3 Multirate Systems and Related Interpolation Problems*

Li Chai and Li Qiu

Department of Electrical and Electronic Engineering
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong
Email: { eelchai, eeqiu} @ee.ust.hk

Abstract. This paper concerns general multirate systems and the constrained two-sided Nevanlinna-Pick interpolation problem. A general multirate system can be converted to an equivalent LTI system with a causality constraint which requires the feedthrough term of its transfer function to belong to a set of nest operators. Motivated by this fact, we propose a multirate version of two-sided Nevanlinna-Pick interpolation problem and give a necessary and sufficient solvability condition based on the matrix positive completion. This constrained interpolation problem is of interest mathematically and has potential applications in control, signal processing and circuit theory.

3.1 Introduction

Recently, much attention has been paid on multirate systems due to its wide applications in signal processing, communication, control and numerical mathematics. Multirate signal processing is now one of the most vibrant areas of research in signal processing, see recent books [24,25] and references therein. The driving force for studying multirate systems in signal processing comes from the need of sampling rate conversion, subband coding, and their ability to generate wavelets. In communication community, multirate sampling is used for multi-channel transmultiplexers [19] and blind system identification and equalization [18]. In control community, there has recently considerable research devoted to multirate controller design, e.g., stabilizing controller design and various types of optimal control [6,23].

The standard technique for treating multirate systems is called lifting in control and blocking in signal processing. It is well-known that a multirate system can be converted to an equivalent single rate LTI system. This LTI system, however, is not arbitrary, but satisfies a causality constraint which is represented by the language of nests and nest operators in a systematic way [4,23]. That is, the feedthrough term of the equivalent LTI system belongs to a set of nest operators. In this paper, we will study a multirate version of a general analytic interpolation problem: constrained two-sided tangential Nevanlinna-Pick (N-P) interpolation problem.

* This work is supported by the Hong Kong Research Grants Council

The theory of interpolation with analytic functions has a very rich history in mathematics [1,9]. Moreover, it is used in a variety of engineering fields such as control, circuit theory and digital filter design [7,14,15]. The N-P interpolation was first brought into system theory by Youla and Saito, who gave a circuit theoretical proof of the Pick criterion [28]. In the early stage of the development of $\mathcal{H}_\infty$ control theory, the analytic function interpolation theory played a fundamental role [10,26]. A detailed review of this connection can be found in [14,16]. Recently, some new methods in high resolution spectral estimation have been presented based on the N-P interpolation with degree constraint [17,11,2]. The analytic function interpolation problems are also used extensively in robust model validation and identification[3,13,20].

In this paper, we study the constrained two-sided tangential N-P interpolation problem, which requires the value of the interpolating functions at the origin to belong to a prescribed set of nest operators. This constrained interpolation problem plays the same role to multirate systems as the unconstrained counterpart do to single rate systems. The necessary and sufficient solvability conditions are given based on the matrix positive completion. The interpolation and distance problems involving analytic function with such structural constraint were first discussed in [14], but the general problem considered in this paper was not given there. This paper is organized as follows. In section II, we show how to convert a general multirate system to its equivalent LTI system with a causality constraint described by the language of nests and nest operator. In section III, we present some preliminary results on analytic interpolation problems and propose the constrained two-sided tangential N-P interpolation problem, which is a multirate version of the standard interpolation. The necessary and sufficient solvability conditions are then presented in Section IV. Finally, the paper is concluded in Section V.

3.2 General Multirate Systems

Consider a general MIMO multirate system shown in Fig. 3.1. Here u_i, $i = 1, 2, \dots, p$, are input signals whose sampling intervals are $m_i h$ respectively, and y_j, $j = 1, 2, \dots, q$, are output signals whose sampling intervals are $n_j h$ respectively, where h is a real number called base sampling interval and m_i, n_j are natural numbers (positive integers). Such systems can result from discretizing continuous time systems using samplers of different rates or they can be found in their own right. Assume that all signals in the system are synchronized at time 0, i.e., the time 0 instances of all signals occur at the same time. In this paper, we will focus on those multirate systems that satisfy certain causal, linear, shift invariance properties which are to be defined below.

Since we need to deal with signals with different rates, it is more convenient and clearer to associate each signal explicitly with its sampling interval.

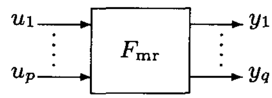

Fig. 3.1. A general multirate system.

Let $\ell^r(\tau)$ denote the space of $\mathbb{R}^r$ valued sequences:

$$\ell^r(\tau) = \{\{\dots, x(-\tau), |x(0), x(\tau), x(2\tau), \dots\} : x(k\tau) \in \mathbb{R}^r\}.$$

The system in Fig. 3.1 is a map from $\oplus_{i=1}^{p} \ell(m_i h)$ to $\oplus_{j=1}^{q} \ell(n_j h)$. It is said to be linear if this map is a linear map.

Let $l \in \mathbb{N}$ be a multiple of m_i and n_j, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$. Let $\bar{m}_i = l/m_i$ and $\bar{n}_j = l/n_j$. Denote the sets $\{m_i\}$ and $\{n_j\}$ by M and N respectively and the sets $\{\bar{m}_i\}$ and $\{\bar{n}_j\}$ by $\bar{M}$ and $\bar{N}$ respectively. Let $S : \ell^r(\tau) \longrightarrow \ell^r(\tau)$ be the forward shift operator, i.e.,

$$S\{\dots, x(-\tau), |x(0), x(\tau), \dots\} = \{\dots, x(-2\tau), |x(-\tau), x(0), x(\tau), \dots\}.$$

Define

$$S_{\bar{M}} = \mathrm{diag}\{S^{\bar{m}_1}, \dots, S^{\bar{m}_p}\}, \qquad S_{\bar{N}} = \mathrm{diag}\{S^{\bar{n}_1}, \dots S^{\bar{n}_q}\}.$$

Then the multirate system in Fig. 3.1 is said to be $(\bar{M}, \bar{N})$-shift invariant or lh periodic in real time if $F_{mr} S_{\bar{M}} = S_{\bar{N}} F_{mr}$. Now let $P_t : \ell^r(\tau) \longrightarrow \ell^r(\tau)$ be the truncation operator, i.e.,

$$\begin{aligned} &P_t\{\dots, x((k-1)\tau), x(k\tau), x((k+1)\tau), \dots\} \\ &= \{\dots, x((k-1)\tau), x(k\tau), 0, \dots\} \end{aligned}$$

if $k\tau \le t < (k+1)\tau$. Extend this definition to spaces $\oplus_{i=1}^{p} \ell(m_i h)$ and $\oplus_{j=1}^{q} \ell(n_j h)$ in an obvious way. Then the multirate system is said to be causal if

$$P_t u = P_t v \Rightarrow P_t F_{mr} u = P_t F_{mr} v$$

for all $t \in \mathbb{R}$. In this paper, we will concentrate on causal linear $(\bar{M}, \bar{N})$-shift invariant systems. Such general multirate system covers many familiar classes of systems as special cases. If m_i, n_j, l are all the same, then this is an LTI single rate system. If m_i, n_j are all the same but l is a multiple of them, then it is a single rate l-periodic system [22,12]. If $p = q = 1$, this becomes the SISO dual rate system studied in [5]. If m_i are the same and n_j are the same, then this becomes the MIMO dual rate system studied in [21]. For systems resulted from discretizing LTI continuous time systems using multirate sample and hold schemes in [4,23], l turns out to be the least common multiple of m_i and n_j. The study of multirate systems in such a generality as indicated above, however, has never been done before.

A standard way for the analysis of multirate systems is to use lifting or blocking. Define a lifting operator $L_r : \ell(\tau) \to \ell^r(r\tau)$ by

$$L_r \{\dots | x(0), x(\tau), \dots\} \to \left\{ \dots | \begin{bmatrix} x(0) \\ \vdots \\ x((r-1)\tau) \end{bmatrix}, \begin{bmatrix} x(r\tau) \\ \vdots \\ x((2r-1)\tau) \end{bmatrix}, \dots \right\}$$

and let

$$L_{\bar{M}} = \operatorname{diag}\left\{ L_{\bar{m}_1}, \dots, L_{\bar{m}_p} \right\}, \qquad L_{\bar{N}} = \operatorname{diag}\left\{ L_{\bar{n}_1}, \dots, L_{\bar{n}_q} \right\}.$$

Then the lifted system $F = L_{\bar{N}} F_{mr} L_{\bar{M}}^{-1}$ is an LTI system in the sense that $FS = SF$. Hence it has transfer function $\hat{F}$ in λ-transform. However, F is not an arbitrary LTI system, instead its direct feedthrough term $\hat{F}(0)$ is subject to a constraint resulted from the causality of F_{mr}. This constraint is best described using the language of nests and nest operators [21,23].

Let $\mathcal{X}$ be a finite dimensional vector space. A nest in $\mathcal{X}$, denoted $\{\mathcal{X}_k\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the non-increasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \dots \supseteq \mathcal{X}_{l-1} \supseteq \mathcal{X}_l = \{0\}.$$

Let $\mathcal{U}, \mathcal{Y}$ be finite dimensional vector spaces. Denote by $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ the set of linear operators $\mathcal{U} \to \mathcal{Y}$. Assume that $\mathcal{U}$ and $\mathcal{Y}$ are equipped respectively with nests $\{\mathcal{U}_k\}$ and $\{\mathcal{Y}_k\}$ which have the same number of subspaces, say, $l+1$ as above. A linear map $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is said to be a nest operator if

$$T\mathcal{U}_k \subseteq \mathcal{Y}_k, \; k = 0, 1, \dots, l. \tag{3.1}$$

Let $\Pi_{\mathcal{U}_k} : \mathcal{U} \to \mathcal{U}_k$ and $\Pi_{\mathcal{Y}_k} : \mathcal{Y} \to \mathcal{Y}_k$ be orthogonal projections. Then (3.1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_k}) T \Pi_{\mathcal{U}_k} = 0, \; k = 0, \dots, l-1. \tag{3.2}$$

The set of all nest operators (with given nests) is denoted $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. If we decompose the spaces $\mathcal{U}$ and $\mathcal{Y}$ in the following way:

$$\mathcal{U} = (\mathcal{U}_0 \ominus \mathcal{U}_1) \oplus (\mathcal{U}_1 \ominus \mathcal{U}_2) \oplus \dots \oplus (\mathcal{U}_{l-1} \ominus \mathcal{U}_l) \tag{3.3}$$

$$\mathcal{Y} = (\mathcal{Y}_0 \ominus \mathcal{Y}_1) \oplus (\mathcal{Y}_1 \ominus \mathcal{Y}_2) \oplus \dots \oplus (\mathcal{Y}_{l-1} \ominus \mathcal{Y}_l) \tag{3.4}$$

then a nest operator $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ has the following block lower triangular form

$$T = \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ T_{21} & T_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ T_{l1} & T_{l2} & \cdots & T_{ll} \end{bmatrix}. \tag{3.5}$$

Write $\underline{u} = L_{\bar{M}} u, \qquad \underline{y} = L_{\bar{N}} y$. Then

$$\underline{u}(0) = \left[u_1(0) \cdots u_1((\bar{m}_1 - 1) m_1 h) \cdots u_p(0) \cdots u_p((\bar{m}_p - 1) m_p h)\right]^T,$$
$$\underline{y}(0) = \left[y_1(0) \cdots y_1((\bar{n}_1 - 1) n_1 h) \cdots y_q(0) \cdots y_q((\bar{n}_q - 1) n_q h)\right]^T.$$

Define for $k = 0, 1, \ldots, l$,

$$\mathcal{U}_k = \{\underline{u}(0) : u_i(r m_i h) = 0 \text{ if } r m_i h < kh\}$$
$$\mathcal{Y}_k = \{\underline{y}(0) : y_j(r n_j h) = 0 \text{ if } r n_j h < kh\}.$$

Then the lifted plant F satisfies

$$\hat{F}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}). \tag{3.6}$$

Now we see that each multirate system has an equivalent single rate LTI system with a causality constraint which is characterized by a nest operator constraint as in (3.6) on its transfer function.

We end this section by showing an example. Consider the system shown in Fig. 3.1. Let $p = q = 2$, $m_1 = 2$, $m_2 = 6$, $n_1 = 4$, $n_2 = 3$ and $l = 12$. Then $\bar{m}_1 = 6$, $\bar{m}_2 = 2$, $\bar{n}_1 = 3$ and $\bar{n}_2 = 4$. Let $\underline{u}$ and $\underline{y}$ be the lifted signals of u and y respectively. Then we have

$$\underline{u}(0) = \begin{bmatrix} u_1(0) \; u_1(2h) \; u_1(4h) \; u_1(6h) \; u_1(8h) \; u_1(10h) \; u_2(0) \; u_2(6h) \end{bmatrix}^T$$
$$\underline{y}(0) = \begin{bmatrix} y_1(0) \; y_1(4h) \; y_1(8h) \; y_2(0) \; y_2(3h) \; y_2(6h) \; y_2(9h) \end{bmatrix}^T.$$

Denote the ith column of 8×8 identity matrix by e_i. Then the nests $\{\mathcal{U}_k\}$ are as follows

$$\begin{aligned}
\mathcal{U}_{12} &= \mathcal{U}_{11} = \{0\} \\
\mathcal{U}_{10} &= \mathcal{U}_9 = \operatorname{span}\{e_6\} \\
\mathcal{U}_8 &= \mathcal{U}_7 = \operatorname{span}\{e_5, e_6\} \\
\mathcal{U}_6 &= \mathcal{U}_5 = \operatorname{span}\{e_4, e_5, e_6, e_8\}, \\
\mathcal{U}_4 &= \mathcal{U}_3 = \operatorname{span}\{e_3, e_4, e_5, e_6, e_8\} \\
\mathcal{U}_2 &= \mathcal{U}_1 = \operatorname{span}\{e_2, e_3, e_4, e_5, e_6, e_8\} \\
\mathcal{U}_0 &= \mathbb{R}^8.
\end{aligned}$$

Similarly, denote jth column of 7×7 identity matrix by d_j, we get $\{\mathcal{Y}_k\}$ as follows

$$\begin{aligned}
&\mathcal{Y}_{12} = \mathcal{Y}_{11} = \mathcal{Y}_{10} = \{0\} \\
&\mathcal{Y}_9 = \operatorname{span}\{d_7\} \\
&\mathcal{Y}_8 = \mathcal{Y}_7 = \operatorname{span}\{d_3, d_7\} \\
&\mathcal{Y}_6 = \mathcal{Y}_5 = \operatorname{span}\{d_3, d_6, d_7\}, \\
&\mathcal{Y}_4 = \operatorname{span}\{d_2, d_3, d_6, d_7\} \\
&\mathcal{Y}_3 = \mathcal{Y}_2 = \mathcal{Y}_1 = \operatorname{span}\{d_2, d_3, d_5, d_6, d_7\} \\
&\mathcal{Y}_0 = \mathbb{R}^7.
\end{aligned}$$

Then $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ consists of matrices of the form

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & * & * & 0 & 0 & 0 & * & 0 \\ * & * & * & * & * & 0 & * & * \\ * & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & * & 0 & 0 & 0 & 0 & * & 0 \\ * & * & * & * & 0 & 0 & * & * \\ * & * & * & * & * & 0 & * & * \end{bmatrix}$$

where "*" represents an arbitrary number. Note that such matrices are not block lower triangular, but can be turned into block lower triangular matrices by permutations of rows and columns.

3.3 Constrained Analytic Interpolation Problems

In this section, we will formulate the two-sided tangential N-P interpolation problem with a nest operator constraint, which can be viewed as a multirate version of the standard interpolation problem.

Let $\mathcal{X}_i$ and $\mathcal{Z}_j$ be finite dimensional Hilbert spaces for $i = 1, \ldots, M$, and $j = 1, \ldots, N$. Also let $\mathcal{U}$ and $\mathcal{Y}$ be finite dimensional Hilbert spaces with nests $\{\mathcal{U}_k\}$ and $\{\mathcal{Y}_k\}$ respectively. Consider the linear bounded operators

$$\begin{aligned} &U_i : \mathcal{X}_i \to \mathcal{U}, \quad Y_i : \mathcal{X}_i \to \mathcal{Y}, i = 1, \cdots, M \\ &V_j : \mathcal{U} \to \mathcal{Z}_j, \quad W_j : \mathcal{Y} \to \mathcal{Z}_j, j = 1, \cdots, N. \end{aligned}$$

Given two sets of complex numbers $\{\alpha_i\}$ and $\{\beta_j\}$ on the open unit disc $\mathbb{D}$, where $\alpha_i \neq \beta_j$ for every i and j. Denote $\mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ the Hardy class of all uniformly bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. The constrained two-sided tangential N-P interpolation problem for the data λ_i, $U_{i,j}$, and $Y_{i,j}$ is to find (if possible) a function $\hat{G}$ in $\mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ which satisfies

(i) the interpolation conditions

$$\hat{G}(\alpha_i) U_i = Y_i \tag{3.7}$$
$$W_j \hat{G}(\beta_j) = V_j \tag{3.8}$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$,

(ii) $\|\hat{G}\|_\infty \leq 1$, and

(iii) the nest operators constraint $\hat{G}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$.

Requiring only conditions (i) and (ii) accounts to the standard two-sided tangential N-P interpolation, the solvability condition of which is well-known [1,9]. We present it here as a lemma for completeness.

Lemma 1. *Given the data $\alpha_i, U_i, Y_i, \beta_j, W_j$ and V_j for $i = 1, \dots, M$ and $j = 1, \dots, N$, where $\alpha_i \neq \beta_j$ for every i and j. The standard two-sided tangential N-P interpolation problem has a solution if and only if*

$$Q = \begin{bmatrix} Q_{11} & Q_{21}^* \\ Q_{21} & Q_{22} \end{bmatrix} \geq 0, \tag{3.9}$$

where

$$Q_{11} = \left[\frac{U_i^* U_m - Y_i^* Y_m}{1 - \alpha_i^* \alpha_m} \right]_{i,m=1}^{M} \tag{3.10}$$

$$Q_{21} = \left[\frac{V_j U_i - W_j Y_i}{\beta_j - \alpha_i} \right]_{\substack{j=1,\dots,N \\ i=1,\dots,M}} \tag{3.11}$$

$$Q_{22} = \left[\frac{W_j W_n^* - V_j V_n^*}{1 - \beta_j \beta_n^*} \right]_{j,n=1}^{N}. \tag{3.12}$$

The right and left tangential N-P interpolation problems can be considered as a two-sided one with only one of the conditions (3.7) and (3.8) respectively. In such cases, Q_{11} and Q_{22} in Lemma 1 are the so-called Pick matrices.

It is natural in the study of multirate systems to require that the interpolating function satisfies the condition (iii) since a multirate system can be converted to an equivalent LTI system with a constraint that its feedthrough term belongs to $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. We end this section by introducing some notations. For the constrained two-sided tangential N-P interpolation data, denote

$$\begin{aligned}
\alpha &= \mathrm{diag}(\alpha_1, \dots, \alpha_M), \\
\beta &= \mathrm{diag}(\beta_1, \dots, \beta_N), \\
U &= \begin{bmatrix} U_1 & U_2 & \cdots & U_M \end{bmatrix}, \\
Y &= \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_M \end{bmatrix}, \\
W &= \begin{bmatrix} W_1 \\ \vdots \\ W_N \end{bmatrix}, \quad V = \begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix}.
\end{aligned}$$

3.4 Solvability Conditions

The purpose of this section is to obtain the necessary and sufficient solvability condition of the constrained two-sided tangential N-P interpolation problem. First, we need a result on matrix positive completion. The matrix positive completion problem is as follows [8]: Given $B_{ij}, |j - i| \leq q$, satisfying $B_{ij} = B_{ji}^*$, find the remaining matrices $B_{ij}, |j - i| > q$, such that the block matrix $B = [B_{ij}]_{i,j=1}^{n}$ is positive definite. The matrix positive completion problem was first proposed by Dym and Gohberg [8], who gave the following result:

Lemma 2. *The matrix positive completion problem has a solution if and only if*

$$\begin{bmatrix} B_{ii} & \cdots & B_{i,i+q} \\ \vdots & & \vdots \\ B_{i+q,i} & \cdots & B_{i+q,i+q} \end{bmatrix} \geq 0, \qquad i = 1, \dots, n-q. \tag{3.13}$$

Reference [27] gave a detailed discussion of such problem and presented an explicit description of the set of all solutions via a linear fractional map of which the coefficients are given in terms of the original data. However, Lemma 2 is enough for us. We are now in a position to state the main result of this section.

Theorem 1. *Given the data $\alpha_i, U_i, Y_i, \beta_j, W_j$ and V_j for $i = 1, \dots, M$ and $j = 1, \dots, N$, where $\alpha_i \neq \beta_j$ for every i and j. In the case when $\beta_j \neq 0$ for all j, the constrained two-sided tangential N-P interpolation problem has a solution if and only if*

$$\begin{aligned} &\begin{bmatrix} Q_{11} & Q_{21}^* \\ Q_{21} & Q_{22} \end{bmatrix} + \begin{bmatrix} Y^* \\ \beta^{-1}W \end{bmatrix} \Pi_{\mathcal{Y}_k} \begin{bmatrix} Y & (\beta^{-1}W)^* \end{bmatrix} \\ &- \begin{bmatrix} U^* \\ \beta^{-1}V \end{bmatrix} \Pi_{\mathcal{U}_k} \begin{bmatrix} U & (\beta^{-1}V)^* \end{bmatrix} \geq 0 \end{aligned} \tag{3.14}$$

for all $k = 1, \dots, l$. In the case when $\alpha_i \neq 0$ for all i, it has a solution if and only if

$$\begin{aligned} &\begin{bmatrix} Q_{11} & Q_{21}^* \\ Q_{21} & Q_{22} \end{bmatrix} + \begin{bmatrix} \alpha^{*-1}U^* \\ V \end{bmatrix} \Pi_{\mathcal{U}_k^\perp} \begin{bmatrix} U\alpha^{-1} & V^* \end{bmatrix} \\ &- \begin{bmatrix} \alpha^{*-1}Y^* \\ W \end{bmatrix} \Pi_{\mathcal{Y}_k^\perp} \begin{bmatrix} Y\alpha^{-1} & W^* \end{bmatrix} \geq 0 \end{aligned} \tag{3.15}$$

for all $k = 0, \dots, l-1$.

Proof: We first give the proof for the case when $\beta_j \neq 0$ for all j. The nest operator constraint can be viewed as an additional interpolation condition

$$\hat{G}(0)I = T$$

for some $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. Set $\alpha_0 = 0, U_0 = I$ and $Y_0 = T$. By Lemma 1, the constrained interpolation problem has a solution if and only if there exists $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that

$$\begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{21}^* \\ \tilde{Q}_{21} & Q_{22} \end{bmatrix} \geq 0,$$

where

$$\tilde{Q}_{11} = \begin{bmatrix} I - T^*T & U - T^*Y \\ U^* - Y^*T & Q_{11} \end{bmatrix}$$

$$\tilde{Q}_{21} = \begin{bmatrix} \beta^{-1}(V - WT) & Q_{21} \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} I - T^*T & U - T^*Y & (V^* - T^*W^*)\beta^{*-1} \\ U^* - Y^*T & Q_{11} & Q_{21}^* \\ \beta^{-1}(V - WT) & Q_{21} & Q_{22} \end{bmatrix} \geq 0. \tag{3.16}$$

Note that the left-hand side of (3.16) can be rewritten as

$$\begin{bmatrix} I & U & V^*\beta^{*-1} \\ U^* & Q_{11} + Y^*Y & Q_{21}^* + Y^*W^*\beta^{*-1} \\ \beta^{-1}V & Q_{21} + \beta^{-1}WY & Q_{22} + \beta^{-1}WW^*\beta^{*-1} \end{bmatrix} - \begin{bmatrix} T^* \\ Y^* \\ \beta^{-1}W \end{bmatrix} \begin{bmatrix} T & Y & W^*\beta^{*-1} \end{bmatrix}.$$

By schur complement, inequality (3.16) is equivalent to

$$\begin{bmatrix} I & U & V^*\beta^{*-1} & T^* \\ U^* & Q_{11} + Y^*Y & Q_{21}^* + Y^*W^*\beta^{*-1} & Y^* \\ \beta^{-1}V & Q_{21} + \beta^{-1}WY & Q_{22} + \beta^{-1}WW^*\beta^{*-1} & \beta^{-1}W \\ T & Y & W^*\beta^{*-1} & I \end{bmatrix} \geq 0. \tag{3.17}$$

If we decompose the space as (3.3-3.4), then a nest operator $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ has a block lower triangular form shown in (3.5). Therefore, the constrained two-sided tangential N-P interpolation problem has a solution if and only if (3.17) holds for a block lower triangular matrix T. This is a matrix positive completion problem. By Lemma 2, there is a block lower triangular matrix T satisfying (3.17) if and only if

$$\begin{bmatrix} I & \Pi_{\mathcal{U}_k} U & \Pi_{\mathcal{U}_k} V^*\beta^{*-1} & 0 \\ (\Pi_{\mathcal{U}_k} U)^* & Q_{11} + Y^*Y & Q_{21}^* + Y^*W^*\beta^{*-1} & (\Pi_{\mathcal{Y}_k^\perp} Y)^* \\ (\Pi_{\mathcal{U}_k} V^*\beta^{*-1})^* & Q_{21} + \beta^{-1}WY & Q_{22} + \beta^{-1}W(\beta^{-1}W)^{*-1} & (\Pi_{\mathcal{Y}_k^\perp} W^*\beta^{*-1})^* \\ 0 & \Pi_{\mathcal{Y}_k^\perp} Y & \Pi_{\mathcal{Y}_k^\perp} W^*\beta^{*-1} & I \end{bmatrix}$$

is positive semi-definite for $k = 0, \ldots, l$. It follows from Schur complement that this is equivalent to that inequality (3.14) holds for $k = 0, \ldots, l$. We claim that (3.14) when $k = l$ implies (3.14) when $k = 0$. In fact, when $k = l$, inequality (3.14) gives (3.9). When $k = 0$, inequality (3.14) gives

$$\begin{bmatrix} Q_{11} & Q_{21}^* \\ Q_{21} & Q_{22} \end{bmatrix} + \begin{bmatrix} Y^* \\ \beta^{-1}W \end{bmatrix} \begin{bmatrix} Y & W^*\beta^{*-1} \end{bmatrix} - \begin{bmatrix} U^* \\ \beta^{-1}V \end{bmatrix} \begin{bmatrix} U & V^*\beta^{*-1} \end{bmatrix} \geq 0. \tag{3.18}$$

By some algebra manipulations, we have (3.18) is equivalent to

$$\begin{bmatrix} \alpha^* & 0 \\ 0 & \beta^{-1} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{21}^* \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta^{*-1} \end{bmatrix} \geq 0. \tag{3.19}$$

It is obvious that (3.9) implies (3.19). Hence, the constrained two-sided tangential N-P interpolation problem has a solution if and only if (3.14) holds for $k = 1, \dots, l$.

On the other hand, assume $\alpha_i \neq 0$ for all i. Note that the nest operator constraint can also be viewed as an additional interpolation condition

$$I\hat{G}(0) = T$$

for some $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. Set $\beta_0 = 0, W_0 = I$ and $V_0 = T$. A similar argument can show that there is a solution if and only if (3.15) holds for $k = 0, \dots, l-1$. This completes the proof. □

The solvability condition for the standard two-sided tangential interpolation problem without constraint stated in Lemma 1 is recovered when $l = 1$.

Corollary 1. *There exists a solution to the right tangential N-P interpolation problem with constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data α_i, U_i, Y_i, $i = 1, \dots, M$, if and only if*

$$\left[\frac{U_i^* U_m - Y_i^* Y_m}{1 - \alpha_i^* \alpha_m} - U_i^* \Pi_{\mathcal{U}_k} U_m + Y_i^* \Pi_{\mathcal{Y}_k} Y_m\right]_{i,m=1}^{M} \geq 0 \tag{3.20}$$

for all $k = 1, \dots, l$.

Proof: Note that the left-hand side of (3.20) is exactly

$$Q_{11} + Y^* \Pi_{\mathcal{Y}_k} Y - U^* \Pi_{\mathcal{U}_k} U,$$

where Q_{11} is given by (3.10). The result is then obvious from Theorem 1. □

Corollary 2. *There exists a solution to the left tangential N-P interpolation problem with constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data β_j, W_j, V_j, $j = 1, \dots, N$, if and only if*

$$\left[\frac{W_j W_n^* - V_j V_n^*}{1 - \beta_j \beta_n^*} + V_j \Pi_{\mathcal{U}_k^\perp} V_n^* - W_j \Pi_{\mathcal{Y}_k^\perp} W_n^*\right]_{j,n=1}^{N} \geq 0$$

for all $k = 0, 1, \dots, l-1$.

3.5 Conclusion

In this paper, we study multirate systems and related analytic function interpolation problems. We show that each multirate system has an equivalent LTI system with a causality constraint which can be formulated in a unified framework via nest operators and a nest algebra. We then propose a multirate version of the two-sided tangential N-P interpolation problem, which requires the value of the interpolating function at the origin to be in a prescribed set of nest operators. A necessary and sufficient solvability condition is given based on the matrix positive completion. This constrained interpolation problem proposed in this paper has potential applications in a variety of issues in control, signal processing, circuit theory and communication.

References

1. Ball, J. A., I. Gohberg, and L. Rodman. (1990) *Interpolation of Rational Matrix Functions.* Birkhäuser, Boston.
2. Byrnes, C. I., Georgiou, T. T., and Lindquist, A. (2000) A generalized entropy criterion for Nevanlinna-Pick interpolation with degree constraint. *IEEE Trans. Automat. Contr.* to appear.
3. Chen, J. (1997) Frequency-domain tests for validation of linear fractional uncertain models. *IEEE Trans. Automat. Contr.* **42**, 748-760.
4. Chen, T. and L. Qiu (1994) $\mathcal{H}_\infty$ design of general multirate sampled-data control systems. *Automatica* **30**, 1139-1152.
5. Chen, T., L. Qiu, and E. Bai (1998) General multirate building blocks and their application in nonuniform filter banks. *IEEE Trans. on Circuits and systems, Part II,* **45** 948-958.
6. Colaneri, P., R. Scattolini, and N. Schiavoni (1990) Stabilization of multirate sampled-data linear systems. *Automatica* **26**, 377-380.
7. Delsarte, P., Y. Genin, and Y. Kamp (1981) On the role of the Nevanlinna-Pick problem in circuit and system theory. *Circuit Theory Application* **9**, 177-187.
8. Dym, H. and I. Gohberg (1981) Extensions of band matrices with band inverses. *Linear Algebra and its Applications* **36**, 1-24.
9. Foias, C. and A. E. Frazho (1990) *The Commutant Lifting Approach to Interpolation Problems.* Birkhäuser, Boston.
10. Francis, B. A. and G. Zames (1984) On $\mathcal{H}^\infty$-optimal sensitivity theory for SISO feedback systems. *IEEE Trans. Automat. Contr.* **29**, 9-16.
11. Georgiou, T. T. (2000) Analytic interpolation and the degree constraint. *CDROM of the 14th International Symposium on MTNS.*
12. Goodwin, G. C. and A. Feuer (1992) Linear periodic control: a frequency domain viewpoint. *Syst. and Contr. Lett.* **19**, 379-390.
13. Gu, G., D. Xiong, and K. Zhou (1993) Identification in $\mathcal{H}_\infty$ using Pick's interpolation. *Syst. and Contr. Lett.* **20**, 263-272.
14. Helton, J. W. (1987) *Operator Theory, Analytic Functions, Matrices, and Electrical Engineering.* Providence, Rhode Island.
15. Kailath, T. (1974) A view of three decades of linear filtering theory. *IEEE Trans. on Information Theory* **20**, 146-181.
16. Kimura, H. (1989) State space approach to the classical interpolation problem and its application. In H. Nijmeijer and J. M. Schumacher, editors, *Three Decades of Mathematical System Theory* 243-275.
17. Lindquist, A. (2000) *Some new methods and concepts in high-resolution spectral estimation.* Plenary Lecture at the Third Asian Control Conference, Shanghai, China.
18. Liu, H., G. Xu, L. Tong, and T. Kailath (1996) Recent developments in blind channel equalization: ¿From cyclostationarity to subspaces. *Signal Processing* **50**, 83-99.
19. Liu, T. and T. Chen (2000) Optimal design of multi-channel transmultiplexers. *Signal Processing,* to appear.
20. Poolla, K., P. P. Khargonekar, A. Tikku, J. Krause, and K. M. Nagpal (1994) A time-domain approach to model validation. *IEEE Trans. Automat. Contr.* **39**, 1088-1096.

21. Qiu, L. and T. Chen (1994) $\mathcal{H}_2$-optimal design of multirate sampled-data systems. *IEEE Trans. Automat. Contr.* **39**, 2506-2511.
22. Qiu, L. and T. Chen (1996) Contractive completion of block matrices and its application to $\mathcal{H}_\infty$ control of periodic systems. In P. L. I. Gohberg and P. N. Shivakumar, editors, *Recent Development in Operator Theory and its Applications*, 263-281, Birkhauser.
23. Qiu, L. and T. Chen (1999) Multirate sampled-data systems: all $\mathcal{H}_\infty$ suboptimal controllers and the minimum entropy controllers. *IEEE Trans. Automat. Contr.* **44**, 537-550.
24. Strang, G. and T. Q. Nguyen (1996) *Wavelets and Filter Banks*, Wellesley-Cambridge Press.
25. Vaidyanathan, P. (1993) *Multirate Systems and Filter Banks*, Prentice-Hall.
26. Vidyasagar, P. (1985) *Control System Synthesis: A Factorization Approach*, MIT press, Cambridge, Massachusetts.
27. Woerdeman, H. J. (1990) Strictly contractive and positive completions for block matrices. *Linear Algebra and its Applications* **136**, 63-105.
28. Youla, D. C. and M. Saito (1967) Interpolation with positive-real functions. *Journal of the Franklin Institute* **284**, 77-108.