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Null controllable region of LTI discrete-time systems with input saturation☆

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Abstract

We present a formula for the extremes of the null controllable region of a general LTI discrete-time system with bounded inputs. For an nth order system with only real poles (not necessarily distinct), the formula is simplified to an elementary matrix function, which in turn shows that the set of the extremes coincides with a set of trajectories of the time-reversed system under bang–bang controls with $n-2$ or less switches. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Two fundamental issues associated to the control of a system are its controllability and stabilizability. Since all the practical control inputs are bounded, the constrained controllability was formulated earlier than the nonconstrained one. While the solution to the latter problem has been well known for several decades, there are still continuing efforts towards obtaining simple and easily implementable solution to the former problem (see, e.g., Bernstein & Michel, 1995; Hu & Lin, 2001 and the references therein).

The null controllable region, denoted as $\mathscr{C}$, also called the controllable set or the reachable set $\mathscr{R}$ (of the time-reversed system), is defined to be the set of states that can be steered to the origin in a finite number of steps by using constrained controls. Clearly, the domain of attraction $\mathscr{S}$ of the origin under any control law must lie within $\mathscr{C}$. A practical control problem is to design a controller such that $\mathscr{S}$ is close to $\mathscr{C}$. But as a first step, we must know how big is $\mathscr{C}$. Efforts to characterize this $\mathscr{C}$ have been made since the 1950s and numerous results have been developed on this topic (see, e.g., D'Alessandra & De Santis, 1992; Fisher & Gayek, 1987; Cwikel & Gutman, 1986; Gutman & Cwikel, 1987; Kalman, 1957; Keerthi & Gilbert, 1987; Lasserre, 1991; LeMay, 1964; Schmitendorf & Barmish, 1980; Sontag, 1984; Til & Schmitendorf, 1986). The situation where $\mathscr{C}$ is the whole state space was made clear, e.g., in Hájek (1991), Lee and Markus (1967), LeMay (1964), Schmitendorf and Barmish (1980) and Sontag (1984). For the case where $\mathscr{C}$ is not the whole state space, there only exist various numerical methods for approximate characterization of $\mathscr{C}$ except for second-order systems with complex eigenvalues. With $\mathscr{C}(K)$ the set of states that can be steered to the origin at step K, $\mathscr{C}$ is typically approximated by $\mathscr{C}(K)$ with K sufficiently large; for a fixed K, $\mathscr{C}(K)$ is characterized in terms of its boundary hyperplanes or vertices which are usually computed via linear programming. As K is increased, the computational burden is more intensive and it is more difficult to implement the control based on on-line computation. An exception is in Lasserre (1991), where a closed-form expression for $\mathscr{C}(K)$ was provided with an algorithm of polynomial complexity. Another nice result was obtained in Fisher and Gayek (1987) where an explicit formula to compute the extremes (or vertices) of $\mathscr{C}$ was provided for second-order systems with complex eigenvalues. An interesting interpretation of this set of extremes is that they form a special trajectory of the system under a periodic bang–bang control.

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Recently, we obtained explicit analytical descriptions of the boundary of the null controllable region for continuous-time systems in Hu, Lin and Qiu (2002). As usual, one might anticipate that the results in the continuous-time setting have their counterparts in the discrete-time setting. Indeed, we will show in this note that through some interesting links, some of the null controllability results in Hu et al. (2002) have natural discrete-time counterparts, though the development is more technically involved.

2. Problem statement and notation

Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $x(k) \in \mathbf{R}^n$ is the state and $u(k) \in \mathbf{R}^m$ is the control. A control signal u is said to be *admissible* if $\|u(k)\|_\infty \leqslant 1$ for all integer $k \geqslant 0$. In this note, we are interested in the control of system (1) by using admissible controls. Our concern is the set of states that can be steered to the origin by admissible controls.

Definition 1.

a. A state x_0 is said to be null controllable at a given step K if there exists an admissible control u such that the time response x of the system satisfies $x(0) = x_0$ and $x(K) = 0$. A state x_0 is said to be null controllable if it is null controllable at some $K < \infty$.
b. The set of all states null controllable at K is called the null controllable region of the system at K and is denoted by $\mathscr{C}(K)$. The set of all null controllable states is called the null controllable region of the system and is denoted by $\mathscr{C}$.

In this note, we say that a matrix A is *semi-stable* if it has no eigenvalues outside of the unit circle and A is *anti-stable* if all of its eigenvalues are outside of the unit circle.

Proposition 1(Hájek, 1991; Lee & Markus, 1967; Sontag, 1984). *Assume that (A,B) is controllable.*

a. *If A is semi-stable, then $\mathscr{C} = \mathbf{R}^n$.*
b. *If A is anti-stable, then $\mathscr{C}$ is a bounded convex open set containing the origin.*
c. *If*
$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$
with $A_1 \in \mathbf{R}^{n_1 \times n_1}$ being anti-stable and $A_2 \in \mathbf{R}^{n_2\times n_2}$ being semi-stable, and B is partitioned as $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ accordingly, then $\mathscr{C} = \mathscr{C}_1 \times \mathbf{R}^{n_2}$, where $\mathscr{C}_1$ is the null controllable region of the anti-stable system $x_1(k+1) = A_1 x_1(k) + B_1 u(k)$.

Because of this proposition, we can concentrate on the study of null controllable regions of anti-stable systems. For such systems, $\mathscr{C}$ can be approximated by $\mathscr{C}(K)$ for sufficiently large K. Since A is nonsingular in this case, by Definition 1, we have

$$\mathscr{C} = \bigcup_{K \in [0,\infty)} \mathscr{C}(K) \tag{2}$$

and

$$\mathscr{C}(K) = \left\{ -\sum_{i=0}^{K-1} A^{-i-1} Bu(i) \colon \|u\|_\infty \leqslant 1 \right\}. \tag{3}$$

If $B = [b_1 \ \cdots \ b_m]$ and the null controllable region of the system $x(k+1) = Ax(k) + b_i u_i(k)$, $i = 1, \ldots, m$, is $\mathscr{C}_i$, then it follows from (2) and (3) that

$$\mathscr{C} = \sum_{i=1}^m \mathscr{C}_i$$
$$= \{x_1 + x_2 + \cdots + x_m \colon x_i \in \mathscr{C}_i,\ i = 1, 2, \ldots, m\}.$$

Hence we can begin our study of the null controllable regions with systems having only one input.

In summary, we will assume in the study of null controllable regions that (A,B) is controllable, A is anti-stable, and $m = 1$.

In many situations, it may be more convenient to study the controllability of a system through the reachability of its time reversed system. The time-reversed system of (1) is

$$z(k+1) = A^{-1} z(k) - A^{-1} Bv(k). \tag{4}$$

Definition 2. For system (4)

a. A state z_f is said to be reachable in a given step K if there exists an admissible control v such that the time response z of system (4) satisfies $z(0)=0$ and $z(K)=z_f$. A state z_f is said to be reachable if it is reachable in some $K < \infty$.
b. The set of all states reachable in K steps is called the reachable region at K and is denoted by $\mathscr{R}(K)$. The set of all reachable states is called the reachable region and is denoted by $\mathscr{R}$.

It is a known result that $\mathscr{C}(K)$ and $\mathscr{C}$ of (1) are the same as $\mathscr{R}(K)$ and $\mathscr{R}$ of (4) (see, e.g., Macki & Strauss, 1982). To avoid confusion, we will reserve the notation x, u, $\mathscr{C}(K)$, and $\mathscr{C}$ for the original system (1), and reserve z, v, $\mathscr{R}(K)$, and $\mathscr{R}$ for the time-reversed system (4).

To proceed we need more notation. For a convex set $\mathscr{X} \subset \mathbf{R}^n$, a point $x_0 \in \bar{\mathscr{X}}$ is said to be an extremal point (or simply, an extreme) of $\mathscr{X}$ if there exists a vector $c \in \mathbf{R}^n$ such that

$$c^{\mathrm{T}} x_0 > c^{\mathrm{T}} x, \quad \forall x \in \bar{\mathscr{X}} \setminus \{x_0\}.$$

We use $\mathrm{Ext}(\mathscr{X})$ to denote the set of all the extremal points of $\mathscr{X}$. If $\mathscr{X}$ has finite number of extremes, then $\mathscr{X}$ is a polyhedron and an extreme is also called a vertex.

With K_1, K_2 integers, for convenience, we use $[K_1, K_2]$ to denote the set of integers $\{K_1, K_1+1, \ldots, K_2\}$.

3. Extremes of the null controllable region

We have assumed in Section 2 that A is anti-stable, (A,B) is controllable, and $m=1$. Since B is now a column vector, we rename it as b for convenience. From (2) and (3), $\mathscr{C}(K)$, $\mathscr{R}(K)$, $\bar{\mathscr{C}}$ and $\bar{\mathscr{R}}$ can be written as

$$\mathscr{C}(K)=\mathscr{R}(K)$$

$$=\left\{-\sum_{\ell=0}^{K-1}A^{-(K-\ell)}bv(\ell):\right.$$

$$\left.|v(\ell)|\leqslant 1,\ \forall \ell\in[0,K-1]\right\}$$

and

$$\bar{\mathscr{C}}=\bar{\mathscr{R}}=\left\{-\sum_{\ell=0}^{\infty}A^{-(K-\ell)}bv(\ell):\ |v(\ell)|\leqslant 1,\ \forall \ell\geqslant 0\right\}.$$

It is well known that $\mathscr{C}(K),\mathscr{R}(K),\bar{\mathscr{C}}$ and $\bar{\mathscr{R}}$ are all convex and that $\mathscr{C}(K)$ and $\mathscr{R}(K)$ are polyhedrons (see, e.g., D'Alessandra & De Santis, 1992). In some special cases, $\bar{\mathscr{R}}$ (or $\bar{\mathscr{C}}$) could also be a polyhedron of finite many extremal points. But in general, $\bar{\mathscr{R}}$ has infinitely many extremal points. In any case, $\bar{\mathscr{R}}$ is the convex hull of $\mathrm{Ext}(\mathscr{R})$, the set of all the extremal points of $\mathscr{R}$. In view of this, it suffices to characterize $\mathrm{Ext}(\mathscr{R})$.

Definition 3. An admissible control v is said to be an extremal control on $[0,K]$ if the response $z(k)$ of system (4), with $z(0)=0$, is in $\mathrm{Ext}(\mathscr{R}(k))$ for all $k\in[0,K]$.

Lemma 1. *If $z_f\in\mathrm{Ext}(\mathscr{R}(K))$, and v is an admissible control that steers the state from the origin to z_f at step K, then v is an extremal control on $[0,K]$.*

This lemma is obvious. Note that if $z(k_1)\notin\mathrm{Ext}(\mathscr{R}(k_1))$ for some $k_1>0$, then $z(k)$ will not be in $\mathrm{Ext}(\mathscr{R}(k))$ for any $k>k_1$ under any admissible control. Denote the set of extremal controls on $[0,K]$ as $\mathscr{E}_c(K)$. It then follows that

$$\mathrm{Ext}(\mathscr{R}(K))=\left\{-\sum_{\ell=0}^{K-1}A^{-(K-\ell)}bv(\ell):\ v\in\mathscr{E}_c(K)\right\}. \qquad (5)$$

Lemma 2 (Qian & Song, 1980). *An admissible control v^* is an extremal control on $[0,K]$ for system (4) if and only if there is a vector $c\in\mathbf{R}^n$ such that*

$$c^{\mathrm{T}}A^kb\neq 0,\quad \forall k\in[0,K-1]$$

and

$$v^*(k)=\mathrm{sgn}(c^{\mathrm{T}}A^kb),\quad \forall k\in[0,K-1].$$

This lemma says that an extremal control is a bang–bang control, i.e., a control only takes value 1 or -1. Because of this lemma, we can write $\mathscr{E}_c(K)$ as

$$\mathscr{E}_c(K)=\{v(k)=\mathrm{sgn}(c^{\mathrm{T}}A^kb):$$

$$c^{\mathrm{T}}A^kb\neq 0,\ \forall k\in[0,K-1]\}.$$

Consequently, it follows from (5) that

$$\mathrm{Ext}(\mathscr{R}(K))=\left\{-\sum_{\ell=0}^{K-1}A^{-(K-\ell)}b\,\mathrm{sgn}(c^{\mathrm{T}}A^{\ell}b):\right.$$

$$\left.c^{\mathrm{T}}A^{\ell}b\neq 0,\ \forall \ell\in[0,K-1]\right\}. \qquad (6)$$

Writing $c^{\mathrm{T}}A^{\ell}$ as $c^{\mathrm{T}}A^KA^{-(K-\ell)}$ and replacing $c^{\mathrm{T}}A^K$ with c^{T} and $K-\ell$ with ℓ, we have

$$\mathrm{Ext}(\mathscr{R}(K))=\left\{-\sum_{\ell=1}^{K}A^{-\ell}b\,\mathrm{sgn}(c^{\mathrm{T}}A^{-\ell}b):\right.$$

$$\left.c^{\mathrm{T}}A^{-\ell}b\neq 0,\ \forall \ell\in[1,K]\right\}.$$

Letting K go to infinity, we arrive at the following result.

Theorem 1.

$$\mathrm{Ext}(\mathscr{C})=\mathrm{Ext}(\mathscr{R})$$

$$=\left\{-\sum_{\ell=1}^{\infty}A^{-\ell}b\,\mathrm{sgn}(c^{\mathrm{T}}A^{-\ell}b):\right.$$

$$\left.c^{\mathrm{T}}A^{-\ell}b\neq 0,\ \forall \ell\geqslant 1\right\}.$$

We note that in the above theorem, the infinite summation always exists since A is anti-stable.

Since $\mathrm{sgn}(c^{\mathrm{T}}A^{-\ell}b)=\mathrm{sgn}(\gamma c^{\mathrm{T}}A^{-\ell}b)$ for any positive number γ, this formula shows that $\mathrm{Ext}(\mathscr{R})$ can be determined from the surface of a unit ball. It should be noted that each extreme corresponds to a region of vectors c in the surface of the unit ball rather than just one point. This formula provides a straightforward method for computing the extremal points of the null controllable region and no optimization is involved. In the following, we will give a more attractive formula for computing the extremal points of the null controllable region for systems with only real eigenvalues.

In comparison with the continuous-time systems, a little more technical consideration is needed here. This difference can be illustrated with a simple example. If $A=-2$, then $c^{\mathrm{T}}A^kb$ changes the sign at each k. Hence, if A has some negative real eigenvalues, an extremal control can have infinitely many switches. This complexity can be avoided through a technical manipulation. Suppose that A has only real eigenvalues including some negative ones. Consider

$$y(k+1)=A^2y(k)+[Ab\quad b]w(k), \qquad (7)$$

where $y(k)=x(2k)$ and

$$w(k)=\begin{bmatrix} u(2k) \\ u(2k+1) \end{bmatrix}.$$

Then the null controllable region of (1) is the same as that of (7), which is the sum of the null controllable regions of the following two subsystems:

$$y(k+1)=A^2y(k)+Abw_1(k)$$

and

$$y(k+1)=A^2y(k)+bw_2(k)$$

both of which have positive real eigenvalues. Therefore, without loss of generality, we further assume that A has only positive real eigenvalues. Under this assumption, it is known that any extremal control can have at most $n-1$ switches (Qian & Song, 1980). Here we will show that the converse is also true. That is, any bang–bang control with $n-1$ or less switches is an extremal control.

Lemma 3. *For system* (4), *suppose that A has only positive real eigenvalues. Then,*

(a) *an extremal control has at most $n-1$ switches*;
(b) *any bang–bang control with $n-1$ or less switches is an extremal control.*

Proof. Since A has only positive real eigenvalues, systems (1) and (4) can be considered as the discretized systems resulting from

$$\dot{x}(t)=A_cx(t)+b_cu(t) \tag{8}$$

and

$$\dot{z}(t)=-A_cz(t)-b_cv(t) \tag{9}$$

with sampling period h, where A_c has only positive real eigenvalues. Thus,

$$A=\mathrm{e}^{A_ch}, \quad b=A_c^{-1}(\mathrm{e}^{A_ch}-I)b_c$$

and

$$c^{\mathrm{T}}A^kb=c^{\mathrm{T}}A_c^{-1}(\mathrm{e}^{A_ch}-I)\mathrm{e}^{A_chk}b_c.$$

By Lemma 2.4.1 of Hu and Lin (2001), the continuous function in t, $c^{\mathrm{T}}A_c^{-1}(I-\mathrm{e}^{A_ch})\mathrm{e}^{A_ct}b_c$, changes sign at most $n-1$ times, it follows that $\mathrm{sgn}(c^{\mathrm{T}}A^kb)$ has at most $n-1$ switches.

To prove (b), let $\mathscr{R}_c(T)$ be the reachable region of the continuous-time system (9) at time T. Suppose that v^* is a discrete-time bang–bang control with $n-1$ or less switches, and that the state of system (4) at step K under the control v^* is z^*; equivalently, a corresponding continuous-time bang–bang control will drive the state of system (9) from the origin to z^* at time Kh. It follows from Theorem 2.6.1 in Hu and Lin (2001) that z^* belongs to $\partial\mathscr{R}_c(Kh)$ of the continuous-time system (9). Recall from the same theorem that $\mathscr{R}_c(Kh)$ is strictly convex, i.e., every boundary point of $\mathscr{R}_c(Kh)$ is an extremal point, it follows that z^* is an extremal point of $\mathscr{R}_c(Kh)$. Since $z^*\in\mathscr{R}(K)\subset\mathscr{R}_c(Kh)$, we must have $z^*\in\mathrm{Ext}(\mathscr{R}(K))$. Therefore, v^* is an extremal control. □

It follows from the above lemma that the set of extremal controls on $[0,K]$ can be described as follows:

$$\mathscr{E}_c(K)=\left\{\pm v\colon v(k)=\begin{cases} 1, & 0\leqslant k<k_1, \\ (-1)^i, & k_i\leqslant k<k_{i+1}, \\ (-1)^{n-1}, & k_{n-1}\leqslant k\leqslant K-1, \end{cases} \quad 0\leqslant k_1\leqslant\cdots\leqslant k_{n-1}\leqslant K-1\right\}.$$

Notice that we allow $k_i=k_{i+1}$ in the above expression to include all the bang–bang controls with $n-1$ or less switches.

For a square matrix X, it can be easily verified that if $I-X$ is nonsingular, then

$$\sum_{k=k_1}^{k_2-1}X^k=(X^{k_1}-X^{k_2})(I-X)^{-1}.$$

By applying this equality we have that, if $v\in\mathscr{E}_c(K)$, then

$$\begin{aligned}\sum_{\ell=0}^{K-1}A^{-(K-\ell)}bv(\ell) &= A^{-K}\left[\sum_{\ell=0}^{k_1-1}A^\ell-\sum_{\ell=k_1}^{k_2-1}A^\ell+\sum_{\ell=k_2}^{k_3-1}A^\ell -\cdots+(-1)^{n-1}\sum_{\ell=k_{n-1}}^{K-1}A^\ell\right]b \\ &= A^{-K}[I-2A^{k_1}+2A^{k_2} -\cdots+(-1)^{n-1}2A^{k_{n-1}}+(-1)^nA^K](I-A)^{-1}b.\end{aligned}$$

It follows from (5) that

$$\begin{aligned}\mathrm{Ext}(\mathscr{R}(K)) &= \left\{-\sum_{\ell=0}^{K-1}A^{-(K-\ell)}bv(\ell)\colon v\in\mathscr{E}_c(K)\right\} \\ &= \left\{\pm\left[A^{-K}+2\sum_{i=1}^{n-1}(-1)^iA^{-\ell_i}+(-1)^nI\right](I-A)^{-1}b\colon K\geqslant\ell_1\geqslant\cdots\geqslant\ell_{n-1}\geqslant 1\right\}.\end{aligned}$$

By letting K go to infinity, we arrive at the following theorem.

Theorem 2. *If A has only real positive eigenvalues, then*

$$\mathrm{Ext}(\mathscr{R})=\left\{\pm\left[2\sum_{i=1}^{n-1}(-1)^iA^{-\ell_i}+(-1)^nI\right](I-A)^{-1}b\colon \infty\geqslant\ell_1\geqslant\cdots\geqslant\ell_{n-1}\geqslant 1\right\}.$$

In particular, for second-order systems, we have

$$\mathrm{Ext}(\mathscr{R}) = \{\pm(2A^{-\ell} - I)(I-A)^{-1}b\colon 1 \leqslant \ell \leqslant \infty\}$$

and for third-order systems

$$\mathrm{Ext}(\mathscr{R}) = \{\pm(2A^{-\ell_1} - 2A^{-\ell_2} + I)(I-A)^{-1}b\colon 1 \leqslant \ell_2 \leqslant \ell_1 \leqslant \infty\}.$$

A more interesting interpretation of $\mathrm{Ext}(\mathscr{R})$ can be obtained after some manipulation. Let $x_e^+ := (I-A)^{-1}b$ be the equilibrium point of system (4) under the constant control $v(k) = 1$. Then for third-order systems.

$$\mathrm{Ext}(\mathscr{R}) = \left\{\pm\left[A^{-k_2}x_e^+ + \sum_{\ell=0}^{k_1-1} A^{-(k_2-\ell-1)}(-A^{-1}b)(-1) + \sum_{\ell=k_1}^{k_2-1} A^{-(k_2-\ell-1)}(-A^{-1}b)(+1)\right]\colon 1 \leqslant k_1 \leqslant k_2 \leqslant \infty\right\}.$$

We see that one half of $\mathrm{Ext}(\mathscr{R})$ is formed by the trajectories of (4) starting from x_e^+, first under the control of $v = -1$, and then switching to $v = +1$ at any step k_1. The other half is symmetric to the first half.

Similarly, for higher-order systems with only positive real eigenvalues, $\mathrm{Ext}(\mathscr{R}) = \mathrm{Ext}(\mathscr{C})$ can be interpreted as the set of points formed by the trajectories of (4) starting from x_e^+ or $-x_e^+$ under any bang–bang control with $n-2$ or less switches.

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