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NEW PERTURBATION BOUNDS FOR THE ROBUST STABILITY OF LINEAR STATE SPACE MODELS*

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Abstract

In this paper, the problem of robust stability of linear time-invariant systems in state space models is considered. Explicit bounds on linear time-invariant perturbations which do not destabilize the system are given for both unstructured and structured perturbations. These bounds are superior to those reported in the recent literature in two senses: i) they are less conservative and ii) they can be applied to a more general class of systems and perturbations. The bounds are easy to compute numerically. Several simple examples are given to demonstrate the new bounds and compare them with results previously reported.

The following notation shall be used throughout the paper.

Notation

$\lambda[\cdot]$	eigenvalues of a square matrix
$\sigma[\cdot]$	singular values of a matrix
$\overline{\sigma}[\cdot]$	largest singular value
$\underline{\sigma}[\cdot]$	smallest singular value
$\|\|\cdot\|\|_p$	matrix norm induced from Holder p-norms of vectors
$\|\|\cdot\|\|_2$	spectral norm which is equal to $\overline{\sigma}[\cdot]$
$\|[\cdot]\|$	modulus matrix, i.e. matrix formed by taking moduli of elements of $[\cdot]$
$[\cdot]_s$	symmetric part of a matrix $= [(\cdot)+(\cdot)']^{1/2}$
$\det[\cdot]$	determinant of a square matrix
$\pi[\cdot]$	perron eigenvalue of a non-negative square matrix
$\mathrm{sp}[\cdot]$	set of eigenvalues of a square matrix

1. Introduction

In the analysis and synthesis of robust control systems, a fundamental problem that arises is the recognition that the mathematical model assumed for the system is always inexact, and that the parameters of the system may deviate away from their nominal values. Thus it is desirable to be able to determine to what extent a nominal system remains stable when subject to a certain class of perturbations. This is called the robust stability problem, e.g. see [1]-[9].

There are two main approaches which have been applied to this problem in the literature: (i) the frequency domain approach, e.g. [2], [3], [7] which is based on the transfer function representation of a system, and (ii) the time domain approach, e.g. [4], [8], [9], [10] which is based on a state space representation of a system. This paper will study the robust stability problem for a state space representation of a system using a frequency domain approach.

2. Development

Assume that a linear time-invariant model of a physical system is described by the following state equation with linear time-invariant perturbations:

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$$\dot{x} = (A + \Delta A)x \tag{1}$$

where $x \in R^n$ is the state, $A \in R^{n \times n}$ is the nominal state matrix which is assumed to be asymptotically stable, and ΔA is a perturbation matrix.

It is assumed that ΔA may be classified into two types of perturbations, namely:

(i) Unstructured perturbations

In this case, only a bound on the norm of the perturbation matrix ΔA is given.

(ii) Structured perturbations

In this case, the structure of perturbations in ΔA is specified and the bounds on such structured perturbations are given.

Given either (i) or (ii), it is desired to determine if the perturbed system (1) remains stable.

The above problem has been extensively studied. For example, the following recent results have been obtained for the unstructured case.

Result 1 [8]: The system (1) is stable if

$$\overline{\sigma}(\Delta A) < \frac{1}{\overline{\sigma}(P)} \triangleq \mu_{PW1} \tag{2}$$

or if A is diagonalizable if

$$\overline{\sigma}(\Delta A) < \min[-\mathrm{Re}\,\lambda(A)] \frac{\underline{\sigma}(T)}{\overline{\sigma}(T)} \triangleq \mu_{PW2}$$

where P satisfies the Lyapunov equation

$$A'P + PA = -2I \tag{3}$$

and T satisfies $T^{-1}AT = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Result 2 [6]: Assume that the orthogonal matrix U in the polar decomposition of A

$$A = UH_R \text{ or } A = H_L U$$

is stable. Then the system (1) is stable if

$$\overline{\sigma}(\Delta A) < -\underline{\sigma}(A)\cos(\theta_{\min}) \triangleq \mu_{LW} \tag{4}$$

where $\theta_{\min}$ is the smallest principal phase of A measured counter-clockwise from the positive real axis.

Result 3 [10]: Assume that A_S is negative definite. Then the system (1) is stable if

$$\overline{\sigma}(\Delta A) < \underline{\sigma}(A_S) \triangleq \mu_{YW} \tag{5}$$

In general, these bounds obtained are quite conservative and they cannot necessarily be applied to arbitrary systems. In general, μ_{PW1} appears to be a better bound than the others. If A is normal, then $\mu_{PW1} = \mu_{PW2} = \mu_{LW} = \mu_{YW} = \min[\mathrm{Re}{-}\lambda(A)]$.

For the structured case, the following result was recently obtained:

Result 4 [10]: Assume that the elements of ΔA are restricted so that

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$$| \Delta A_{ij} | \leq \epsilon_{ij}$$

and let $\epsilon \triangleq \max_{i,j} \epsilon_{ij}$. Then the system (1) is stable if

$$\epsilon < \frac{1}{\overline{\sigma}[(|P| U)_S]} \triangleq \mu_Y \tag{6}$$

where P satisfies (3) and U is a matrix with elements $U_{ij} = \frac{\epsilon_{ij}}{\epsilon}$.

This paper will consider both the structured and unstructured case of ΔA perturbations. The new bounds obtained are better than the above results in two senses: (i) they are tighter and (ii) they encompass a wider class of systems and perturbations. In the unstructured case, the new bound requires no conditions on the system matrix A and for a normal A, it gives the exact bound. In the structured case, a characterization of permissible perturbations is given which includes the results of [10] as a special case.

3. Main Results

First two lemmas which form the foundation of the later development will be given. Then the stability robustness will be discussed for both the unstructured and structured perturbation cases.

Lemma 1: Let $A \in \mathcal{C}^{n\times n}$ and $\Delta A \in \mathcal{C}^{n\times n}$. Let $s \notin \mathrm{sp}(A)$ be a point in the complex plane such that there exists a nonsingular matrix $R \in \mathcal{C}^{n\times n}$ so that

$$||R^{-1}\Delta A\,(sI-A)^{-1}R\,||_p < 1 \text{ or } ||R^{-1}(sI-A)^{-1}\Delta A R\,||_p < 1$$

Then s cannot be an eigenvalue of $A+\Delta A$.

Proof: See Appendix 1.

Lemma 1 can be called an eigenvalue exclusive lemma. For given A, ΔA, R, it is easy to compute such an exclusive region in the complex plane.

Lemma 2: Let $A \in R^{n\times n}$ be stable and assume that $\Delta A \in R^{n\times n}$ belongs to a set $\mathbf{S}$ which has the property that if $\Delta A \in \mathbf{S}$ then this implies that $\alpha\Delta A \in \mathbf{S}$, $\forall \alpha \in [0,1]$. Then $A+\Delta A$ is stable $\forall \Delta A \in \mathbf{S}$ if for some nonsingular $R(j\omega) \in R^{n\times n}$

$$||R\,\Delta A\,(j\omega I-A)^{-1}R^{-1}||_p < 1 \text{ or } ||R\,(j\omega I-A)^{-1}\Delta A R\,||_p < 1$$

$\forall \Delta A \in \mathbf{S}$, $\forall \omega \geq 0$.

Proof: See Appendix 2.

3.1 Unstructured Perturbations

Consider the system (1). In this case, only the norm of ΔA is known. Suppose it is $||\Delta A\,||_p$. The following result is obtained.

Theorem 1: The perturbed state matrix $A+\Delta A$ is stable if

$$||\Delta A\,||_p < \frac{1}{\sup_{\omega\geq 0} ||(j\omega I-A)^{-1}||_p} \triangleq \mu_{QW} \tag{7}$$

Proof: Let $R=I$ and use lemma 2 directly. Then $A+\Delta A$ is stable if

$$\sup_{\omega\geq 0} ||\Delta A\,(j\omega I-A)^{-1}||_p < 1$$

which is satisfied if

$$||\Delta A\,||_p \cdot \sup_{\omega\geq 0} ||(j\omega I-A)^{-1}||_p < 1$$

from which the result immediately follows.

Remarks

1. As $\omega\to\infty$, $||(j\omega I-A)^{-1}||_p \to 0$ which implies that the supremum in (7) only needs to be determined in a finite interval. When $p=2$, the supremum can be taken in the frequency interval $\omega \in [0, 2\overline{\sigma}(A)]$.
2. There are no extra requirements for A needed in theorem 1, unlike the case for the bounds μ_{PW2}, μ_{LW}, μ_{YW}, and the computations required in theorem 1 are numerically well defined.
3. If the spectral norm is used in theorem 1 (i.e. $p=2$), the condition becomes

$$\overline{\sigma}(\Delta A) < \inf_{\omega\geq 0} \underline{\sigma}(j\omega I-A) \triangleq \mu_{QW} \tag{8}$$

The following result shows that the bound μ_{QW} is tighter than μ_{PW} when the spectral norm is used.

Theorem 2: Let P be the solution of the Lyapunov equation (3); then

$$\mu_{QW} = \inf_{\omega\geq 0} \underline{\sigma}(j\omega I-A) \geq \mu_{PW} = \frac{1}{\overline{\sigma}(P)}$$

Proof: See Appendix 3.

For some special cases, the condition in theorem 1 becomes necessary and sufficient. The following is obtained.

Theorem 3: Assume A is a normal matrix and that ΔA is bounded by its spectral norm; then $A+\Delta A$ is stable if and only if

$$\overline{\sigma}(\Delta A) < \inf_{\omega\geq 0} \underline{\sigma}(j\omega I-A) = \min[-\mathrm{Re}\,\lambda(A)] \triangleq \mu_{QW}$$

Proof: See Appendix 4.

This implies that for the special case when A is normal that theorem 1 gives the same bounds as μ_{PW1}, μ_{PW2}, μ_{YW}, and μ_{LW} which in fact are necessary. This was not recognized in [6], [8], [10].

3.2 Structured Perturbations

Consider the system (1) and assume ΔA has the structure

$$\Delta A = S_1 \Delta E S_2 \tag{9}$$

where $S_1 \in R^{n\times p}$, $\Delta E \in R^{p\times q}$, $S_2 \in R^{q\times n}$, $p\leq n$, $q\leq n$, and S_1, S_2 are known constant matrices. With no loss of generality, assume that rank $S_1=p$ or/and rank $S_2=q$, and let the elements of the perturbation matrix be denoted by $\{\Delta E_{ij}\}$, and assume that

$$| \Delta E_{ij} | \leq \epsilon_{ij}\epsilon \tag{10}$$

where $\epsilon_{ij} \geq 0$ are given, and $\epsilon > 0$ is unknown.

Such a structured class of perturbation matrices includes those of [9], [10] as a special case, and occurs in the analysis of control systems. For example, perturbations of sensors/actuators of a closed loop system can be represented in the form of (9).

The following bound on ϵ such that $A+\Delta A$ remains stable for all perturbations ΔA of the type (9) is given by the following theorem, whose proof is given in Appendix 5.

Theorem 4: Given the class of perturbations ΔA described by (9), (10), then $A+\Delta A$ is stable if

$$\epsilon < \frac{1}{\sup_{\omega\geq 0} \pi[\,| S_2(j\omega-A)^{-1}S_1 |\, U]} \triangleq \mu_Q \tag{11}$$

where $U \in R^{q\times p}$ is a matrix with elements given by $u_{ij} = \epsilon_{ij}$.

Remarks

1. As $\omega\to\infty$, $||(j\omega I-A)^{-1}||_p \to 0$, which implies the supremum only needs to be considered over a finite interval.
2. The computations required to determine μ_Q are numerically well defined.

3. If the choice of $S_1=I_n$, $S_2=I_n$, $\epsilon_{ij}=1$, $\forall i,j$ is made in (11), then this case may be considered as an unstructured case with a norm defined to be equal to the maximum modulus of the elements of ΔA.
4. Theorem 4 is a generalization of the class of perturbations considered in Result 4 [10], and the following examples show that the bounds μ_Q obtained are less conservative than the bounds obtained by using Result 4.

4. Numerical Examples

Example 1: The following matrix was considered in [10]:

$$A = \begin{bmatrix} -1 & -0.25 \\ 0.5 & -1.2 \end{bmatrix}$$

It is required to compute the stability robust bound for unstructured perturbations. [10] gave $\mu_{PW1}=1.0025$, $\mu_{LW}=1.0025$, $\mu_{YW}=1.0$. Theorem 1 gives $\mu_{QW}=1.0281$ which is a tighter bound than the previously reported ones.

Example 2: Patel and Toda [8] considered a system obtained by linear quadratic optimal control theory and examined the stability robust bound for unstructured perturbations of the following matrix:

$$A = \begin{bmatrix} -0.201 & 0.755 & 0.351 & -0.075 & 0.033 \\ -0.149 & -0.696 & -0.160 & 0.110 & -0.048 \\ 0.081 & 0.004 & -0.189 & -0.003 & 0.001 \\ -0.173 & 0.802 & 0.251 & -0.804 & 0.056 \\ 0.092 & -0.467 & -0.127 & 0.075 & -1.162 \end{bmatrix}$$

They computed μ_{PW1} and the result is $\mu_{PW1}=0.077$. Using theorem 1, we obtain $\mu_{QW}=0.1116$. The improvement is 45%.

Example 3: Yedavalli [10] considered the structured perturbation problem for the matrix:

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

Table 1 gives bounds of perturbations when the possible perturbed elements of A have different combinations. Table 1 also gives exact bounds which provide necessary and sufficient conditions for stability robustness. These exact bounds are difficult to compute generally but in the 2×2 case, they can be obtained by observation. Table 1 shows that the new bounds are a significant improvement over the old ones and are close to the exact bounds.

Example 4: This example shows that it is possible for the new bound μ_Q to reach infinity, whereas the previous bounds remain finite. Consider the matrix

$$A = \begin{bmatrix} -8 & 0 \\ 0 & -1 \end{bmatrix}$$

subject to structured perturbations. Then Table 2 provides a comparison between the results of the new bound μ_Q and the previous bound μ_Y [10]. Note that all the results obtained by the new bound are exact bounds.

Example 5: This example demonstrates the application of theorem 3 to a system which is controlled. Given the system [4]:

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 7 & 8 \\ 12 & 14 \end{pmatrix} u\ ,\ \ y = \begin{pmatrix} 7 & -8 \\ -6 & 7 \end{pmatrix} x \qquad (12)$$

with control $u=\begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix} y$ where $k_1=1$, $k_2=1$, assume that the controller gains are subject to perturbations such that $|\Delta k_1|<\frac{\epsilon}{2}$, $|\Delta k_1|<\epsilon$. It is desired to determine a value of ϵ which guarantees closed loop stability. In this case, for the closed loop sytem

$$A_c = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 7 & 8 \\ 12 & 14 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & -8 \\ -6 & 7 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \qquad (13)$$

$$\Delta A_c = \begin{pmatrix} 7 & 8 \\ 12 & 14 \end{pmatrix} \begin{pmatrix} \Delta k_1 & 0 \\ 0 & \Delta k_2 \end{pmatrix} \begin{pmatrix} 7 & -8 \\ -6 & 7 \end{pmatrix}$$

and so on letting $U=\begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$, $S_1=\begin{pmatrix} 7 & 8 \\ 12 & 14 \end{pmatrix}$, $S_2=\begin{pmatrix} 7 & -8 \\ -6 & 7 \end{pmatrix}$, the application of theorem 4 gives $\mu_Q=0.0816$. Thus the closed loop system remains stable $\forall \Delta k_1$, $\forall \Delta k_2$ such that $|\Delta k_1|<0.0408$, $|\Delta k_2|<0.0816$, i.e. the closed loop system can tolerate 4% and 8% gain changes respectively. This result is consistent with the type of results obtained in [4], which showed that the system (12) has a very small gain-margin tolerance.

4. Conclusions

Two improved bounds on perturbations of asymptotically stable linear time-invariant systems are obtained; both structured and unstructured perturbations are considered. The new bounds are shown to be a significant improvement over recent ones reported, and apply to more general situations. For some special cases, e.g. when the nominal state matrix is normal, the bound for unstructured perturbations is actually an exact bound.

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Appendix 1: Proof of Lemma 2

If $||R^{-1}\Delta A(sI-A)^{-1}R||_p<1$. Then

$$\det[I-R^{-1}\Delta A(sI-A)^{-1}R]\neq 0$$

which is equivalent to

$$\det[I-\Delta A(sI-A)^{-1}]\neq 0$$

which can be rewritten as

$$\det[sI-A-\Delta A]\neq 0$$

i.e., s is not an eigenvalue of $A+\Delta A$.

The same proof can be applied to the case when $||R^{-1}(sI-A)^{-1}\Delta AR||<1$.

Appendix 2: Proof of Lemma 2

Lemma 2 can be proved by contradiction.

The eigenvalues of a matrix are continuous in its elements. So that as α goes from 0 to 1 continuously, the eigenvalues of $A+\alpha\Delta A$ will vary from the eigenvalues of A to those of $A+\Delta A$ continuously. In the case of A being stable, suppose $A+\Delta A$ has an eigenvalue in the closed right half complex plane; then there exists a $\alpha:0<\alpha\leq 1$, such that $A+\alpha\Delta A$ has an eigenvalue on the imaginary axis. By lemma 1, this contradicts the condition in lemma 2. Also since A, ΔA, R are real, then the conditions of lemma 2 only need to be satisfied for $\omega\geq 0$.

Appendix 3: Proof of Theorem 2

From the Lyapunov equation (3), we obtain

$$(-j\omega I-A')P+P(j\omega I-A)=2I \tag{1a}$$

Pre- and post-multiplying by $(-j\omega I-A')^{-1}$ and $(j\omega I-A)^{-1}$ respectively, (1a) becomes:

$$P(j\omega I-A)^{-1}+(-j\omega I-A')^{-1}P=2(-j\omega I-A')^{-1}(j\omega I-A)^{-1} \tag{2a}$$

Let $(j\omega I-A)^{-1}=G(j\omega)$, then $(-j\omega I-A')^{-1}=G^*(j\omega)$, so that (2a) becomes:

$$PG(j\omega)+G^*(j\omega)P=2G^*(j\omega)G(j\omega)$$

which implies that:

$$2\bar{\sigma}^2(G(j\omega))\leq 2\bar{\sigma}(P)\cdot\bar{\sigma}[G(j\omega)]$$

or that

$$\bar{\sigma}[G(j\omega)]\leq\bar{\sigma}(P)$$

Hence it is concluded that:

$$\inf_{\omega\geq 0}\underline{\sigma}(j\omega I-A)=\inf_{\omega\geq 0}\frac{1}{\bar{\sigma}[G(j\omega)]}\geq\frac{1}{\bar{\sigma}(P)}$$

Appendix 4: Proof of Theorem 3

If A is normal, it can be represented as $A=T^{-1}\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)T$ where T is unitary and λ_i, $i=1,\ldots,n$ are the eigenvalues of A. Then

$$\begin{aligned}\inf_{\omega\geq 0}\underline{\sigma}[j\omega I-A]&=\inf_{\omega\geq 0}\underline{\sigma}\{T^{-1}\mathrm{diag}[j\omega-\lambda_1,j\omega-\lambda_2,\ldots,j\omega-\lambda_n]T\}\\&=\inf_{\omega\geq 0}\underline{\sigma}\{\mathrm{diag}[j\omega-\lambda_1,j\omega-\lambda_2,\ldots,j\omega-\lambda_n]\}\\&=\min_{i=1,\ldots,n}[|\mathrm{Re}\,\lambda_i|]=\min[-\mathrm{Re}\,\lambda(A)]\end{aligned}$$

which proves sufficiency of the result.

If $||\Delta A||_p=\mu_{QW}=\min[-\mathrm{Re}\,\lambda(A)]$, let $\Delta A=\min[-\mathrm{Re}\,\lambda(A)]\cdot I$. Then $A+\Delta A=T^{-1}[\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)+\min|\mathrm{Re}\,\lambda(A)|\cdot I]T$ which is unstable. This proves necessity of the result.

Appendix 5: Proof of Theorem 4

The following definition and preliminary results are required in order to prove theorem 4.

A matrix $A\in R^{n\times n}$ is called non-negative (denoted by $A\geq 0$) if all the elements of A are non-negative. The following result is obtained from [11].

Lemma A1: A non-negative matrix $A\in R^{n\times n}$ always has a non-negative eigenvalue (denoted by $\pi(A)$) that is greater than or equal to the moduli of all the other eigenvalues of A. The right and left eigenvector of A corresponding to $\pi(A)$ are non-negative.

The following two results are obtained from Bauer [12].

Lemma A2: Let $B\in R^{n\times m}$, $C\in R^{m\times n}$ be non-negative matrices and let $D_1\in R^{n\times n}$, $D_2\in R^{m\times m}$ be diagonal matrices. Then

$$\inf_{D_1,D_2}\{||D_1BD_2||_p\cdot||D_2^{-1}CD_1^{-1}||_p\}=\pi(BC)=\pi(CB) \tag{3a}$$

Bauer proved this result for the case when $B>0$, $C>0$ and "min" is used instead of "inf"; however his proof can be modified to allow $B\geq 0$, $C\geq 0$.

Lemma A3: Let $A\in\mathbb{C}^{m\times n}$; then in general $||A||_p\leq|||A|||_p$, but the equality holds for the following cases:

i) $\exists E_1\in\mathbb{C}^{m\times m}$, $E_2\in\mathbb{C}^{n\times n}$ with $|E_1|=I_m$, $|E_2|=I_n$ such that $E_1AE_2=|A|$. Such a matrix A is called a checkerboard matrix.

ii) $p=1$ or $p=\infty$.

From lemma A2 and lemma A3, the following corollary is obtained:

Corollary 1: Let $B\in\mathbb{C}^{m\times n}$, $C\in\mathbb{C}^{n\times m}$ and D_1, D_2 be diagonal matrices; then

$$\begin{aligned}\inf_{D_1,D_2}\{||D_1BD_2||_p||D_2^{-1}CD_1^{-1}||_p\}&\leq\pi(|B||C|)\\&=\pi(|C||B|)\end{aligned} \tag{4a}$$

The equalities hold if B and C are checkerboard matrices or if a 1-norm or ∞-norm is used.

Proof of Theorem 4: From lemma 2, $A+\Delta A$ is stable, if $\exists R(j\omega)$ such that

$$||RS_1\Delta ES_2(j\omega I-A)^{-1}R^{-1}||_p<1,\quad\forall\Delta E\in\mathbf{E},\ \forall\omega\geq 0 \tag{5a}$$

where $\mathbf{E}\triangleq\{\Delta E\mid|\Delta E_{ij}|\leq\epsilon_{ij}\epsilon\}$.

Assume initially that S_1 is left-invertible; then there exists a nonsingular matrix T such that

$$TS_1=\begin{bmatrix}I\\0\end{bmatrix} \tag{6a}$$

Let $R=D_1(j\omega)T$, where $D_1(j\omega)$ is a diagonal matrix. Then (5a) is true if $\exists D_1(j\omega)$ such that

$$||D_1\begin{bmatrix}\Delta E\\0\end{bmatrix}S_2(j\omega I-A)^{-1}T^{-1}D_1^{-1}||_p<1,\ \forall\Delta E\in\mathbf{E},\ \forall\omega\geq 0 \tag{7a}$$

which is true if $\exists D_1(j\omega),D_2(j\omega)$ which are diagonal such that:

$$||D_1\begin{pmatrix}\Delta E\\0\end{pmatrix}D_2||_p\ ||D_2^{-1}S_2(j\omega I-A)^{-1}T^{-1}D_1^{-1}||_p<1, \tag{8a}$$

$$\forall\Delta E\in\mathbf{E},\ \forall\omega\geq 0$$

Since $\exists E_1,E_2$ with $|E_1|=I$, $|E_2|=I$ so that

$$E_1D_1\begin{bmatrix}\epsilon U\\0\end{bmatrix}D_2E_2=|D_1\begin{bmatrix}\epsilon U\\0\end{bmatrix}D_2| \tag{9a}$$

then from lemma A3, we conclude that

$$||D_1 \begin{bmatrix}\epsilon U\\ 0\end{bmatrix} D_2||_p = ||\, |D_1 \begin{bmatrix}\epsilon U\\ 0\end{bmatrix} D_2|\, ||_p \tag{10a}$$

which implies that:

$$||D_1 \begin{bmatrix}\epsilon U\\ 0\end{bmatrix} D_2||_p = ||\, |D_1 \begin{bmatrix}\epsilon U\\ 0\end{bmatrix} D_2|\, ||_p \geq ||D_1 \begin{bmatrix}\Delta E\\ 0\end{bmatrix} D_2||_p \tag{11a}$$

is true $\forall \Delta E \in \mathbf{E}$. Thus condition (8a) is satisfied if $\exists$ diagonal matrices $D_1(j\omega)$, $D_2(j\omega)$ such that

$$||D_1 \begin{bmatrix}U\\ 0\end{bmatrix} \epsilon D_2||_p\, ||D_2^{-1} S_2(j\omega I - A)^{-1} T^{-1} D_1^{-1}||_p < 1\,,\ \forall \omega \geq 0 \tag{12a}$$

which is true if $\exists$ diagonal $D_1(j\omega)$, $D_2(j\omega)$ such that:

$$\epsilon \cdot \inf_{D_1, D_2} ||D_1 \begin{bmatrix}U\\ 0\end{bmatrix} D_2||_p\, ||D_2^{-1} S_2(j\omega I - A)^{-1} T^{-1} D_1^{-1}||_p < 1\,, \quad \forall \omega \geq 0 \tag{13a}$$

which will be true, from corollary 1, if the following condition is satisfied:

$$\epsilon \cdot \pi \left\{ \begin{bmatrix}U\\ 0\end{bmatrix} |S_2(j\omega I - A)^{-1} T^{-1}| \right\} < 1\,,\ \forall \omega \geq 0 \tag{14a}$$

Now let $T^{-1} = [V_1, V_2]$; then this implies from (6a) that $V_1 = S_1$ which implies that

$$\pi \left\{ \begin{bmatrix}U\\ 0\end{bmatrix} |S_2(j\omega I - A)^{-1} T^{-1}| \right\} = \pi \left\{ \begin{bmatrix} U\, |S_2(j\omega I - A)^{-1} [S_1, V_2]| \\ 0 \end{bmatrix} \right\}$$

$$= \pi[U\, |S_2(j\omega I - A)^{-1} S_1|] = \pi[|S_2(j\omega I - A)^{-1} S_1|\, U]$$

Thus condition (14a) is equivalent to the condition

$$\epsilon < 1/\sup_{\omega \geq 0} \pi[|S_2(j\omega I - A)^{-1} S_1|\, U] \triangleq \mu_Q \tag{15a}$$

which proves theorem 4 for the case S_1 is left-invertible. Assume now that S_1 is not left invertible, but that S_2 is right-invertible; then the proof can be repeated in exactly the same way, on noting from lemma 2, that $A + \Delta A$ is stable, if $\exists R(j\omega)$ such that:

$$||R(j\omega I - A)^{-1} S_1 \Delta E S_2 R^{-1}||_p < 1\,,\ \forall \Delta E \in \mathbf{E},\ \forall \omega \geq 0$$

Table 1: Comparison of Results Obtained for the Stability Robust Bound for Structured Perturbations of Example 3 for the Case when $A = \begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix}$

Perturbed Elements of A	$a_{11}a_{12}$ $a_{21}a_{22}$	a_{11}	a_{12}	a_{21}	a_{22}	$a_{11}a_{12}$	$a_{11}a_{22}$	$a_{11}a_{21}$	$a_{12}a_{21}$	$a_{12}a_{22}$	$a_{21}a_{22}$	$a_{11}a_{12}$ a_{21}	$a_{11}a_{12}$ a_{22}	$a_{11}a_{21}$ a_{22}	$a_{12}a_{21}$ a_{22}
U	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	$\begin{bmatrix}0&1\\1&1\end{bmatrix}$
μ_y	0.236	1.657	1.657	0.655	0.396	1.0	0.382	0.48	0.5	0.324	0.3027	0.317	0.311	0.273	0.256
μ_Q	0.3295	3.0000	2.0000	1.0000	0.6667	1.5201	0.5612	0.9150	0.8108	0.5000	0.4000	0.6848	0.4486	0.3714	0.3528
exact bounds	0.3333	3	2	1	0.6667	2	0.5616	1	1	0.5	0.4	1	0.4495	0.3723	0.3542

Table 2: Comparison of bounds for example 4

Perturbed elements	$a_{11}a_{12}$ $a_{21}a_{22}$	$a_{11}a_{22}$	$a_{12}a_{21}$	a_{11}	a_{22}	a_{12}	a_{21}
μ_Y	0.784	1	1.777	8	1	16	2
μ_Q	0.889	1	2.8284	8	1	∞	∞