NEST ALGEBRAS, CAUSALITY CONSTRAINTS, AND MULTIRATE ROBUST CONTROL*

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Abstract. Nest operators and nest algebras present a natural framework for studying causality constraints in multirate control systems [8]. In this article, we first give a tutorial on this framework and then look at robust stabilization of analog plants via multirate controllers and provide an explicit solution to the problem.

1. Introduction. The main advantage of a multirate sampling and hold scheme is to achieve better trade-off between performance and implementation cost. Generally speaking, faster A/D and D/A conversions improve performance in digital control systems; but these also mean higher cost in implementation. For signals with different bandwidths, better trade-offs between performance and implementation cost are possible using A/D and D/A converters at different rates.

The study of multirate systems began as early as 1950's [24,21,22]; recent interests are reflected in the LQG/LQR designs [5,1,27], the controller parametrization [25,31], and the work in [28,2,18,9,32]. The controller parametrization in [25,31] provides a basis for designing optimal multirate systems. However, the special structure due to causality presents a constraint in design; treating this causality constraint is the new feature in multirate optimal design.

Causality constraints also arise in discrete-time periodic control [23], where after lifting, the feedthrough terms in controllers must be block lower-triangular. Treatment of causality constraints in this setup is carried out in [13,16,36,10].

Our objective in this article is to reflect our recent work on multirate sampled-data control systems [30,8] and to introduce a proposed framework for handling causality issues in multirate design. The framework is based on nset operators and nest algebras [3,12]. As an application, a multirate robust stabilization problem is solved explicitly.

Before we attack the robust stabilization problem, it is beneficial to look at multirate systems from a general viewpoint and review some dis-

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cussions in [8] about properties of multirate systems.

A general multirate sampled-data system is shown in Figure 1.1. We have used continuous arrows for continuous signals and dotted arrows for discrete signals. Here, G is the continuous-time generalized plant with two inputs, the exogenous input w and the control input u, and two outputs,

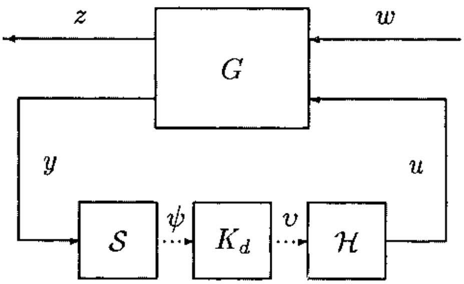

FIG. 1.1. *A general multirate system*

the signal z to be controlled and the measured signal y. $\mathcal{S}$ and $\mathcal{H}$ are multirate sampling and hold operators and are defined as follows:

$$\mathcal{S} = \begin{bmatrix} S_{m_1 h} & & \\ & \ddots & \\ & & S_{m_p h} \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} H_{n_1 h} & & \\ & \ddots & \\ & & H_{n_q h} \end{bmatrix}.$$

Here $S_{m_i h}$ and $H_{n_j h}$ are periodic samplers and zero-order holds with periods $m_i h$ and $n_j h$ respectively; all samplers and holds are synchronized at $t = 0$. The setup in Figure 1.1 samples p channels of y with periods $m_i h, i = 1, \cdots, p$, respectively and holds q channels of v with periods $n_j h, j = 1, \cdots, q$, respectively. If we partition the signals accordingly

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_p \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_q \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_q \end{bmatrix},$$

then

$$\begin{aligned} \psi_i(k) &= y_i(km_i h), \quad i = 1, \cdots, p \\ u_j(t) &= v_j(k), \quad kn_j h \leq t < (k+1)n_j h, \quad j = 1, \cdots, q. \end{aligned}$$

We shall allow each channel in y and v to be vector-valued. In Figure 1.1, K_d is the discrete-time multirate controller, implemented via a microprocessor; it is synchronized with $\mathcal{S}$ and $\mathcal{H}$ in the sense that it inputs a value from the i-th channel at times $k(m_i h)$ and outputs a value to the j-th channel at $k(n_j h)$.

Figure 1.1 gives a compact way of describing multirate systems. It is clear that this model captures all multirate systems in which the rates are rationally related, i.e., the ratio of any two rates is rational. Note that any common factor among m_i and n_j can be absorbed into h; thus we can assume without loss of generality that the greatest common factor among m_i and n_j is 1. With this condition, for any multirate system in which rates are rationally related, there exists a unique number h and a unique set of integers m_i and n_j so that the system can be put into the framework of Figure 1.1. For ease of reference, h is termed the *base period.*

The article is organized as follows. In Section 2 we shall start by introducing the basic concepts and some results on nest operators and nest algebras. These form the groundwork for our subsequent study of causality issues in multirate systems.

Section 3 treats multirate controllers as operators on the space of sequences and study three basic properties which are generalizations of the well-known (single-rate) discrete-time concepts: time-invariance, causality, and finite dimensionality. The causality of multirate controllers is defined via operators between appropriate nests. This provides an effective way to handle causality in design.

In Section 4, we present an application by looking at multirate robust stabilization: Perturbations are modeled naturally in continuous time and controllers are designed with causality constraints considered.

Finally, some concluding remarks are given in Section 6.

Now some words about notation. In this article, we choose to use in discrete time λ-transforms instead of the more traditional z-transforms, where $\lambda = z^{-1}$; in this case, discrete-time Hardy spaces, $\mathcal{H}_2$ and $\mathcal{H}_\infty$, are defined on the open unit disk. If G is a linear time-invariant (LTI) system, we use $\hat{G}$ for its transfer function.

2. Nest operators and nest algebras. In this section we collect some concepts and facts about nests and nest algebras. Following [8], we shall restrict our attention to finite-dimensional spaces; more general treatment can be found in [3,12].

Let $\mathcal{X}$ be a finite-dimensional space. A *nest* in $\mathcal{X}$, denoted $\{\mathcal{X}_i\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the nonincreasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{n-1} \supseteq \mathcal{X}_n = \{0\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be both finite-dimensional inner-product spaces with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ respectively. Assume the two nests have the same number of subspaces, say, $n+1$ as above. A linear map $T : \mathcal{X} \to \mathcal{Y}$ is *nest operator* if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_i, \quad i = 0, 1, \cdots, n. \tag{2.1}$$

Let $\Pi_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i$ and $\Pi_{\mathcal{Y}_i} : \mathcal{Y} \to \mathcal{Y}_i$ be orthogonal projections. Then the condition in (2.1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i} = 0, \qquad i = 0, 1, \cdots, n.$$

The set of all such operators is denoted $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and abbreviated $\mathcal{N}(\{\mathcal{X}_i\})$ if $\{\mathcal{X}_i\} = \{\mathcal{Y}_i\}$. The following properties follows easily.

LEMMA 2.1.
(a) *If* $T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ *and* $T_2 \in \mathcal{N}(\{\mathcal{Y}_i\}, \{\mathcal{Z}_i\})$, *then* $T_2T_1 \in \mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Z}_i\})$.
(b) $\mathcal{N}(\{\mathcal{X}_i\})$ *forms an algebra, called* nest algebra.
(c) *If* $T \in \mathcal{N}(\{\mathcal{X}_i\})$ *and* T *is invertible, then* $T^{-1} \in \mathcal{N}(\{\mathcal{X}_i\})$.

It is a fact that every operator on $\mathcal{X}$ can be factored as the product of a unitary operator and an operator in $\mathcal{N}(\{\mathcal{X}_i\})$.

LEMMA 2.2. *Let* T *be an operator on* $\mathcal{X}$.
(a) *There exist a unitary operator* U_1 *on* $\mathcal{X}$ *and an operator* R_1 *in* $\mathcal{N}(\{\mathcal{X}_i\})$ *such that* $T = U_1R_1$.
(b) *There exist an operator* R_2 *in* $\mathcal{N}(\{\mathcal{X}_i\})$ *and a unitary operator* U_2 *on* $\mathcal{X}$ *such that* $T = R_2U_2$.

Computing such factorizations can be done as follows. Consider part (a) in the lemma: Since $\mathcal{X}_i \supseteq \mathcal{X}_{i+1}$, we write $(\mathcal{X}_{i+1})^{\perp}_{\mathcal{X}_i}$ as the orthogonal complement of $\mathcal{X}_{i+1}$ in $\mathcal{X}_i$. Decompose $\mathcal{X}$ into

$$\mathcal{X} = (\mathcal{X}_1)^{\perp}_{\mathcal{X}_0} \oplus (\mathcal{X}_2)^{\perp}_{\mathcal{X}_1} \oplus \cdots \oplus (\mathcal{X}_n)^{\perp}_{\mathcal{X}_{n-1}}.$$

It follows that under this decomposition any operator R belongs to $\mathcal{N}(\{\mathcal{X}_i\})$ iff its matrix is block lower-triangular, all the diagonal blocks being square. Do a QR type of factorization for square matrices: $T = U_1R_1$ with U_1 orthogonal and R_1 lower-triangular. This factorization serves our purpose under the decomposition of $\mathcal{X}$ described above.

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional inner-product spaces with nests $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$. It is readily seen that $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ is a subspace in the normed space of operators mapping $\mathcal{X}$ to $\mathcal{Y}$. What is the distance (via induced norms) from an operator $T : \mathcal{X} \to \mathcal{Y}$ to $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$, abbreviated $\mathcal{N}$? Or how to compute:

$$\operatorname{dist}(T, \mathcal{N}) := \inf_{Q \in \mathcal{N}} \|T - Q\|. \tag{2.2}$$

It is clear that

$$\operatorname{dist}(T, \mathcal{N}) \geq \max_i \|(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i}\|.$$

THEOREM 2.3.

$$\operatorname{dist}(T, \mathcal{N}) = \max_i \|(I - \Pi_{\mathcal{Y}_i}) T \Pi_{\mathcal{X}_i}\|.$$

This is Corollary 9.2 in [12] specialized to operators on finite-dimensional spaces; it is an extension of a result in [29,11] on norm-preserving dilation of operators, which is specialized to matrices below. We denote the Moore-Penrose generalized inverse of a matrix by $(\cdot)^\dagger$.

LEMMA 2.4. *Assume that A, B, C are fixed matrices of appropriate dimensions. Then*

$$\inf_X \left\| \begin{bmatrix} C & A \\ X & B \end{bmatrix} \right\| = \max\left\{ \left\| \begin{bmatrix} C & A \end{bmatrix} \right\|, \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\| \right\} := \alpha.$$

Moreover, a minimizing X is given by

$$X = -BA^*(\alpha I - AA^*)^\dagger C.$$

It will be of interest to us how to compute a Q to achieve the infimum in (2.2); this can be done by sequentially applying Lemma 2.4:

Step 1 Decompose the spaces $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{aligned} \mathcal{X} &= (\mathcal{X}_1)^\perp_{\mathcal{X}_0} \oplus (\mathcal{X}_2)^\perp_{\mathcal{X}_1} \oplus \cdots \oplus (\mathcal{X}_n)^\perp_{\mathcal{X}_{n-1}} \\ \mathcal{Y} &= (\mathcal{Y}_1)^\perp_{\mathcal{Y}_0} \oplus (\mathcal{Y}_2)^\perp_{\mathcal{Y}_1} \oplus \cdots \oplus (\mathcal{Y}_n)^\perp_{\mathcal{Y}_{n-1}}. \end{aligned}$$

We get matrix representations for T and Q:

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & 0 & \cdots & 0 \\ Q_{21} & Q_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{bmatrix},$$

Q being block lower-triangular.

Step 2 Define $X_{ij} = T_{ij} - Q_{ij}$, $i \geq j$, and

$$P = \begin{bmatrix} X_{11} & T_{12} & \cdots & T_{1n} \\ X_{21} & X_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix}.$$

The problem reduces to

$$\min_{X_{ij}} \|P\|,$$

where T_{ij}, $i < j$, are fixed. Minimizing X_{ij} can be selected as follows. First, set $X_{11}, X_{21}, \cdots, X_{n1}$ and $X_{n2}, \cdots, X_{nn}$ to zero. Second, choose X_{22} be Lemma 2.4 such that the norm of the matrix $(I - \Pi_{\mathcal{Y}_2})P\Pi_{\mathcal{X}_1}$ (obtained by retaining the first 2 block rows and the last $n-1$ block columns in P) is minimized:

$$\|(I - \Pi_{\mathcal{Y}_2})P\Pi_{\mathcal{X}_1}\| = \max\{\|(I - \Pi_{\mathcal{Y}_1})T\Pi_{\mathcal{X}_1}\|, \|(I - \Pi_{\mathcal{Y}_2})T\Pi_{\mathcal{X}_2}\|\}.$$

Fix this X_{22}. Third, choose $[X_{32}X_{33}]$ again by Lemma 2.4 to minimize

$$\|(I - \Pi_{\mathcal{Y}_3})P\Pi_{\mathcal{X}_2}\| = \max\{\|(I - \Pi_{\mathcal{Y}_2})T\Pi_{\mathcal{X}_1}\|, \|(I - \Pi_{\mathcal{Y}_3})T\Pi_{\mathcal{X}_3}\|\}.$$

In this way, we can find all X_{ij} such that

$$\min_{X_{ij}} \|P\| = \max_i \|(I - \Pi_{\mathcal{Y}_i})T\Pi_{\mathcal{X}_i}\|.$$

3. Multirate systems. In this section we shall discuss desirable properties for multirate controllers and then look at internal stability of Figure 1.1. The materials are taken from [8].

The first topic to be examined is periodicity of K_d. Define

$$l = \mathrm{LCM}\,\{m_1, \cdots, m_p, n_1, \cdots, n_q\},$$

where LCM means least common multiple. Thus $\sigma := lh$ is the least time interval in which the sampling and hold schedule repeats itself. K_d can be chosen so that $\mathcal{H}K_d\mathcal{S}$ is σ-periodic in continuous time. For this, we need a few definitions.

Let ℓ be the space of sequences, perhaps vector-valued, defined on the time set $\{0, 1, 2, \cdots\}$. Let U be the unit time delay on ℓ and U^* the unit time advance. Define the integers

$$\begin{aligned} \bar{m}_i &= \tfrac{l}{m_i}, \quad i = 1, 2, \cdots, p \\ \bar{n}_j &= \tfrac{l}{n_j}, \quad j = 1, 2, \cdots, q. \end{aligned}$$

We say K_d is (m_i, n_j)*-periodic* if

$$\begin{bmatrix} (U^*)^{\bar{n}_1} & & \\ & \ddots & \\ & & (U^*)^{\bar{n}_q} \end{bmatrix} K_d \begin{bmatrix} U^{\bar{m}_1} & & \\ & \ddots & \\ & & U^{\bar{m}_p} \end{bmatrix} = K_d.$$

This means shifting the i-th input channel by $\bar{m}_i$ time units ($\bar{m}_i m_i h = \sigma$) corresponds to shifting the j-th output channel by $\bar{n}_j$ units ($\bar{n}_j n_j h = \sigma$).

Now we lift K_d to get an LTI system. For an integer $m > 0$, define the discrete lifting operator L_m via $\underline{v} = L_m v$:

$$\{v(0), v(1), \cdots\} \mapsto \left\{ \begin{bmatrix} v(0) \\ \vdots \\ v(m-1) \end{bmatrix}, \begin{bmatrix} v(m) \\ \vdots \\ v(2m-1) \end{bmatrix}, \cdots \right\}.$$

L_m maps ℓ to ℓ^m, the external direct sum of m copies of ℓ. Define the lifted controller K_d by

$$\underline{K_d} := \begin{bmatrix} L_{\bar{n}_1} & & \\ & \ddots & \\ & & L_{\bar{n}_q} \end{bmatrix} K_d \begin{bmatrix} L_{\bar{m}_1}^{-1} & & \\ & \ddots & \\ & & L_{\bar{m}_p}^{-1} \end{bmatrix}.$$

LEMMA 3.1.
(a) $\mathcal{H}K_d\mathcal{S}$ is σ-periodic in continuous time iff K_d is (m_i, n_j)-periodic.
(b) $\underline{K_d}$ is time-invariant iff K_d is (m_i, n_j)-periodic.

The proof is straightforward. Part (b) was stated in [26]. Normally, we assume G is LTI. Then the closed-loop system in Figure 1.1 is σ-periodic if K_d is (m_i, n_j)-periodic. We shall call σ the *system period*.

Next is causality. For K_d to be implementable in real time, $\mathcal{H}K_d\mathcal{S}$ must be causal in continuous time. This implies that $\underline{K_d}$, as a single-rate system, must be causal; and moreover, the feedthrough term $\underline{D}$ in $\underline{K_d}$ must satisfy a certain constraint, that is, some blocks in $\underline{D}$ must be zero [25,31]. Now let us characterize this constraint on $\underline{D}$ using nest operators.

Write $\underline{v} = \underline{K_d}\,\underline{\psi}$; then $\underline{v}(0) = \underline{D}\underline{\psi}(0)$, where by definitions

$$\begin{aligned}\underline{\psi}(0) &= \left(\begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_p \end{bmatrix} \right)(0) \\ &= \begin{bmatrix} \psi_1(0)' & \cdots & \psi_1(\bar{m}_1 - 1)' & \cdots & \psi_p(0)' & \cdots & \psi_p(\bar{m}_p - 1)' \end{bmatrix}'\end{aligned}$$

Note that $\psi_i(k)$ is sampled at $t = km_ih$. Similarly,

$$\underline{v}(0) = \begin{bmatrix} v_1(0)' & \cdots & v_1(\bar{n}_1 - 1)' & \cdots & v_p(0)' & \cdots & v_p(\bar{n}_q - 1)' \end{bmatrix}'$$

and $v_j(k)$ occurs at $t = kn_jh$. Let Σ be the set of sampling and hold instants in the interval $[0, \sigma)$ (modulo the base period h), i.e.,

$$\Sigma := \left(\bigcup_i \{0, m_i, 2m_i, \cdots, l - m_i\} \right) \bigcup \left(\bigcup_j \{0, n_j, 2n_j, \cdots, l - n_j\} \right).$$

This is a finite set of, say, $n + 1$ elements (not counting repetitions); order Σ increasingly ($\sigma_r < \sigma_{r+1}$):

$$\Sigma = \{\sigma_r : \ r = 0, 1, \cdots, n\}.$$

Let $\underline{\psi}(0)$ and $\underline{v}(0)$ live in the finite-dimensional spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively. For $r = 0, 1, \cdots, n$, define

$$\begin{aligned}\mathcal{X}_r &= \operatorname{span}\{\underline{\psi}(0) : \ \psi_i(k) = 0 \text{ if } km_i < \sigma_r\} \\ \mathcal{Y}_r &= \operatorname{span}\{\underline{v}(0) : \ v_j(k) = 0 \text{ if } kn_j < \sigma_r\}.\end{aligned}$$

$\mathcal{X}_r$ and $\mathcal{Y}_r$ correspond to, respectively, the inputs and outputs occurred after and including time $\sigma_r h$. It is easily checked that $\{\mathcal{X}_r\}$ and $\{\mathcal{Y}_r\}$ are nests and that the causality condition on $\underline{D}$ (the output at time $\sigma_r h$ depends only on inputs up to $\sigma_r h$) is exactly

$$\underline{D}\mathcal{X}_r \subseteq \mathcal{Y}_r, \qquad r = 0, 1, \cdots, n.$$

Thus we define $\underline{D}$ to be (m_i, n_j)-*causal* if $\underline{D} \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$. This is the same causality constraint in [25,31] defined in terms of the elements of $\underline{D}$.

For later benefit, we define $\underline{D}$ to be (m_i, n_j)-*strictly causal* if

$$\underline{D}\mathcal{X}_r \subseteq \mathcal{Y}_{r+1}, \qquad r = 0, 1, \cdots, n-1.$$

This means that the output at time $\sigma_{r+1} h$ depends only on inputs up to time $\sigma_r h$.

The following lemma, which is straightforward to prove, justifies our use of terminology from a continuous-time viewpoint.

LEMMA 3.2.
(a) *$\mathcal{H}K_d\mathcal{S}$ is causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-causal.*
(b) *$\mathcal{H}K_d\mathcal{S}$ is strictly causal in continuous time iff $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-strictly causal.*

Some conclusions on causality issues [25] are transparent under this new formulation.

LEMMA 3.3.
(a) *If $\underline{D}_1$ is (m_i, p_k)-causal and $\underline{D}_2$ is (p_k, n_j)-causal, then $\underline{D}_2\underline{D}_1$ is (m_i, n_j)-causal; furthermore, if $\underline{D}_1$ or $\underline{D}_2$ is strictly causal, then $\underline{D}_2\underline{D}_1$ is also strictly causal.*
(b) *If $\underline{D}$ is (m_i, m_j)-causal and invertible, then $\underline{D}^{-1}$ is (m_j, m_i)-causal.*
(c) *If $\underline{D}$ is (m_i, m_i)-strictly causal, then $(I - \underline{D})^{-1}$ exists and is (m_i, m_i)-causal.*

The proof is easy under the current framework, see, e.g., [8]. Let us define K_d to be (m_i, n_j)-*causal* if $\underline{K_d}$ is causal and $\underline{D}$ is (m_i, n_j)-causal.

We assume K_d is (m_i, n_j)-periodic and -causal. Then $\underline{K_d}$ is LTI and causal. To get finite-dimensional difference equations for $\underline{K_d}$, we further assume $\underline{K_d}$ is finite-dimensional. Thus $\underline{K_d}$ has a state model

$$\hat{\underline{K_d}}(\lambda) = \left[\begin{array}{c|ccc} A & B_1 & \cdots & B_p \\ \hline C_1 & D_{11} & \cdots & D_{1p} \\ \vdots & \vdots & & \vdots \\ C_q & D_{q1} & \cdots & D_{qp} \end{array}\right].$$

Let the state for $\underline{K_d}$ be η. The corresponding equations for $\underline{K_d}$ ($\underline{v} = \underline{K_d}\,\underline{\psi}$) are

$$\begin{aligned} \eta(k+1) &= A\eta(k) + \sum_{i=1}^{p} B_i \underline{\psi}_i(k) \\ \underline{v}_j(k) &= C_j\eta(k) + \sum_{i=1}^{p} D_{ji}\underline{\psi}_i(k), \quad j = 1, 2, \cdots, q. \end{aligned}$$

Note that $\underline{\psi}_i = L_{\bar{m}_i}\psi_i$ and $\underline{v}_j = L_{\bar{n}_j}v_j$. Partitioning the matrices accordingly

$$\begin{aligned} B_i &= \begin{bmatrix} (B_i)_0 & \cdots & (B_i)_{\bar{m}_i-1} \end{bmatrix}, \\ C_j &= \begin{bmatrix} (C_j)_0 \\ \vdots \\ (C_j)_{\bar{n}_j-1} \end{bmatrix}, \quad D_{ji} = \begin{bmatrix} c(D_{ji})_{00} & \cdots & (D_{ji})_{0,\bar{m}_i-1} \\ \vdots & & \vdots \\ (D_{ji})_{\bar{n}_j-1,0} & \cdots & (D_{ji})_{\bar{n}_j-1,\bar{m}_i-1} \end{bmatrix} \end{aligned}$$

(some blocks in D_{ji} must be zero for causality), we get the difference equations for K_d ($v = K_d\psi$):

$$(3.1)\qquad \begin{aligned} \eta(k+1) &= A\eta(k) + \sum_{i=1}^{p}\sum_{s=0}^{\bar{m}_i-1} (B_i)_s\psi_i(k\bar{m}_i + s) \\ v_j(k\bar{n}_j + r) &= (C_j)_r\eta(k) + \sum_{i=1}^{p}\sum_{s=0}^{\bar{m}_i-1} (D_{ji})_{rs}\psi_i(k\bar{m}_i + s), \end{aligned}$$

where the indices in (3.1) go as follows: $j = 1, 2, \cdots, q$ and $r = 0, 1, \cdots, \bar{n}_j - 1$. These are the equations for implementing K_d on microprocessors and they require only finite memory. Note that the state vector η for K_d is updated every system period σ.

In summary, we are interested in the class of multirate K_d which are (m_i, n_j)-periodic and -causal and finite-dimensional; this class is called the *admissible* class of K_d and can be modeled by difference equations (3.1-3.1) with $\underline{D}$ (m_i, n_j)-causal. The corresponding admissible class of $\underline{K_d}$ is characterized by LTI, causal, and finite-dimensional $\underline{K_d}$ with the same constraint on $\underline{D}$.

Finally, we conclude this section by looking at internal stability of Figure 1.1. We assume the continuous G is LTI, causal, and finite-dimensional; partition G as follows:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}.$$

G has a state model

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

Let the plant state be x and the controller state be η (K_d is admissible). Note that the system in Figure 1.1 is σ-periodic. Define the continuous-time vector

$$x_{sd}(t) := \begin{bmatrix} x(t) \\ \eta(k) \end{bmatrix}, \quad k\sigma \le t < (k+1)\sigma .$$

The (autonomous) system in Figure 1.1 is *internally stable*, or K_d *internally stabilizes* G, if for any initial value $x_{sd}(t_0)$, $0 \le t_0 < \sigma$, $x_{sd}(t) \to 0$ as $t \to \infty$.

This stability notion can be related to stability of the discrete-time system in Figure 3.1, where

$$\underline{G_{22d}} := \begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix} \mathcal{S} G_{22} \mathcal{H} \begin{bmatrix} L_{\bar{n}_1}^{-1} & & \\ & \ddots & \\ & & L_{\bar{n}_q}^{-1} \end{bmatrix} .$$

Because G_{22} is LTI and strictly causal, $\mathcal{S}G_{22}\mathcal{H}$, the multirate discretization of G_{22}, is (n_j, m_i)-periodic and -strictly causal. Thus $\underline{G_{22d}}$ is LTI and causal with $\underline{D_{22d}}$ (n_j, m_i)-strictly causal. So Figure 3.1 gives an LTI discrete system. In fact, a state model for $\underline{G_{22d}}$ can be obtained [26]; its state being $\xi := S_\sigma x$, or $\xi(k) = x(k\sigma)$.

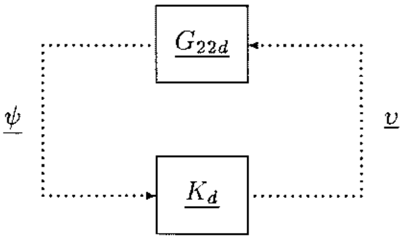

FIG. 3.1. *The lifted system for stability*

Let us see that Figure 3.1 is well-posed, i.e., the matrix $I - \underline{D_{22d}}\underline{D}$ is invertible, where $\underline{D}$ is the feedthrough term of $\underline{K_d}$. This follows from Lemma 3.3: $\underline{D_{22d}}\underline{D}$ is (m_i, m_i)-strictly causal [Lemma 3.3 (a)] and so $I - \underline{D_{22d}}\underline{D}$ is invertible [Lemma 3.3 (c)]. This also implies that the multirate system of Figure 1.1 is well-posed.

The system in Figure 3.1 is *internally stable*, or $\underline{K_d}$ *internally stabilizes* $\underline{G_{22d}}$ if for any initial states $\xi(0)$ and $\eta(0)$,

$$\begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \to 0 \text{ as } k \to \infty.$$

THEOREM 3.4. *K_d internally stabilizes G iff $\underline{K_d}$ internally stabilizes $\underline{G_{22d}}$.*

A proof of this result is contained in [8]. Sufficient conditions for the internal stability to be achievable are that (A, B_2) and (C_2, A) are stabilizable and detectable respectively and that the system period σ is nonpathological in a certain sense, see, e.g., [15,30].

4. Multirate robust stabilization. The sampled-data robust stabilization problem was treated in the single-rate setting in [7,20]; though the problem is a special case of the general $\mathcal{H}_\infty$ control problem [19,35,4,33,34,20], the reduction to discrete-time problem requires no iteration on the performance bounds; this greatly simplifies computation of near optimal solutions.

Our goal in this section is to extend the single-rate result in [7] to a general multirate setup. In this case, the design of robust controllers is subject to the causality constraints discussed in the preceding section.

The multirate setup is shown in Figure 4.1. Here, P is the analog plant

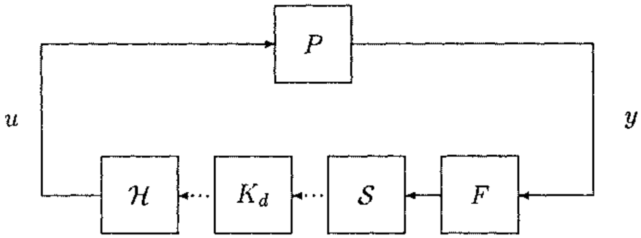

FIG. 4.1. *A multirate system*

modeled by a nominal plant P_n with an additive perturbation

$$P = P_n + \Delta W,$$

where Δ is an unknown perturbation due to unmodeled dynamics or parameter variations and W is a fixed frequency weighting system. We shall assume that P_n and W are both LTI, causal, and finite-dimensional and that Δ is linear and bounded $\mathcal{L}_2 \to \mathcal{L}_2$. The multirate sampling operator $\mathcal{S}$, hold operator $\mathcal{H}$, and controller K_d in Figure 4.1 are as before and we shall require that K_d be admissible. F in Figure 4.1 is the (analog) anti-aliasing filter and is assumed to be LTI, strictly causal, and finite-dimensional.

For any positive number γ, define the set of perturbed plants

$$\mathcal{P}_\gamma := \{P_n + \Delta W : \quad \|\Delta\| < \gamma\}.$$

So γ is a measure of the size of the perturbation Δ. The following question will be considered: Given a positive number γ, how to design an admissible controller K_d to stabilize all the plants in $\mathcal{P}_\gamma$?

For K_d to stabilize all the plants in $\mathcal{P}_\gamma$, it must stabilize the nominal plant P_n. Putting Figure 4.1 into the general setup of Figure 1.1 with

$\Delta = 0$, we obtain that the $(2,2)$ block in the plant is FP_n. Define $\underline{K_d}$ as in Section 3 and

$$\underline{P_{nd}} = \begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix} \mathcal{S} F P_n \mathcal{H} \begin{bmatrix} L_{\bar{n}_1}^{-1} & & \\ & \ddots & \\ & & L_{\bar{n}_q}^{-1} \end{bmatrix}.$$

It follows as before that $\underline{K_d}$ and $\underline{P_{nd}}$ are both LTI, causal, and finite-dimensional with the feedthrough term in $\underline{P_{nd}}$ being (n_j, m_i)-strictly causal (since F is strictly causal). Thus by Theorem 3.4, K_d internally stabilizes P_n iff $\underline{K_d}$ internally stabilizes $\underline{P_{nd}}$ in discrete time.

Introduce the *discrete sampling operator* $S_m : \ell \to \ell$ defined via

$$\psi = S_m \phi \Longleftrightarrow \psi(k) = \phi(km)$$

and the *discrete hold operator* $H_n : \ell \to \ell$ via

$$v = H_n \phi \Longleftrightarrow v(kn + r) = \phi(k), \quad r = 0, 1, \cdots, n-1.$$

It is easily checked that $S_{m_i h} = S_{m_i} S_h$ and $H_{n_j h} = H_h H_{n_j}$. Defining the discrete multirate sampling and hold operators

$$\mathcal{S}_d = \begin{bmatrix} S_{m_1} & & \\ & \ddots & \\ & & S_{m_p} \end{bmatrix}, \quad \mathcal{H}_d = \begin{bmatrix} H_{n_1} & & \\ & \ddots & \\ & & H_{n_q} \end{bmatrix},$$

we have that the multirate $\mathcal{S}$ and $\mathcal{H}$ can be factored as

$$\mathcal{S} = \mathcal{S}_d S_h, \quad \mathcal{H} = \mathcal{H}_d H_h.$$

Now bring in the two useful factorizations studied in [6]:

$$\begin{aligned} (WH_h)^*(WH_h) &= G_1^* G_1, \\ (S_h F)(S_h F)^* &= G_2 G_2^*, \end{aligned}$$

with the operators G_1 and G_2 both LTI, causal, and finite-dimensional in discrete time. Define

$$\begin{aligned} \underline{G_1} &:= L_l G_1 \mathcal{H}_d \begin{bmatrix} L_{\bar{n}_1}^{-1} & & \\ & \ddots & \\ & & L_{\bar{n}_q}^{-1} \end{bmatrix} \\ \underline{G_2} &:= \begin{bmatrix} L_{\bar{m}_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} \end{bmatrix} \mathcal{S}_d G_2 L_l^{-1} \end{aligned}$$

Let $\underline{T_d}$ be the discrete map $\underline{v} \mapsto \underline{\zeta}$ in Figure 4.2. It is not hard to see that both $\underline{G_1}$ and $\underline{G_2}$ are LTI, causal, and finite-dimensional because G_1 and G_2

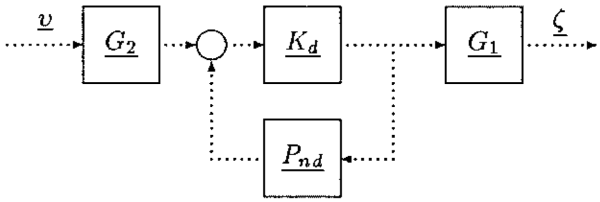

FIG. 4.2. *A lifted discrete-time system*

are. So Figure 4.2 represents an LTI system in discrete time and moreover, $\underline{T_d}$ belongs to $\mathcal{RH}_\infty$ if $\underline{K_d}$ internally stabilizes $\underline{P_{nd}}$. We are set up to state the main result.

THEOREM 4.1. *The multirate K_d stabilizes all the plants in $\mathcal{P}_\gamma$ if $\underline{K_d}$ internally stabilizes $\underline{P_{nd}}$ in discrete time and achieves $\|\hat{\underline{T_d}}\|_\infty \leq 1/\gamma$.*

Proof. Suppose $\underline{K_d}$ internally stabilizes $\underline{P_{nd}}$, or equivalently, K_d internally stabilizes P_n. The perturbed system configuration is shown in Figure 4.3. Reconfigure the diagram in Figure 4.3 into that in Figure 4.4, where T can be read as

$$T = W(I - \mathcal{H}K_d\mathcal{S}FP_n)^{-1}\mathcal{H}K_d\mathcal{S}F.$$

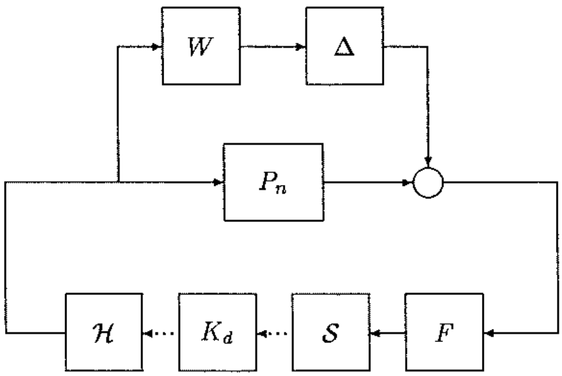

FIG. 4.3. *The perturbed system*

Since $\mathcal{S} = \mathcal{S}_dS_h$ and $\mathcal{H} = H_h\mathcal{H}_d$, by some algebra

$$T = WH_h\mathcal{H}_d(I - K_d\mathcal{S}FP_n\mathcal{H})^{-1}K_d\mathcal{S}_dS_hF. \tag{4.1}$$

Thus the perturbed system is stable if the small-gain condition is satisfied:

$$\|\Delta\| \cdot \|T\| < 1.$$

Therefore K_d stabilizes all the plants in $\mathcal{P}_\gamma$ if $\|T\| \leq 1/\gamma$, the norm being on $\mathcal{L}_2$.

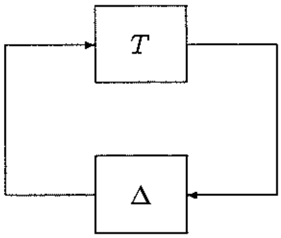

FIG. 4.4. *Reconfigured diagram*

Define the discrete operator

$$G_d := \mathcal{H}_d(I - K_d \mathcal{S} F P_n \mathcal{H})^{-1} K_d \mathcal{S}_d.$$

Then the continuous-time operator T becomes (via (4.1))

$$T = W H_h G_d S_h F.$$

Thus by Proposition 1 in [6], $\|T\|$ equals the ℓ_2 induced norm of $T_d := G_1 G_d G_2$. Now it can be verified from the definitions that

$$\underline{T_d} = L_l T_d L_l^{-1},$$

which is time-invariant as we commented before. Since the lifting operator L is norm-preserving,

$$\|T\| = \|T_d\| = \|\underline{T_d}\| = \|\hat{\underline{T_d}}\|_\infty.$$

The proof is completed. □

To solve the multirate robust stabilization problem, from Theorem 4.1 we arrive at an LTI discrete-time $\mathcal{H}_\infty$ problem; but the feedthrough term in the controllers must be (m_i, n_j)-causal. Such problems can be solved using the results in Section 2 [8]; this will be discussed later.

Note that the discrete system in Figure 4.2 does not depend on γ. If one wants to compute near optimal controllers, one needs to iterate on the achievable γ; but the fixed parts in Figure 4.2, namely, $\underline{G_1}$, $\underline{G_2}$, and $\underline{P_{nd}}$, remain the same in each iteration. To compute the lifted systems in Figure 4.2, we need the following useful lemma.

Let G be a discrete-time system with state ξ and transfer function

$$\hat{G}(\lambda) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Let $m, n, \bar{m}, \bar{n}, l$ be positive integers such that

$$m\bar{m} = n\bar{n} = l.$$

Define

$$\underline{G} := L_{\bar{m}} S_m G H_n L_{\bar{n}}^{-1}$$

and the characteristic function on integers

$$\chi_{[p,q)}(r) = \begin{cases} 1, & p \leq r < q \\ 0, & \text{else.} \end{cases}$$

LEMMA 4.2. *A state model for $\underline{G}$ is*

$$\underline{\hat{G}}(\lambda) = \left[\begin{array}{c|cccc} A^l & \sum_{r=0}^{n-1} A^{l-1-r}B & \sum_{r=n}^{2n-1} A^{l-1-r}B & \cdots & \sum_{r=l-n}^{l-1} A^{l-1-r}B \\ \hline C & D_{00} & D_{01} & \cdots & D_{0,\bar{n}-1} \\ CA^m & D_{10} & D_{11} & \cdots & D_{1,\bar{n}-1} \\ \vdots & \vdots & \vdots & & \vdots \\ CA^{l-m} & D_{\bar{m}-1,0} & D_{\bar{m}-1,1} & \cdots & D_{\bar{m}-1,\bar{n}-1} \end{array}\right],$$

where

$$D_{ij} = D\chi_{[jn,(j+1)n)}(im) + \sum_{r=jn}^{(j+1)n-1} CA^{im-1-r}B\chi_{[0,im)}(r).$$

The corresponding state vector is $\underline{\xi} = S_l \xi$.

All the lifted systems in Figure 4.2, namely, $\underline{P_{nd}}$, $\underline{G_1}$, and $\underline{G_2}$, can be computed from this lemma. For example, let us see how to compute $\underline{P_{nd}}$. We start with a state model for FP_n:

$$\hat{F}(s)\hat{P}_n(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

Compute the single-rate discretization $P_{ndh} := S_h F P_n H_h$:

$$\hat{P}_{ndh}(\lambda) = \left[\begin{array}{c|c} e^{hA} & \int_0^h e^{tA}\,dt\,B \\ \hline C & 0 \end{array}\right].$$

Then by definition of $\underline{P_{nd}}$ and the factorizations $\mathcal{S} = \mathcal{S}_d S_h$ and $\mathcal{H} = H_h \mathcal{H}_d$, we have

$$\underline{P_{nd}} = \begin{bmatrix} L_{\bar{m}_1} S_{m_1} & & \\ & \ddots & \\ & & L_{\bar{m}_p} S_{m_p} \end{bmatrix} P_{ndh} \begin{bmatrix} H_{n_1} L_{\bar{n}_1}^{-1} & & \\ & \ddots & \\ & & H_{n_q} L_{\bar{n}_q}^{-1} \end{bmatrix}.$$

The transfer function $\underline{\hat{P}_{nd}}$ is a $p \times q$ block matrix; each block can be computed exactly by Lemma 4.2. Let us note that the feedthrough term $\underline{D}_{nd}$ of $\underline{P_{nd}}$ is (n_j, m_i)-strictly causal and therefore the feedback loop in Figure 4.2 is well-posed.

Now we use the controller parametrization [25,31] to reduce the problem further to a model-matching problem.

Bring in a doubly-coprime factorization for $\underline{\hat{P}_{nd}}$:

$$\underline{\hat{P}_{nd}} = \hat{N}\hat{M}^{-1} = \hat{\tilde{M}}^{-1}\hat{\tilde{N}}$$

$$\begin{bmatrix} \hat{\tilde{X}} & -\hat{\tilde{Y}} \\ -\hat{\tilde{N}} & \hat{\tilde{M}} \end{bmatrix} \begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = I$$

with the conditions:

$$\begin{aligned} &\hat{M}(0) = I, \quad \hat{\tilde{M}}(0) = I, \\ &\hat{N}(0) = \hat{\tilde{N}}(0) = \underline{D}_{nd}, \\ &\hat{X}(0) = I, \quad \hat{\tilde{X}}(0) = I, \\ &\hat{Y}(0) = \hat{\tilde{Y}}(0) = 0. \end{aligned}$$

The standard procedure in [14] yields such a factorization. Since $\underline{D}_{nd}$ is (n_j, m_i)-strictly causal, it follows from [25,31] that the set of admissible $\underline{K_d}$ that provide internal stability is parametrized by

$$\underline{\hat{K}_d} = (\hat{Y} - \hat{M}\hat{Q})(\hat{X} - \hat{N}\hat{Q})^{-1}, \quad \hat{Q} \in \mathcal{RH}_\infty, \quad \hat{Q}(0) \ \ (m_i, n_j)\text{-causal}.$$

Define

$$\begin{aligned} \hat{T}_1 &= \underline{\hat{G}_1}\hat{Y}\hat{\tilde{M}}\underline{\hat{G}_2} \\ \hat{T}_2 &= \underline{\hat{G}_1}\hat{M} \\ \hat{T}_3 &= \hat{\tilde{M}}\underline{\hat{G}_2}. \end{aligned}$$

It follows that

$$\underline{\hat{T}_d} = \hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3.$$

Recall in Section 3 that $\hat{Q}(0)$ is (m_i, n_j)-causal iff $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$, where the nests $\{\mathcal{X}_r\}$ and $\{\mathcal{Y}_r\}$ were defined in Section 3. In this way we arrive at the constrained $\mathcal{H}_\infty$ model-matching problem: Find $\hat{Q} \in \mathcal{RH}_\infty$ with $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ such that

$$\|\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3\|_\infty < 1, \tag{4.2}$$

where γ is absorbed into $\hat{T}_1$ and $\hat{T}_2$.

This latter problem is studied in detail in [30,8]; the solution is summarized below.

For regularity, we need the following assumption:

For every λ on the unit circle, $\hat{T}_2(\lambda)$ and $\hat{T}_3(\lambda^{-1})'$ are both injective.

Dropping the causality constraint on $\hat{Q}(0)$, we get a standard unconstrained $\mathcal{H}_\infty$ problem: Find a $\hat{Q} \in \mathcal{RH}_\infty$ such that

$$\|\hat{T}_1 - \hat{T}_2\hat{Q}\hat{T}_3\|_\infty < 1\,. \tag{4.3}$$

Assume this unconstrained problem is solvable; this is necessary for the solvability of the constrained problem in (4.2). Then we can parametrize all $\hat{Q}$ in $\mathcal{RH}_\infty$ achieving (4.3) via a powerful result in [17]: There exists an $\mathcal{RH}_\infty$ matrix

$$\hat{K} = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix}$$

with $\hat{K}_{12}^{-1}, \hat{K}_{21}^{-1} \in \mathcal{RH}_\infty$ and $\|\hat{K}_{22}\|_\infty < 1$ such that all $\hat{Q} \in \mathcal{RH}_\infty$ satisfying (4.3) are characterized by

$$\hat{Q} = \hat{K}_{11} + \hat{K}_{12}\hat{Q}_1(I - \hat{K}_{22}\hat{Q}_1)^{-1}\hat{K}_{21}, \quad \hat{Q}_1 \in \mathcal{RH}_\infty, \quad \|\hat{Q}_1\|_\infty < 1\,.$$

We refer to [17] for the details of checking the solvability condition for the unconstrained problem and the expression of $\hat{K}$.

By an argument used in [30,8], we can assume without loss of generality that $\hat{K}_{22}(0) = 0$. Thus

$$\hat{Q}(0) = \hat{K}_{11}(0) + \hat{K}_{12}(0)\hat{Q}_1(0)\hat{K}_{21}(0)\,. \tag{4.4}$$

This is an affine function $\hat{Q}_1(0) \mapsto \hat{Q}(0)$.

Now we bring in the causality constraint on $\hat{Q}(0)$. Our goal is to find a $\hat{Q}_1 \in \mathcal{RH}_\infty$ with $\|\hat{Q}_1\|_\infty < 1$ such that $\hat{Q}(0)$ in (4.4) lies in $\mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$. Since $\hat{Q}(0)$ depends only on $\hat{Q}_1(0)$ and in general $\|\hat{Q}_1\|_\infty \geq \|\hat{Q}_1(0)\|$, the equivalent problem is to find a constant matrix $\hat{Q}_1(0)$ with $\|\hat{Q}_1(0)\| < 1$ such that $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$.

Using Lemma 2.2, we can reduce the problem to a distance problem. Introduce matrix factorizations (Lemma 2.2)

$$\hat{K}_{12}(0) = R_1U_1, \quad \hat{K}_{21}(0) = -U_2R_2\,,$$

where R_1, R_2, U_1, U_2 are all invertible, U_1, U_2 are orthogonal, and R_1, R_2 belongs to the nest algebras $\mathcal{N}(\{\mathcal{Y}_r\}), \mathcal{N}(\{\mathcal{X}_r\})$ respectively. Substitute the factorizations into (4.4) and pre- and post-multiply by R_1^{-1} and R_2^{-1} respectively to get

$$R_1^{-1}\hat{Q}(0)R_2^{-1} = R_1^{-1}\hat{K}_{11}(0)R_2^{-1} - U_1\hat{Q}_1(0)U_2\,.$$

Define

$$V := R_1^{-1}\hat{Q}(0)R_2^{-1}, \quad E := R_1^{-1}\hat{K}_{11}(0)R_2^{-1}, \quad Z := U_1\hat{Q}_1(0)U_2\,.$$

It follows that $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ iff $V \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ (Lemma 2.1) and $\|\hat{Q}_1(0)\| < 1$ iff $\|Z\| < 1$. Therefore, we arrive at the following equivalent matrix problem: Given E, find Z with $\|Z\| < 1$ such that $V = E - Z \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$; or equivalently, find $V \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ such that $\|E - V\| < 1$. This can be solved via the distance problem studied in Theorem 2.3: There exists a matrix $V \in \mathcal{N}(\{\mathcal{X}_r\}, \{\mathcal{Y}_r\})$ such that $\|E - V\| < 1$ iff

$$\mu := \max_r \{\|(I - \Pi_{\mathcal{Y}_r})E\Pi_{\mathcal{X}_r}\|\} < 1 .$$

Moreover, a V achieving $\|E - V\| = \mu$ can be computed by the procedure given at the end of Section 2.

5. Conclusions. We have introduced a framework based on nest algebras for treating causality issues in multirate design. The usefulness of this framework is illustrated by solving a robust stabilization problem via multirate digital controllers. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control designs in multirate systems can also be studied using this framework, yielding explicit solutions to the problems [30,8].

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