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ON ENTROPY OF CONTINUOUS-TIME PERIODIC SYSTEMS

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Abstract. A continuous-time linear periodic system has an operator-valued (lifted) transfer function, for which an entropy function can be defined. This paper relates this entropy to some linear-exponential-quadratic-Gaussian function defined in the time domain. Also addressed is the problem of minimum entropy $\mathcal{H}_\infty$ sampled-data control: For a sampled-data control system, design a digital controller to achieve a certain $\mathcal{H}_\infty$ norm bound and minimize the entropy of the closed-loop system.

Keywords. $\mathcal{H}_\infty$ control, periodic systems, sampled-data control, entropy, white noise.

1. INTRODUCTION

Sampled-data control systems have recently been viewed as periodic systems in continuous time. This viewpoint allows analysis and synthesis to be accomplished, taking into consideration of intersample behavior. Along this direction, optimal sampled-data control design was studied in the $\mathcal{H}_\infty$ framework (Bamieh and Pearson, 1992*b*; Toivonen, 1992; Kabamba and Hara, 1993; Sun *et al.*, 1993; Tadmor, 1992) and $\mathcal{H}_2$ framework (Khargonekar and Sivashankar, 1991; Bamieh and Pearson, 1992*a*) (among other references). For motivation of our work in this paper, let us observe the following two points from the recent sampled-data control literature. First, a powerful technique called lifting was developed in connection with the study of the $\mathcal{H}_\infty$ sampled-data problem; this technique applies in general to linear periodic continuous-time systems and converts them into *equivalent*, linear time-invariant (LTI), discrete-time systems, which have infinite-dimensional input and output spaces and thus can be described by operator-valued discrete-time transfer functions. Second, the solution of the sampled-data $\mathcal{H}_\infty$ control problem is obtained by converting it into an equivalent discrete-time $\mathcal{H}_\infty$ control problem, from which one can, for example, characterize the set of all sampled-data controllers achieving a certain $\mathcal{H}_\infty$ norm bound. This set of controllers in general contains infinite elements. The question arising is: How to select one controller which satisfies further good properties?

In the LTI case, the so-called central controller further minimizes a certain entropy function of the closed-loop transfer function (Mustafa and Glover, 1991; Iglesias *et al.*, 1990; Iglesias and Mustafa, 1993); this entropy function has certain interpretation in the time domain and is an upper bound for the $\mathcal{H}_2$ norm of the system. This paper is aimed at addressing these issues for general continuous-time periodic systems, treating sampled-data systems as special cases. In particular, we answer the following questions:

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- What is an entropy function for a general periodic system?
- How to interpret this entropy in the time domain?
- For sampled-data control systems, how to find the minimum entropy controller among the class of controllers that achieve a certain $\mathcal{H}_\infty$ norm bound?

Briefly, the paper is organized as follows. In the next section we define the entropy for a general periodic system and relate it to some linear-exponential-quadratic-Gaussian (LEQG) cost function in the time domain. In Section 3 we discuss minimum entropy $\mathcal{H}_\infty$ control for sampled-data systems. Finally, some concluding remarks are given in Section 4.

2. ENTROPY AND LEQG FUNCTION OF PERIODIC SYSTEMS

A sampled-data system with an LTI analog plant and LTI digital controller with sampling period σ can be viewed as a continuous-time σ-periodic system. Such systems can be lifted to get lifted transfer functions. In this section, we define an entropy for general periodic systems and then establish its stochastic interpretation in the time domain.

First, we introduce some definitions on operator-valued sequences and the associated λ-transforms. Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces and $f = \{f(k) : k = 0, 1, 2, \ldots\}$ be a sequence of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. Then

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

is an operator-valued function on some subset of $\mathcal{C}$. We say that $\hat{F}$ belongs to $\mathcal{H}_\infty(\mathcal{X}, \mathcal{Y})$ if $\hat{F}$ is analytic in $\mathcal{D}$, the open unit disk, and

$$\sup_{\lambda \in \mathcal{D}} \|\hat{F}(\lambda)\| < \infty.$$

In this case, the above left-hand side is defined to be the $\mathcal{H}_\infty$ norm of $\hat{F}$, denoted by $\|\hat{F}\|_\infty$; the operator $\hat{F}(e^{j\omega})$ is bounded for almost every $\omega \in [-\pi, \pi)$, and

$$\operatorname{ess}\sup_{\omega \in [-\pi,\pi)} \|\hat{F}(e^{j\omega})\| = \|\hat{F}\|_\infty.$$

Let $f = \{f(k) : k = 1, 2, \ldots\}$ be a sequence of Hilbert-Schmidt operators from $\mathcal{X}$ to $\mathcal{Y}$. The set of Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, $\|\cdot\|_{\mathrm{HS}}$, is a Hilbert space (Gohberg and Kreĭn, 1969). Then

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

is a Hilbert-space vector-valued function on some subset of $\mathcal{C}$. We say that $\hat{F}$ belongs to $\mathcal{H}_2(\mathcal{X}, \mathcal{Y})$ if

$$\left(\sum_{k=0}^{\infty} \|f(k)\|_{\mathrm{HS}}^2\right)^{1/2} < \infty,$$

In this case, the left-hand side above is defined to be the $\mathcal{H}_2$ norm of $\hat{F}$, denoted by $\|\hat{F}\|_2$, the operator $\hat{F}(e^{j\omega})$ is Hilbert-Schmidt for almost every $\omega \in [-\pi, \pi)$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\hat{F}(e^{j\omega})\|_{\mathrm{HS}}^2 d\omega = \|\hat{F}\|_2^2.$$

Assume $\hat{F} \in \mathcal{H}_\infty(\mathcal{X}, \mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X}, \mathcal{Y})$ and $\|\hat{F}\|_\infty < 1$. Extending the entropy definition for matrix valued analytic functions (Iglesias *et al.*, 1990; Iglesias and Mustafa, 1993), we define the entropy of $\hat{F}$ as

$$\mathcal{I}(\hat{F}) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det[I - \hat{F}^*(e^{j\omega})\hat{F}(e^{j\omega})]d\omega.$$

This entropy is well defined. Since $\hat{F}(e^{j\omega})$ is a Hilbert-Schmidt operator at almost every $\omega \in [-\pi, \pi)$, its singular values form a square-summable sequence $\{\sigma_k(e^{j\omega})\}$. Hence

$$\det[I - \hat{F}^*(e^{j\omega})\hat{F}(e^{j\omega})] = \prod_{k=1}^{\infty}[1 - \sigma_k^2(e^{j\omega})],$$

which converges to some number in $(0, 1)$ due to square-summability of $\{\sigma_k(e^{j\omega})\}$ and the fact that $\|\hat{F}\|_\infty < 1$. This also shows that $\mathcal{I}(\hat{F})$ is nonnegative.

Lemma 1 *Assume* $\hat{F} \in \mathcal{H}_\infty(\mathcal{X}, \mathcal{Y}) \cap \mathcal{H}_2(\mathcal{X}, \mathcal{Y})$ *and* $\|\hat{F}\|_\infty < 1$. *Then* $\|\hat{F}\|_2^2 \le \mathcal{I}(\hat{F})$.

The proof of this lemma follows from exactly the same argument as for the finite-dimensional case (Iglesias *et al.*, 1990).

Now let us turn back to our periodic system. Let F_a be a continuous-time, σ-periodic, causal system described by the following integral operator

$$(F_a w)(t) = \int_0^t f_a(t, \tau)w(\tau)\,d\tau.$$

We assume that f_a, the matrix-valued impulse response of F_a, is locally square-integrable, i.e., every element is square-integrable on any compact subset of $\mathcal{R}^2$. The periodicity of F_a implies $f_a(t + \sigma, \tau + \sigma) = f_a(t, \tau)$, and the causality implies that $f_a(t, \tau) = 0$ if $\tau > t$.

Lifting F_a as in (Bamieh and Pearson, 1992*b*) we get an operator sequence $\{f(k)\}$, where every $f(k)$ maps

$\mathcal{L}_2[0,\sigma)$ to $\mathcal{L}_2[0,\sigma)$, and an operator-valued transfer function $\hat{F}(\lambda)$:

$$\hat{F}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k. \qquad (1)$$

Related to $\mathcal{H}_\infty$ control, the $\mathcal{L}_2$-induced norm is used. It is a fact that the $\mathcal{L}_2$-induced norm of F_a equals the $\mathcal{H}_\infty$ norm of the lifted transfer function $\hat{F}$. Similarly, the $\mathcal{H}_2$ measure for F_a defined in (Khargonekar and Sivashankar, 1991; Bamieh and Pearson, 1992*a*) equals the $\mathcal{H}_2$ norm of $\hat{F}$ defined earlier. This $\mathcal{H}_2$ norm also has interpretations in terms of impulse responses or white noise responses (Khargonekar and Sivashankar, 1991; Bamieh and Pearson, 1992*a*) of the system. What is the interpretation of the entropy of $\hat{F}$ in terms of the time-domain model F_a? This is what we look at now.

To avoid unnecessary technicality, we will concentrate on finite-dimensional periodic systems, i.e., those F_a with finite-dimensional realizations, or equivalently, the lifted transfer functions with only finite number of poles.

Continuing with our discussion of σ-periodic, causal F_a and its lifted transfer function $\hat{F}(\lambda)$, let w be a Gaussian white noise with zero mean and unit covariance on the time interval $[0,\infty)$ and z the corresponding response: $z = F_a w$. Define an LEQG cost function for F_a as

$$\Omega_T = \frac{2}{T}\ln \mathbf{E}\left\{\exp\left[\frac{1}{2}\int_0^T z'(t)z(t)dt\right]\right\}$$

where $\mathbf{E}(\cdot)$ means the expectation.

Theorem 1 *Given a finite-dimensional σ-periodic system F_a, assume its lifted transfer function $\hat{F}$ satisfies $\hat{F} \in \mathcal{H}_\infty \cap \mathcal{H}_2$ and $\|\hat{F}\|_\infty < 1$. Then $\lim_{T\to\infty}\Omega_T = \mathcal{I}(\hat{F})/\sigma$.*

We first need to establish the operator-valued Szegö formula (Grenander and Szegö, 1984, pp. 64–65). The proof presented here resembles that suggested by (Brillinger, 1981, pp. 84) for the scalar case but is much more technical.

Lemma 2 *Let $\hat{G}$ be a selfadjoint nuclear operator-valued analytic function on the unit circle satisfying*

$$\max_{\omega\in[-\pi,\pi)} \|\hat{G}(e^{j\omega})\| < 1.$$

Let $g(k)$, $k = 0, \pm1, \pm2, \ldots$, be its Fourier coefficients. Define the Toeplitz matrix

$$G_K = \begin{bmatrix} g(0) & g(-1) & \cdots & g(-K+1) \\ g(1) & g(0) & \cdots & g(-K+2) \\ \vdots & \vdots & \ddots & \vdots \\ g(K-1) & g(K-2) & \cdots & g(0) \end{bmatrix}.$$

Then

$$\lim_{K\to 0} \ln\det(I - G_K) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det[I - \hat{G}(e^{j\omega})]d\omega.$$

Proof: Consider the circulant matrix

$$\tilde{G}_K = \begin{bmatrix} g(0) & g(-1)+g(K-1) & \cdots & g(-K+1)+g(1) \\ g(-K+1)+g(1) & g(0) & \cdots & g(-K+2)+g(2) \\ \vdots & \vdots & \ddots & \vdots \\ g(-1)+g(K-1) & g(-2)+g(K-2) & \cdots & g(0) \end{bmatrix}.$$

Define the block Vandermonde matrix

$$V_K = \begin{bmatrix} I & I & \cdots & I \\ I & e^{j2\pi/K}I & \cdots & e^{j2(K-1)\pi/K}I \\ \vdots & \vdots & \cdots & \vdots \\ I & e^{j2(K-1)\pi/K}I & \cdots & e^{j2(K-1)(K-1)\pi/K}I \end{bmatrix}.$$

It can be verified that

$$V_K^{-1}\tilde{G}_K V_K = \operatorname{diag}\left(\sum_{k=-K+1}^{K-1} g(k), \cdots, \sum_{k=-K+1}^{K-1} g(k)e^{j2(K-1)k\pi/K}\right).$$

For $l = 0, 1, \ldots, K-1$, denote by μ_{li}, $i = 1, 2, \ldots$, the eigenvalues of $\sum_{k=-\infty}^{\infty} g(k)e^{j2lk\pi/K}$; then there is a way to order the eigenvalues of $\sum_{k=-K+1}^{K-1} g(k)e^{j2lk\pi/K}$, denoted by ν_{li}, $i = 1, 2, \ldots$, such that

$$\begin{aligned} &\sum_{i=1}^{\infty} |\mu_{li} - \nu_{li}| \\ &\le \|\sum_{k=-\infty}^{-K} g(k)e^{j2lk\pi/K} + \sum_{k=K}^{\infty} g(k)e^{j2lk\pi/K}\|_{\mathrm{tr}} \\ &\le 2\sum_{k=K}^{\infty} \|g(k)\|_{\mathrm{tr}}, \end{aligned}$$

where $\|\cdot\|_{\mathrm{tr}}$ means the trace (or nuclear) norm. Furthermore, notice that

$$\begin{aligned} \|\tilde{G}_K - G_K\|_{\mathrm{tr}} &= \left\| \begin{bmatrix} 0 & g(K-1) & \cdots & g(1) \\ g(-K+\ ) & 0 & \cdots & g(2) \\ \vdots & \vdots & \ddots & \vdots \\ g(-1) & g(-2) & \cdots & 0 \end{bmatrix} \right\|_{\mathrm{tr}} \\ &\le \sum_{k=-\infty}^{\infty} |k|\|g(k)\|_{\mathrm{tr}}. \end{aligned}$$

Hence there is a way to order the eigenvalues of G_K, denoted by $\zeta_{li}, l = 0, 1, \ldots, K-1, i = 1, 2, \ldots$, such that

$$\sum_{l=0}^{K-1}\sum_{i=1}^{\infty}|\zeta_{li} - \nu_{li}| \leq \sum_{k=-\infty}^{\infty}|k|\|g(k)\|_{\mathrm{tr}}.$$

Notice that the analyticity of $\hat{G}$ implies that

$$\hat{G}(e^{j2l\pi/K}) = \sum_{k=-\infty}^{\infty} g(k)e^{j2lk\pi/K},$$

$$\sum_{k=-\infty}^{\infty}|k|\cdot\|g(k)\|_{\mathrm{tr}} < \infty,$$

and

$$\lim_{K\to\infty}\sum_{k=K}^{\infty}\|g(k)\|_{\mathrm{tr}} = 0.$$

Consequently,

$$\begin{aligned}
&\left|\ln\det(I - G_K) - \sum_{l=0}^{K-1}\ln\det[I - \hat{G}(e^{j2l\pi/K})]\right| \\
&= \left|\sum_{l=0}^{K-1}\sum_{i=1}^{\infty}\ln(1-\zeta_{li}) - \sum_{l=0}^{K-1}\sum_{i=1}^{\infty}\ln(1-\mu_{li})\right| \\
&= \left|\sum_{l=0}^{K-1}\sum_{i=1}^{\infty}\frac{-1}{1-\xi_{li}}(\zeta_{li}-\mu_{li})\right|
\end{aligned}$$

for some $\xi_{li} \in [-\|G\|_\infty, \|\hat{G}\|_\infty]$. This shows that

$$\begin{aligned}
&\left|\ln\det(I - G_K) - \sum_{l=0}^{K-1}\ln\det[I - \hat{G}(e^{j2l\pi/K})]\right| \\
&\leq \frac{1}{1-\|\hat{G}\|_\infty}\sum_{l=0}^{K-1}\sum_{i=1}^{\infty}(|\zeta_{li}-\nu_{li}| + |\nu_{li}-\mu_{li}|) \\
&\leq \frac{1}{1-\|\hat{G}\|_\infty}\left(\sum_{k=-\infty}^{\infty}|k|\|g(k)\|_{\mathrm{tr}} + 2K\sum_{k=K}^{\infty}\|g(k)\|_{\mathrm{tr}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{K\to\infty}\frac{1}{K}\ln\det(I - G_K) \\
&= \frac{1}{2\pi}\lim_{K\to\infty}\frac{2\pi}{K}\sum_{k=0}^{K-1}\ln\det[I - \hat{G}(e^{j2k\pi/K})] \\
&= \frac{1}{2\pi}\int_0^{2\pi}\ln\det[I - \hat{G}(e^{j\omega})]d\omega
\end{aligned}$$

This completes the proof.

Proof of Theorem 1: The proof follows from the idea in (Glover and Doyle, 1988) but has two complications: (1) operator-valued transfer functions are treated, which requires dealing with random variables in Hilbert spaces (Vakhania *et al.*, 1987); (2) signals are defined on time $[0,\infty)$ instead of $(-\infty,\infty)$, which requires treating non-stationary stochastic processes. Since F_a is linear, it follows that z is a Gaussian process. Define z_T as the stochastic process on $[0,T]$ such that $z_T(t) = z(t)$ for $t \in [0,T]$. Then z_T can be considered as a Gaussian random variable in the Hilbert space $\mathcal{L}_2[0,T]$. The covariance operator $V_T : \mathcal{L}_2[0,T] \to \mathcal{L}_2[0,T]$ is then given by (for $t \in [0,T]$)

$$\begin{aligned}
&(V_T x)(t) \\
&= \mathbf{E}\left[z_T(t)\int_0^T z_T'(\tilde{t})x(\tilde{t})d\tilde{t}\right] = \int_0^T \mathbf{E}[z_T(t)z_T'(\tilde{t})]x(\tilde{t})d\tilde{t} \\
&= \int_0^T \mathbf{E}\left[\int_0^T f_a(t,\tau)w(\tau)d\tau\int_0^T w'(\tilde{\tau})f_a'(\tilde{t},\tilde{\tau})d\tilde{\tau}\right]x(\tilde{t})d\tilde{t} \\
&= \int_0^T\int_0^T\int_0^T f_a(t,\tau)\mathbf{E}[w(\tau)w'(\tilde{\tau})]f_a'(\tilde{t},\tilde{\tau})x(\tilde{t})d\tau d\tilde{\tau}d\tilde{t} \\
&= \int_0^T\int_0^T\int_0^T f_a(t,\tau)\delta(\tau-\tilde{\tau})f_a'(\tilde{t},\tilde{\tau})x(\tilde{t})d\tau d\tilde{\tau}d\tilde{t} \\
&= \int_0^T\int_0^T f_a(t,\tau)f_a'(\tilde{t},\tilde{\tau})x(\tilde{t})d\tau d\tilde{t} = (F_aF_a^* x)(t).
\end{aligned}$$

This shows that

$$V_T = \Pi_{\mathcal{L}_2[0,T]}F_aF_a^*|_{\mathcal{L}_2[0,T]}.$$

Since $\Pi_{\mathcal{L}_2[0,T]}F_a|_{\mathcal{L}_2[0,T]}$ is a contractive Hilbert-Schmidt operator and F_a is causal, it follows that V_T is a self-adjoint contractive nuclear operator. Let the Schmidt expansion of V_T be

$$V_T = \sum_{i=1}^{\infty}\sigma_i\langle\cdot, v_i\rangle v_i.$$

Then z_T can be expressed as

$$z_T = \sum_{i=1}^{\infty}\alpha_i v_i$$

and α_i, $i = 1, 2, \ldots$, are independent scalar Gaussian random variables with covariance σ_i. Hence

$$\mathbf{E}\left\{\exp\left[\frac{1}{2}\int_0^T z'(t)z(t)dt\right]\right\} = \mathbf{E}\left\{\exp\left[\frac{1}{2}\langle z_T, z_T\rangle\right]\right\}$$

$$= \mathbf{E}\left\{\exp\left[\frac{1}{2}\sum_{i=1}^{\infty}\alpha_i^2\right]\right\} = \prod_{i=1}^{\infty}\mathbf{E}\{\exp(\alpha_i^2/2)\}$$

$$= \prod_{i=1}^{\infty}(1-\sigma_i)^{-1/2} = [\det(I - V_T)]^{-1/2}.$$

Now lift w to get ω and lift z to get ζ. Then $z = F_a w$ is equivalent to $\zeta = F\omega$ and F has a matrix representation

$$F = \begin{bmatrix} f(0) & 0 & & \\ f(1) & f(0) & 0 & \\ f(2) & f(1) & f(0) & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

Let F_K be the leading K by K submatrix of F. Then

$$\mathbf{E}\left\{\exp\left[\frac{1}{2}\int_0^{K\sigma} z'(t)z(t)dt\right]\right\} = \det(I - F_K F_K^*)^{-1/2}.$$

Since $\hat{F}$ has only finite number of poles, the infinite Hankel matrix

$$H = \begin{bmatrix} f(1) & f(2) & f(3) & \cdots \\ f(2) & f(3) & f(4) & \cdots \\ f(3) & f(4) & f(5) & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

has finite rank. Let H_K be the first K block rows of H and define

$$W_K = F_K F_K^* + H_K H_K^*.$$

Notice that W_K is a selfadjoint Toeplitz matrix

$$W_K = \begin{bmatrix} w(0) & w(-1) & \cdots & w(-K+1) \\ w(1) & w(0) & \cdots & w(-K+2) \\ \vdots & \vdots & \ddots & \vdots \\ w(K-1) & w(K-2) & \cdots & w(0) \end{bmatrix}$$

and $w(i)$ is the i-th Fourier coefficient of $\hat{F}\hat{F}^{\sim}$, where $\hat{F}^{\sim}(\lambda) = \hat{F}(\bar{\lambda}^{-1})^*$. Denote by $\sigma_i(W_K)$ and $\sigma_i(F_K F_K^*)$, $i = 1, 2, \ldots$, the singular values of W_K and $F_K F_K^*$, respectively, assuming nondecreasing order. Then

$$\sum_{i=1}^{\infty}|\sigma_i(W_K) - \sigma_i(F_K F_K^*)| \le \mathrm{tr} H_K H_K^* \le \mathrm{tr} H H^* < \infty.$$

Since $\sigma_i(W_K)$ and $\sigma_i(F_K F_K^*)$ are all contained in the interval $[-\|\hat{F}\|_\infty^2, \|\hat{F}\|_\infty^2]$, it follows that

$$|\ln\det(I - F_K F_K^*) - \ln\det(I - W_K)|$$
$$= \left|\sum_{i=1}^{\infty}\ln[1 - \sigma_i(F_K F_K^*)] - \sum_{i=1}^{\infty}\ln[1 - \sigma_i(W_K)]\right|$$
$$= \left|\sum_{i=1}^{\infty}\frac{-1}{1-\xi_i}[\sigma_i(F_K F_K^*) - \sigma_i(W_K)]\right|$$

for some $\xi_i \in [-\|\hat{F}\|_\infty^2, \|\hat{F}\|_\infty^2]$. This shows that

$$|\ln\det(I - F_K F_K^*) - \ln\det(I - W_K)|$$
$$\le \frac{1}{1 - \|\hat{F}\|_\infty^2}\sum_{i=1}^{\infty}|\sigma_i(W_K) - \sigma_i(F_K F_K^*)|$$
$$\le \frac{1}{1 - \|\hat{F}\|_\infty^2}\mathrm{tr} H H^*.$$

Hence by using Lemma 2,

$$\lim_{K\to\infty}\Omega_{K\sigma} = -\lim_{K\to\infty}\frac{1}{K\sigma}\ln\det(I - F_K F_K^*)$$
$$= -\lim_{K\to\infty}\frac{1}{K\sigma}\ln\det(I - W_K)$$
$$= -\frac{1}{2\pi\sigma}\int_{-\pi}^{\pi}\ln\det[I - \hat{F}(e^{j\omega})\hat{F}^*(e^{j\omega})]d\omega$$
$$= \frac{1}{\sigma}\mathcal{I}(\hat{F}).$$

Notice that for $K\sigma < T < (K+1)\sigma$,

$$\frac{K}{K+1}\Omega_{K\sigma} \le \Omega_T \le \frac{K+1}{K}\Omega_{(K+1)\sigma}.$$

Therefore, $\lim_{T\to\infty}\Omega_T = \mathcal{I}(\hat{F})/\sigma$.

This result establishes a stochastic interpretation of the entropy introduced for periodic systems. Next, we shall consider a control application for sampled-data systems.

3. MINIMUM ENTROPY $\mathcal{H}_\infty$ CONTROL

The standard sampled-data system is shown in Figure 1, where G is an analog plant, K_d is a digital controller, S and H are the periodic sampler and zero-order hold both with period σ. The closed-loop system, $w \mapsto z$, is given by the linear fractional transformation $\mathcal{F}(G, HK_dS)$. The sampled-data $\mathcal{H}_\infty$ control problem is as follows: Design K_d to stabilize G and achieve $\|\mathcal{F}(G, HK_dS)\| < 1$.

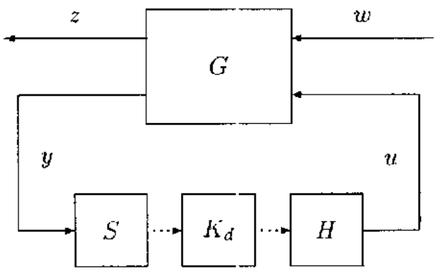

Fig. 1. The sampled-data setup

Here, the norm is $\mathcal{L}_2$-induced and the bound is normalized.

This problem is solved in, e.g., (Bamieh and Pearson, 1992*b*). Under some mild conditions on G, the sampled-data problem is equivalent to a certain discrete-time $\mathcal{H}_\infty$ control problem, which then admits solutions by standard methods, see, e.g., (Iglesias and Glover, 1991; Green *et al.*, 1990). Briefly, the solutions of the problem, if solvable, can be characterized by a linear fractional transformation with a free parameter:

$$\hat{K}_d = \mathcal{F}(\hat{L}, \hat{Q}), \quad \hat{Q} \in \mathcal{H}_\infty, \quad \|\hat{Q}\|_\infty < 1, \qquad (2)$$

where $\hat{L}$ is in $\mathcal{RH}_\infty$ and satisfies $\hat{L}_{22}(0) = 0$.

The sampled-data $\mathcal{H}_\infty$ problem we are interested is the minimum entropy one: Design K_d to stabilize G, achieve $\|\mathcal{F}(G, HK_dS)\| < 1$, and minimize $\mathcal{I}[\mathcal{F}(G, HK_dS)]$. The entropy is well-defined because the sampled-data system is σ-periodic.

It was observed in (Qiu and Chen, 1994) that the reduction process in (Bamieh and Pearson, 1992*b*) has the property that the entropy of the sampled-data system equals that of the equivalent discrete-time system plus some constant. In other word, the equivalent problem is now a discrete-time minimum entropy $\mathcal{H}_\infty$ problem, which is studied in (Mustafa and Glover, 1991; Iglesias *et al.*, 1990; Iglesias and Mustafa, 1993). Hence, under some mild assumptions, we reach the same conclusion: The minimum entropy $\mathcal{H}_\infty$ controller is the central controller in (2), obtained by setting $\hat{Q}$ to zero.

4. CONCLUSIONS

In this paper we established the connection between the entropy of periodic systems and some LEQG quantity in the time domain. This connection involves some interesting generalization of the Szegö formula. With this, we can formulate and solve the minimum entropy $\mathcal{H}_\infty$ problem for sampled-data control systems.

Extension of this work to multirate sampled-data systems (Chen and Qiu, 1994) is reported in (Qiu and Chen, 1994).

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