat{Q}(0) = \hat{M}_{11}(0) + \hat{M}_{12}(0)\hat{\Phi}(0)\hat{M}_{21}(0).$$

Pre- and post-multiply this by $\hat{M}_{12}(0)^{-1}$ and $\hat{M}_{21}(0)^{-1}$ respectively to get

$$\begin{aligned} &\hat{M}_{12}(0)^{-1}\hat{Q}(0)\hat{M}_{21}(0)^{-1} \\ &\quad = \hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \hat{\Phi}(0). \end{aligned}$$

It follows from Theorem 1 that in order to have $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$ and $\|\hat{\Phi}(0)\| < 1$, we must have (3). Conversely, if (3) is true, then there exists a constant matrix Φ with $\|\Phi\| < 1$ such that

$$\hat{M}_{12}(0)^{-1}\hat{M}_{11}(0)\hat{M}_{21}(0)^{-1} + \Phi \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\}).$$

Hence $\hat{Q} = \mathcal{F}(\hat{M}, \Phi)$ achieves $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$.

□

For solvability of the constrained $\mathcal{H}_\infty$ problem, by Theorem 2, we need to check the solvability of the corresponding unconstrained problem and the condition (3). Solvability of the unconstrained problem is standard and is given in, e.g., [12, 15]; condition (3) involves only matrix norm computations.

If the conditions in Theorem 2 are satisfied, then there exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\}, \{\mathcal{U}_r \oplus \mathcal{Y}_r\})$$

with P_{12} and P_{21} invertible such that

$$W = \begin{bmatrix} -\hat{M}_{12}^{-1}(0)\hat{M}_{11}(0)\hat{M}_{21}^{-1}(0) + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary. Define $\hat{N} = \hat{F} \star \hat{M} \star W$. It is easy to check that $\hat{N}(0) \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\}, \{\mathcal{U}_r \oplus \mathcal{Y}_r\})$ and $\hat{N}_{12}(0)$, $\hat{N}_{21}(0)$ are invertible.

Theorem 3: Assume solvability of the constrained $\mathcal{H}_\infty$ problem. Then the set of all controllers solving the problem is given by

$$\{\hat{K} = \mathcal{F}(\hat{N}, \hat{\Phi}) : \quad \hat{\Phi} \in \mathcal{RH}_\infty, \|\hat{\Phi}\|_\infty < 1, \quad \hat{\Phi}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})\}.$$

Proof: Since $\hat{N}(0) \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\}, \{\mathcal{U}_r \oplus \mathcal{Y}_r\})$ and $\hat{N}_{12}(0)$, $\hat{N}_{21}(0)$ are invertible, it follows that $\mathcal{F}(\hat{N}, \hat{\Phi}) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$ iff $\hat{\Phi} \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$. Then the result follows immediately. □

This theorem lays the groundwork for design of multirate control systems involving multiple/mixed design specifications, with one of them being a certain $\mathcal{H}_\infty$ norm bound.

6. Minimum Entropy Control

Under certain solvability conditions, we have shown in the preceding section that all $\mathcal{H}_\infty$ suboptimal controllers are characterized by

$$\{\hat{K} = \mathcal{F}(\hat{N}, \hat{\Phi}) : \quad \hat{\Phi} \in \mathcal{RH}_\infty, \quad \|\hat{\Phi}\|_\infty < 1, \quad \hat{\Phi}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})\},$$

where

$$\hat{N}(0) \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\}, \{\mathcal{U}_r \oplus \mathcal{Y}_r\}).$$

Hence all $\mathcal{H}_\infty$ suboptimal closed-loop transfer functions are

$$\{\mathcal{F}(\hat{G}, \hat{K}) = \mathcal{F}(\hat{J}, \hat{\Phi}) : \quad \hat{\Phi} \in \mathcal{RH}_\infty, \|\hat{\Phi}\|_\infty < 1, \quad \hat{\Phi}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})\},$$

where

$$\hat{J} = \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \end{bmatrix} = \hat{G} \star \hat{N} \in \mathcal{N}(\{\mathcal{Y}_r \oplus \mathcal{U}_r\}, \{\mathcal{U}_r \oplus \mathcal{Y}_r\})$$

and

$$\hat{J}_{22}(0) = \mathcal{N}_s(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}).$$

Because of internal stability and the norm requirement, $\hat{J}$ belongs to $\mathcal{RH}_\infty$ and is contractive.

The purpose of this section is to choose one particular $\mathcal{H}_\infty$ suboptimal controller which minimizes the entropy of the closed-loop transfer function.

Since $\hat{J}$ is contractive, we can find $\mathcal{RH}_\infty$ functions $\hat{J}_{13}$, $\hat{J}_{23}$, $\hat{J}_{31}$, $\hat{J}_{32}$, $\hat{J}_{33}$ such that

$$\hat{J}_{aug} = \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} & \hat{J}_{13} \\ \hat{J}_{21} & \hat{J}_{22} & \hat{J}_{23} \\ \hat{J}_{31} & \hat{J}_{32} & \hat{J}_{33} \end{bmatrix}$$

is para-unitary. Then another way to characterize the $\mathcal{H}_\infty$ suboptimal closed-loop transfer functions is

$$\{\mathcal{F}(\hat{J}_{aug}, \begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix}) : \quad \hat{\Phi} \in \mathcal{RH}_\infty, \|\hat{\Phi}\|_\infty < 1, \quad \hat{\Phi}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})\}.$$

By Lemma 3,

$$\begin{aligned}
&\mathcal{I}[\mathcal{F}(\hat{G},\hat{K})] \\
&= \mathcal{I}(\begin{bmatrix} \hat{\Phi} & 0 \\ 0 & 0 \end{bmatrix}) + \mathcal{I}(\hat{J}_{11}) \\
&\quad +2\ln\det(I - \begin{bmatrix} \hat{J}_{22}(0) & \hat{J}_{23}(0) \\ \hat{J}_{32}(0) & \hat{J}_{33}(0) \end{bmatrix} \begin{bmatrix} \hat{\Phi}(0) & 0 \\ 0 & 0 \end{bmatrix}) \\
&= \mathcal{I}(\hat{\Phi}) + \mathcal{I}(\hat{J}_{11}) + 2\ln\det(I - \hat{J}_{22}(0)\hat{\Phi}(0)).
\end{aligned}$$

The last equality is due to $\hat{J}_{22}(0)\hat{\Phi}(0) \in \mathcal{N}_s(\{\mathcal{Y}_r\})$. Therefore, the minimum of $\mathcal{I}[\mathcal{F}(\hat{G},\hat{K})]$ is achieved at $\hat{\Phi} = 0$. The following theorem is thus obtained.

Theorem 4: The minimum entropy controller is given by $\hat{K} = \hat{N}_{11}$.

7. References

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