# On Performance Limitation in Tracking a Sinusoid

Weizhou Su, Li Qiu, and Jie Chen

*Abstract*—**This note studies the performance limitation of a feedback system with a given linear time-invariant (LTI) plant in tracking a sinusoidal signal. It continues and goes beyond some recent studies in the same topic in which it is assumed that the controller can access all the past and future values of the reference signal. In this note, we consider the more realistic (and more difficult) situation where the controller only accesses the current and past values of the reference. An explicit formula of the best attainable performance is obtained for a single-input–single-output (SISO) system which depends on the nonminimum phase zeros of the plant and the frequency of the reference sinusoid. Compared to the previously studied case when the future of the reference is available, this formula clearly shows the extra effort one has to pay due to the lack of the reference information. A partial result for a multiple-input–multiple-output (MIMO) system is also given.**

*Index Terms*—**Linear system structure, nonminimum phase, optimal control, performance limitation, tracking.**

## I. Introduction

In this note, we study the performance limitation of a feedback system with a given linear time-invariant (LTI) plant in tracking a sinusoidal signal. The main issue in such a study is to find the analytical relationship, hopefully simple and insightful, between the best tracking error attainable by designing the controller and the properties of the plant and the reference. In our previous study [12], it was assumed that the dynamic controller not only had the access of the instantaneous values of the reference signal and hence its past history, but also the instantaneous values of all state variables of the exogenous reference generator and, hence, all the past and future values of the reference. Under this complete or full information assumption, the best attainable tracking error over all possible controller designs was given in terms of the inherent properties, mainly the nonminimum phase zeros, of the plant and the frequency of the reference signal. Although this best attainable performance, called the performance limit, obtained under the complete reference information assumption is more fundamental than that under any other incomplete or partial information assumption where the controller does not have all the past and future values of the reference, it is an ideal case. In real applications, however, it is often the case that the controller can only access the current value of the reference signal. It is then of interest to consider the performance limit under this information constraint. It will be shown that for a single-input–single-output (SISO) plant this performance limit can also be expressed in terms of the nonminimum phase zeros of the plant and the frequency of the reference in a rather simple way. Compared with the performance limit in the complete reference information case,

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W. Su is with the College of Automation Science and Engineering, South China University of Technology, Guangzhou 510641, China (e-mail: wzhsu@scut.edu.cn; weizhousu@hotmail.com).

L. Qiu is with the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China (e-mail: eeqiu@ust.hk).

J. Chen is with the Department of Electrical Engineering, the University of California, Riverside, CA 92521-0425 USA (e-mail: jchen@ee.ucr.edu).

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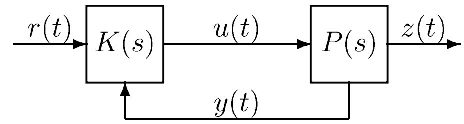


Fig. 1. Two-parameter control structure with incomplete reference information.

the limiting tracking error contains an extra nonnegative term which is the price we need to pay for the lack of enough information. For a multiple-input–multiple-output (MIMO) plant, the same problem is also addressed with less generality. Only a MIMO system with at most two nonminimum phase zeros will be studied. The performance limit in this special MIMO case exhibits in one hand some interesting insightful features and on the other hand the difficulty in deriving a performance limit for a general MIMO plant.

The studies on performance limitation of feedback systems provide deep understandings to inherent constraints on the best achievable performance of the systems due to the structures and characteristics of the plants. It has been attracting a growing amount of interest in the control community. The type of works related to our study can be traced back to the early 1970s when optimal cheap LQ control was studied by Kwakernaak and Sivan in [8] and later by Francis in [6]. It was shown that perfect regulation can be achieved for right-invertible minimum phase systems but not for general nonminimum phase systems. The performance limitation in tracking/disturbance rejection was first studied by Davison and Scherzinger in [4] where it was shown that perfect tracking/disturbance rejection can be achieve for right-invertible minimum phase systems but not for general nonminimum phase systems. The recent trend is more on the quantitative limits in the achievable performance for nonminimum phase systems. Morari and Zafiriou [9], Qiu and Davison [10] gave simple expressions of the performance limits in tracking step signals for a right invertible plant. A more refined study for multivariable plants was given in [1]. These works have since been extended to [2], [3], [10], [12], discrete time systems [7], [13], [15], nonlinear systems [11], and systems with uncertainties [5], [14].

The organization of this note is as follows. In Section II, the problem under consideration in this note is precisely formulated based on our previous works. In Section III, we present our main result for a SISO LTI system and the its proof. Then we discuss the relationship between the main result and our previous results. Section IV extends the main result for the SISO system in Section III to a special class of MIMO systems with at most two nonminimum phase zeros. Section V is the conclusion.

The notation used throughout this note is fairly standard. For any complex number, vector and matrix, denote their conjugate, conjugate transpose, real, and imaginary parts by $\bar{(\cdot)}$, $(\cdot)^*$, $\mathrm{Re}(\cdot)$, and $\mathrm{Im}(\cdot)$, respectively. The phase or argument of a nonzero complex number is denoted by $\angle(\cdot)$. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open right- and left-half plane be denoted by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. $\mathcal{L}_2$ is the standard frequency domain Lebesgue space. $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are subspaces of $\mathcal{L}_2$ containing functions that are analytical in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. It is well-known that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ constitute orthogonal complements in $\mathcal{L}_2$. $\mathcal{RH}_\infty$ is the set of all stable, rational transfer matrices. Finally, the inner product between two complex vectors $u$, $v$ is defined by $\langle u, v\rangle := u^* v$.

## II. Problem Statements

The system under consideration in this note is shown in Fig. 1. Here, $P(s)$ is the transfer function of a given plant whose output $z(t)$ and measurement $y(t)$ may not be the same, $K(s)$ is the transfer function of

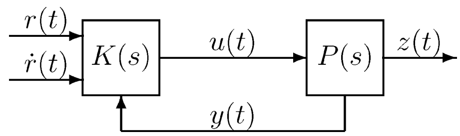


Fig. 2. Two-parameter control structure with complete reference information.

a two-degrees-of-freedom (2DOF) controller (for details, see, e.g., [16, pp. 141–150]) which is to be designed. We write $P(s) = \begin{bmatrix} G(s) \\ H(s) \end{bmatrix}$ where $G(s)$ is the transfer function from $u(t)$ to $z(t)$ and $H(s)$ is the transfer function from $u(t)$ to $y(t)$. One typical sinusoidal tracking problem is to design a controller $K(s)$ so that the closed-loop system is internally stabilized and the plant output $z(t)$ asymptotically tracks a sinusoidal reference signal $r(t)$ of the form

$$r(t) = \bar{v}e^{-j\omega t} + ve^{j\omega t} = 2\mathrm{Re}(v)\cos\omega t + 2\mathrm{Im}(v)\sin\omega t. \quad (1)$$

In [12], a more general version of the sinusoidal tracking problem is studied in which the reference might be a linear combination of a step and several sinusoidal waves of different frequencies. More importantly, in [12], the controller is assumed to know the magnitude and phase information of all harmonics of the reference $r(t)$ in advance. Such a case will be called the complete reference information case. Specialized to the single frequency sinusoid tracking where the reference signal is given in (1), the complete reference information case is equivalent to the case when the controller $K(s)$ also takes the derivative of $r(t)$, in addition to the reference $r(t)$ itself, as one of its input, as shown in Fig. 2. In this note, we will assume that the controller does not know the magnitude and phase of $r(t)$, i.e., the vector $v$, and it can only access the instantaneous values of $r(t)$. If the controller finds that the information on vector $v$ is needed, it has to spend time and effort to estimate it. This latter case will be called the incomplete reference information case. The intuition tells us that the lack of complete information in the incomplete reference information case would likely result in performance deterioration, but how much deterioration will be resulted exactly? This is precisely the question that we try to answer in this note.

The transient tracking error is measured by its energy

$$J(v) = \int_0^\infty \|r(t) - z(t)\|^2\,dt = \int_0^\infty \|e(t)\|^2\,dt. \quad (2)$$

In order for the tracking problem to be meaningful and solvable, we make the following assumptions throughout this note.

*Assumption 1:*

1) $P(s)$, $G(s)$ and $H(s)$ have the same unstable poles.
2) $G(s)$ has no zero at $-j\omega, j\omega$.

The first item in the assumption means that the measurement can be used to stabilize the system and at the same time does not introduce any additional unstable modes. It is satisfied in the special cases of output feedback, where $y(t) = z(t)$, and state feedback, where $y(t)$ is the state vector of system $G(s)$. A more precise way of stating this is that if $P(s) = \begin{bmatrix} N(s) \\ L(s) \end{bmatrix} M^{-1}(s)$ is a coprime factorization, then we assume that $N(s)M^{-1}(s)$ and $L(s)M^{-1}(s)$ are also coprime factorizations. The second item is of course necessary for the solvability of the tracking problem.

In the complete reference information case, $J(v)$ can be minimized for each individual $v$. The best achievable performance is then given by

$$J_{\mathrm{opt}}(v) = \inf_K J(v)$$

which depends on $v$ of course. One possible assessment and the standard practice [8] of the overall performance limitation is given by the average of $J_{\mathrm{opt}}(v)$ when $v$ is taken as a random vector with zero mean, unit covariance, and uncorrelated conjugate

$$J_{\mathrm{opt}} = \boldsymbol{E}\left\{J_{\mathrm{opt}}(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I, \boldsymbol{E}(vv^T) = 0\right\}.$$

The statistical properties of the coefficient $v$ mean that the terms $\sin\omega t$ and $\cos\omega t$ in (1) are uncorrelated and have the normalized magnitude variances. The explicit expressions for $J_{\mathrm{opt}}(v)$ and $J_{\mathrm{opt}}$ were obtained in [12].

In the incomplete reference information case, since the magnitude and phase of the reference are not available to the feedback controller, it is only meaningful to consider the averaged tracking performance of the system over a reasonable set of magnitudes and phases. Here, we again take the average when $v$ is considered as a random vector with zero mean, unit covariance, and uncorrelated conjugate. Hence, the averaged performance is given by

$$E = \boldsymbol{E}\left\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I, \boldsymbol{E}(vv^T) = 0\right\}. \quad (3)$$

The limit of $E$, under any controller design, is given by

$$E_{\mathrm{opt}} = \inf_K E. \quad (4)$$

Mathematically, the difference between $J_{\mathrm{opt}}$ and $E_{\mathrm{opt}}$ lies in the order of the expectation over $v$ and the infimum over the controller $K$. Immediately, we know $E_{\mathrm{opt}} \geq J_{\mathrm{opt}}$ from their definitions. It is the purpose of this note to derive an explicit formula for $E_{\mathrm{opt}}$, hence a good understanding of the exact amount of $E_{\mathrm{opt}}$ in excess of $J_{\mathrm{opt}}$.

To find an explicit formula for $E_{\mathrm{opt}}$ and compare it with $J_{\mathrm{opt}}$, some preliminary results in [12] are reviewed. Let us consider the frequency $\omega$ of the reference signal. We first find a unit zero vector $\eta_{\omega 1}$ of $G(s)$ corresponding to $z_1$ and define

$$G_{\omega 1}(s) = I - \eta_{\omega 1}\frac{2\mathrm{Re}(z_1)}{z_1 - j\omega}\frac{s - j\omega}{z_1^* + s}\eta_{\omega 1}^* = U_{\omega 1}\begin{bmatrix} \frac{z_1^*+j\omega}{z_1-j\omega}\frac{z_1-s}{z_1^*+s} & 0 \\ 0 & I \end{bmatrix}U_{\omega 1}^*$$

where $U_{\omega 1}$ is a unitary matrix with the first column equal to $\eta_{\omega 1}$. Here, $G_{\omega 1}(s)$ is so constructed that it is inner, has the only zero at $z_1$ with $\eta_{\omega 1}$ as a corresponding zero vector, and $G_{\omega 1}(j\omega) = I$. Since $G_{\omega 1}(s)$ is a generalization of the standard scalar Blaschke factor, we call it a matrix Blaschke factor at the frequency $\omega$ and $\eta_{\omega 1}$ a corresponding Blaschke vector. Also notice that the choice of other columns in $U_{\omega 1}$ is immaterial. Now, $G_{\omega 1}^{-1}(s)G(s)$ has zeros $z_2, z_3, \ldots, z_m$. Find a zero vector $\eta_{\omega 2}$ of $G_{\omega 1}^{-1}(s)G(s)$ corresponding to $z_2$ and define

$$G_{\omega 2}(s) = I - \eta_{\omega 2}\frac{2\mathrm{Re}(z_2)}{z_2 - j\omega}\frac{s - j\omega}{z_2^* + z}\eta_{\omega 2}^* = U_{\omega 2}\begin{bmatrix} \frac{z_2^*+j\omega}{z_2-j\omega}\frac{z_2-s}{z_2^*+s} & 0 \\ 0 & I \end{bmatrix}U_{\omega 2}^*$$

where $U_{\omega 2}$ is a unitary matrix with the first column equal to $\eta_{\omega 2}$. Then, $G_{\omega 2}^{-1}(s)G_{\omega 1}^{-1}(s)G(s)$ has zeros $z_3, z_4, \ldots, z_m$. Continue this process until Blaschke vectors $\eta_{\omega 1}, \ldots, \eta_{\omega m}$ and factors $G_{\omega 1}(s), \ldots, G_{\omega m}(s)$ are all obtained. This procedure shows that $G(s)$ can be factorized as

$$G(s) = G_{\omega 1}(s)\cdots G_{\omega m}(s)G_{\omega 0}(s) \quad (5)$$

where

$$G_{\omega i}(s) = I - \eta_{\omega i}\frac{2\mathrm{Re}(z_i)}{z_i - j\omega}\frac{s - j\omega}{z_i^* + s}\eta_{\omega i}^* = U_{\omega i}\begin{bmatrix} \frac{z_i^*+j\omega}{z_i-j\omega}\frac{z_i-s}{z_i^*+s} & 0 \\ 0 & I \end{bmatrix}U_{\omega i}^* \quad (6)$$

and $G_{\omega 0}(s)$ has no nonminimum phase zero. In this factorization, each Blaschke vector and factor correspond to one nonminimum phase zero. Keep in mind that these vectors and factors depend on the order of the nonminimum zeros, as well as on the frequency $\omega$. Moreover, it is obtained in [12] that, for given unit directional vectors $\eta_{\omega i}, i = 1, \ldots, m$, there exist $\eta_{-\omega i}, i = 1, \ldots, m$, such that

$$\eta_{-\omega i} = \eta_{\omega i} \quad \text{and} \quad \eta_{-\omega i} = G_{\omega 1}(-j\omega) \cdots G_{\omega i-1}(-j\omega)\eta_{\omega i}, \quad i = 2, \ldots, m. \tag{7}$$

Consequently, $N(s)$ can be factorized into

$$N(s) = G_{\omega 1}(s) \cdots G_{\omega m}(s) N_0(s) \tag{8}$$

where $N_0(s)$ is an outer function, i.e., it belongs to $\mathcal{RH}_\infty$ and has no zero in $\mathbb{C}_+$ (for details see, e.g., [16]).

The results in [12], when specialized to the single frequency reference given by (1), give the explicit expressions for $J_{\text{opt}}(v)$ and $J_{\text{opt}}$.

*Lemma 1:* [12] Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$. Then, the tracking performance limit is given by

$$J_{\text{opt}}(v) = \sum_{i=1}^{m} 2\text{Re}(z_i) \left| \frac{\langle \eta_{-\omega i}, \bar{v} \rangle}{z_i + j\omega} + \frac{\langle \eta_{\omega i}, v \rangle}{z_i - j\omega} \right|^2$$

and

$$J_{\text{opt}} = 2 \sum_{i=1}^{m} \left( \frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega} \right).$$

## III. SISO Systems

In this section, we give a rather complete answer for the case when $G(s)$ is a SISO system. In this case, item 1 in Assumption 1 simply means that $G(s)$ and $H(s)$ have the same unstable poles.

*Theorem 1:* Let $G(s)$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$. Then

$$E_{\text{opt}} = 2 \sum_{i=1}^{m} \left( \frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega} \right) + \frac{2}{\omega} \sin^2 \left[ 2 \sum_{i=1}^{m} \angle (z_i - j\omega) \right]. \tag{9}$$

*Proof:* Let $G(s) = N(s)M^{-1}(s)$ be a coprime factorization. Then by using the parameterization of all stabilizing 2DOF controllers as in [16], we see that the achievable transfer function from $r(t)$ to $z(t)$ is $N(s)Q(s)$ where $Q(s)$ is an arbitrary $\mathcal{H}_\infty$ transfer function which can be designed. Hence, for a fixed $v$, the tracking performance $J(v)$ defined in (2) is written to

$$\begin{aligned} J(v) &= \| [1 - N(s)Q(s)] R(s) \|_2^2 \\ &= \left\| [1 - N(s)Q(s)] \left[ \frac{1}{s + j\omega} \, \frac{1}{s - j\omega} \right] \begin{bmatrix} \bar{v} \\ v \end{bmatrix} \right\|_2^2. \end{aligned}$$

The averaged cost function $E$ is then given by

$$\begin{aligned} E &= \left\| [1 - N(s)Q(s)] \left[ \frac{1}{s + j\omega} \, \frac{1}{s - j\omega} \right] \right\|_2^2 \\ &= \left\| [1 - N(s)Q(s)] \frac{\sqrt{2}(s + \omega)}{s^2 + \omega^2} \right. \\ &\qquad \left. \times \left[ \frac{s - j\omega}{\sqrt{2}(s + \omega)} \, \frac{s + j\omega}{\sqrt{2}(s + \omega)} \right] \right\|_2^2 \\ &= \left\| [1 - N(s)Q(s)] \frac{\sqrt{2}(s + \omega)}{s^2 + \omega^2} \right\|_2^2. \end{aligned}$$

The last equality follows from the fact that $[(s - j\omega)/\sqrt{2}(s + \omega) \; (s + j\omega)/\sqrt{2}(s + \omega)]$ is co-inner. Hence, the averaged tracking performance $E$ is equal to the performance of the system in tracking the signal

$$\begin{aligned} r(t) &= \frac{\sqrt{2}}{2}(1 + j)e^{-j\omega t} + \frac{\sqrt{2}}{2}(1 - j)e^{j\omega t} \\ &= \sqrt{2}\cos \omega t + \sqrt{2} \sin \omega t \end{aligned}$$

i.e., $E = J((\sqrt{2}/2)(1 - j))$. It follows from Lemma 1 that the performance limit is given by

$$\begin{aligned} E_{\text{opt}} &= J_{\text{opt}} \left( \frac{\sqrt{2}}{2}(1 - j) \right) = \sum_{i=1}^{m} 2\text{Re}(z_i) \\ &\quad \times \left| \frac{\left\langle \eta_{-\omega i}, \frac{\sqrt{2}}{2}(1 + j) \right\rangle}{z_i + j\omega} + \frac{\left\langle \eta_{\omega i}, \frac{\sqrt{2}}{2}(1 - j) \right\rangle}{z_i - j\omega} \right|^2. \end{aligned} \tag{10}$$

For the SISO system $G(s)$, we select the unit directional vectors $\eta_{\omega i}$ and the inner functions $G_{\omega i}(s)$, $i = 1, \ldots, m$, associated with $z_i$, $i = 1, \ldots, m$, as follows:

$$\eta_{\omega i} = 1 \quad \text{and} \quad G_{\omega i}(s) = \frac{z_i^* + j\omega}{z_i - j\omega} \frac{z_i - s}{z_i^* + s}, \qquad i = 1, \ldots, m.$$

Then, following (7), we have

$$\eta_{-\omega i} = \frac{z_1^* + j\omega}{z_1 - j\omega} \frac{z_1 + j\omega}{z_1^* - j\omega} \cdots \frac{z_{i-1}^* + j\omega}{z_{i-1} - j\omega} \frac{z_{i-1} + j\omega}{z_{i-1}^* - j\omega}, \quad i = 2, \ldots, m.$$

Expanding (10) gives $E_{\text{opt}} = E_a + E_b$ where

$$\begin{aligned} E_a &= \sum_{i=1}^{m} \left[ \frac{2\text{Re}(z_i)}{(z_i^* - j\omega)(z_i + j\omega)} + \frac{2\text{Re}(z_i)}{(z_i^* + j\omega)(z_i - j\omega)} \right] \\ &= 2 \sum_{i=1}^{m} \left( \frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega} \right) \end{aligned}$$

and

$$E_b = \sum_{i=1}^{m} \left[ -\frac{j2\text{Re}(z_i)\eta_{-\omega i}\eta_{\omega i}^*}{(z_i^* - j\omega)(z_i - j\omega)} + \frac{j2\text{Re}(z_i)\eta_{\omega i}\eta_{-\omega i}^*}{(z_i^* + j\omega)(z_i + j\omega)} \right].$$

In the remaining part of this proof, induction is used.

First of all, denote $\angle(z_i^* - j\omega)(z_i - j\omega)$ by $\phi_i$. Then

$$(z_i^* - j\omega)(z_i - j\omega) = |z_i^* - j\omega| |z_i - j\omega| e^{j\phi_i}$$

and

$$-2\text{Re}(z_i)\omega = |z_i^* - j\omega| |z_i - j\omega| \sin \phi_i.$$

The first term of $E_b$ can then be written as

$$\begin{aligned} &-\frac{j2\text{Re}(z_1)}{(z_1^* - j\omega)(z_1 - j\omega)} + \frac{j2\text{Re}(z_1)}{(z_1^* + j\omega)(z_1 + j\omega)} \\ &\qquad = \frac{j \sin \phi_1}{\omega} \left( e^{-j\phi_1} - e^{j\phi_1} \right) = \frac{2}{\omega} \sin^2 \phi_1. \end{aligned} \tag{11}$$

Assume that

$$\sum_{i=1}^{k-1} \left[ -\frac{j2\text{Re}(z_i)\eta_{-\omega i}\eta_{\omega i}^*}{(z_i^* - j\omega)(z_i - j\omega)} + \frac{j2\text{Re}(z_i)\eta_{\omega i}\eta_{-\omega i}^*}{(z_i^* + j\omega)(z_i + j\omega)} \right] = \frac{2}{\omega} \sin^2(\phi_1 + \cdots + \phi_{k-1}).$$

Notice the fact that $\eta_{-\omega k}\eta_{\omega k}^* = e^{-j2(\phi_1+\cdots+\phi_{k-1})}$ and $\eta_{\omega k}\eta_{-\omega k}^* = e^{j2(\phi_1+\cdots+\phi_{k-1})}$. Then, it holds

$$\begin{aligned}\sum_{i=1}^{k}&\left[-\frac{j2\mathrm{Re}(z_i)\eta_{-\omega i}\eta_{\omega i}^*}{(z_i^*-j\omega)(z_i-j\omega)}+\frac{j2\mathrm{Re}(z_i)\eta_{\omega i}\eta_{-\omega i}^*}{(z_i^*+j\omega)(z_i+j\omega)}\right]\\
&=\frac{2}{\omega}\sin^2(\phi_1+\cdots+\phi_{k-1})\\
&\quad-\frac{j}{\omega}\frac{2\mathrm{Re}(z_k)\omega e^{-j2(\phi_1+\cdots+\phi_{k-1})}}{(z_k^*-j\omega)(z_k-j\omega)}\\
&\quad+\frac{j}{\omega}\frac{2\mathrm{Re}(z_k)\omega e^{j2(\phi_1+\cdots+\phi_{k-1})}}{(z_k^*+j\omega)(z_k+j\omega)}\\
&=\frac{2}{\omega}\sin^2(\phi_1+\cdots+\phi_{k-1})\\
&\quad+\frac{j\sin\phi_k}{\omega}\left[e^{-j2(\phi_1+\cdots+\phi_{k-1})-\phi_k]}\right.\\
&\qquad\left.-e^{j2(\phi_1+\cdots+\phi_{k-1})+\phi_k]}\right]\\
&=\frac{2}{\omega}\sin^2(\phi_1+\cdots+\phi_{k-1})\\
&\quad+\frac{2}{\omega}\sin\left[2(\phi_1+\cdots+\phi_{k-1})+\phi_k\right]\sin\phi_k\\
&=\frac{2}{\omega}\sin^2(\phi_1+\cdots+\phi_k).\end{aligned}$$

Here, in the last step, we used elementary trigonometrical identities. Therefore

$$E_b=\frac{2}{\omega}\sin^2(\phi_1+\cdots+\phi_m)=\frac{2}{\omega}\sin^2 2\left[\sum_{i=1}^{m}\angle(z_i-j\omega)\right].$$

This completes the proof. □

Notice that in the complete reference information case we have the following performance limit, as stated in Lemma 1:

$$J_{\mathrm{opt}}=2\sum_{i=1}^{m}\left(\frac{1}{z_i^*+j\omega}+\frac{1}{z_i-j\omega}\right).$$

Theorem 1 gives an exact picture on how the lack of the reference state information affects the best tracking performance. Compared with the performance limit in the complete reference information case, the performance limit in the incomplete reference information case has an extra nonnegative term which is caused by the constraint on the reference information structure. As it is known that $J_{\mathrm{opt}}$ is the energy of the error between the reference and the output generated by the best possible control input, the extra term is the energy of the error due to the mismatch between the best control input and ones which can be generated by an incomplete information controller. In the complete information case, this mismatch is zero since the best control input can be produced by the optimal complete information controller while in the incomplete information case the mismatch is nonzero since the best control input and ones which can be generated by any controller under this information constraint are not equal in general.

Finally, we present an extended version of Theorem 1 to the case when $G(s)$ contains a time delay.

*Theorem 2:* Let $G(s)=e^{-\tau s}G_r(s)$ where $G_r(s)$ is a real rational transfer function with nonminimum phase zeros $z_1,\ldots,z_m$. Then

$$E_{\mathrm{opt}}=2\tau+2\sum_{i=1}^{m}\left(\frac{1}{z_i^*+j\omega}+\frac{1}{z_i-j\omega}\right)+\frac{2}{\omega}\sin^2\left[-\omega\tau+2\sum_{i=1}^{m}\angle(z_i-j\omega)\right].$$

The proof is omitted since it is just a minor modification of that of Theorem 1.

## IV. MIMO Systems

It appears that extending the SISO result in the last section to the case when $G(s)$ is MIMO is difficult in general. Here, we consider a special case of MIMO systems with no more than two nonminimum phase zeros $z_1$ and $z_2$. This special case is manageable and the result reveals some interesting insights and also the possible difficulties in the general case. The directional vectors associated with $z_1$ and $z_2$ are denoted by $\eta_{\omega 1}$ and $\eta_{\omega 2}$, respectively. Assume that $P(s), G(s), H(s)$ satisfy Assumption 1.

*Theorem 3:* Let $G(s)$ have two nonminimum phase zeros $z_1, z_2$ and let $\theta$ be the angle between the associated directional vectors $\eta_{\omega 1}$ and $\eta_{\omega 2}$. Then

$$\begin{aligned}E_{\mathrm{opt}}&=2\sum_{i=1}^{2}\left(\frac{1}{z_i^*+j\omega}+\frac{1}{z_i-j\omega}\right)\\
&\quad+\frac{2}{\omega}\sin^2\theta\sum_{i=1}^{2}\sin^2\left[\angle(z_i-j\omega)(z_i^*-j\omega)\right]\\
&\quad+\frac{2}{\omega}\cos^2\theta\sin^2\left[\sum_{i=1}^{2}\angle(z_i-j\omega)(z_i^*-j\omega)\right].\end{aligned}$$

*Proof:* By the same procedure as that used for a SISO LTI system in the last section, we have

$$E=\left\|\left[I-N(s)Q(s)\right]\frac{\sqrt{2}(s+\omega)}{s^2+\omega^2}\right\|_2^2. \tag{12}$$

Suppose that the output dimension is $n$. Denote the $i$th column of the $n\times n$ identify matrix by $e_i, i=1,\ldots,n$. It follows from (12) that

$$E=\sum_{l=1}^{n}\left\|\left[I-N(s)Q(s)\right]e_l\frac{\sqrt{2}(s+\omega)}{s^2+\omega^2}\right\|_2^2. \tag{13}$$

From (13), we can see that the averaged tracking performance $E$ is equal to a sum of the performances of the system in tracking $n$ different references

$$r(t)=e_l\left[\frac{\sqrt{2}}{2}(1+j)e^{-j\omega t}+\frac{\sqrt{2}}{2}(1-j)e^{j\omega t}\right],\quad l=1,\ldots,n.$$

Since the terms in (13) depend on different columns of $Q(s)$, the overall optimum over $Q(s)$ is equal to the sum of the optimal values of the individual terms. Applying Lemma 1, we get

$$\begin{aligned}E_{\mathrm{opt}}&=\sum_{l=1}^{n}J_{\mathrm{opt}}\left(\frac{\sqrt{2}}{2}(1-j)e_l\right)\\
&=\sum_{l=1}^{n}\sum_{i=1}^{2}2\mathrm{Re}(z_i)\left|\frac{\left\langle\eta_{-\omega i},\frac{\sqrt{2}}{2}(1+j)e_l\right\rangle}{z_i+j\omega}\right.\\
&\qquad\left.+\frac{\left\langle\eta_{\omega i},\frac{\sqrt{2}}{2}(1-j)e_l\right\rangle}{z_i-j\omega}\right|^2. \end{aligned}\tag{14}$$

Expanding (14) and noticing that $\sum_{l=1}^{n}e_le_l^T=I$, we have

$$\begin{aligned}E_{\mathrm{opt}}&=\sum_{i=1}^{2}2\mathrm{Re}(z_i)\\
&\quad\times\left[\frac{\langle\eta_{-\omega i},\eta_{-\omega i}\rangle}{(z_i+j\omega)(z_i^*-j\omega)}+\frac{\langle\eta_{\omega i},\eta_{\omega i}\rangle}{(z_i-j\omega)(z_i^*+j\omega)}\right.\\
&\qquad\left.+\frac{-j\langle\eta_{\omega i},\eta_{-\omega i}\rangle}{(z_i-j\omega)(z_i^*-j\omega)}+\frac{j\langle\eta_{-\omega i},\eta_{\omega i}\rangle}{(z_i+j\omega)(z_i^*+j\omega)}\right].\end{aligned}$$

Denote

$$E_a = \sum_{i=1}^{2} 2\mathrm{Re}(z_i)\left[\frac{\langle \eta_{-\omega i}, \eta_{-\omega i}\rangle}{(z_i + j\omega)(z_i^* - j\omega)} + \frac{\langle \eta_{\omega i}, \eta_{\omega i}\rangle}{(z_i - j\omega)(z_i^* + j\omega)}\right]$$

and

$$E_b = \sum_{i=1}^{2} 2\mathrm{Re}(z_i)\left[\frac{-j\langle \eta_{\omega i}, \eta_{-\omega i}\rangle}{(z_i - j\omega)(z_i^* - j\omega)} + \frac{j\langle \eta_{-\omega i}, \eta_{\omega i}\rangle}{(z_i + j\omega)(z_i^* + j\omega)}\right].$$

It is clear that

$$E_a = 2\left(\frac{1}{z_1^* + j\omega} + \frac{1}{z_1 - j\omega} + \frac{1}{z_2^* + j\omega} + \frac{1}{z_2 - j\omega}\right).$$

Due to $\eta_{-\omega 1} = \eta_{\omega 1}$, it holds that

$$\langle \eta_{\omega 1}, \eta_{-\omega 1}\rangle = \langle \eta_{-\omega 1}, \eta_{\omega 1}\rangle = 1. \tag{15}$$

It follows from (7) that the vector $\eta_{-\omega 2}$ is given by

$$\eta_{-\omega 2} = \left[I + \eta_{\omega 1}\eta_{\omega 1}^* \left(\frac{z_1^* + j\omega}{z_1 - j\omega}\frac{z_1 + j\omega}{z_1^* - j\omega} - 1\right)\right]\eta_{\omega 2}.$$

Define $\phi_i = \angle(z_i^* - j\omega)(z_i - j\omega)$. Then, we have

$$\begin{aligned}\langle \eta_{\omega 2}, \eta_{-\omega 2}\rangle &= 1 + \cos^2\theta\left(\frac{z_1^* + j\omega}{z_1 - j\omega}\frac{z_1 + j\omega}{z_1^* - j\omega} - 1\right)\\ &= \sin^2\theta + e^{-j2\phi_1}\cos^2\theta. \end{aligned}\tag{16}$$

Consequently, it holds that

$$\langle \eta_{-\omega 2}, \eta_{\omega 2}\rangle = \sin^2\theta + e^{j2\phi_1}\cos^2\theta. \tag{17}$$

It follows from (11) and (15) that

$$-\frac{j2\mathrm{Re}(z_1)\langle \eta_{\omega 1}, \eta_{-\omega 1}\rangle}{(z_1^* - j\omega)(z_1 - j\omega)} + \frac{j2\mathrm{Re}(z_1)\langle \eta_{-\omega 1}, \eta_{\omega 1}\rangle}{(z_1^* + j\omega)(z_1 + j\omega)} = \frac{2}{\omega}\sin^2\phi_1.$$

Following (16), (17), and the discussion in the proof of Theorem 1, we have

$$\begin{aligned}&-\frac{j2\mathrm{Re}(z_2)\langle \eta_{\omega 2}, \eta_{-\omega 2}\rangle}{(z_2^* - j\omega)(z_2 - j\omega)} + \frac{j2\mathrm{Re}(z_2)\langle \eta_{-\omega 2}, \eta_{\omega 2}\rangle}{(z_2^* + j\omega)(z_2 + j\omega)}\\ &\quad = \frac{2}{\omega}\sin^2\theta\sin^2\phi_2 + \frac{2}{\omega}\cos^2\theta\sin(2\phi_1 + \phi_2)\sin\phi_2.\end{aligned}$$

Consequently, it holds that

$$E_b = \frac{2}{\omega}\sin^2\theta(\sin^2\phi_1 + \sin^2\phi_2) + \frac{2}{\omega}\cos^2\theta\sin^2(\phi_1 + \phi_2).$$

Plugging in the definitions of $\phi_1$ and $\phi_2$ gives the expression to be proved. □

This theorem shows that, in the incomplete reference information case, the tracking performance limit $E_{\mathrm{opt}}$ depends on not only the phases of $z_1 - j\omega$ and $z_2 - j\omega$ but also the angle $\theta$ between $\eta_{\omega 1}$ and $\eta_{\omega 2}$. There are two extreme cases. One is that $\eta_{\omega 1}$ and $\eta_{\omega 2}$ are in a common one-dimensional subspace, i.e., $\theta = 0$ while the other is that $\eta_{\omega 1}$ and $\eta_{\omega 2}$ are orthogonal, i.e., $\theta = \pi/2$. In the first case, the two nonminimum phase zeros can be considered to appear in the same channel and the performance limit is given by

$$\begin{aligned}E_{\mathrm{opt}} = 2\sum_{i=1}^{2}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right)\\ + \frac{2}{\omega}\sin^2\left[\sum_{i=1}^{2}\angle(z_i - j\omega)(z_i^* - j\omega)\right].\end{aligned}$$

In the second case, the two nonminimum phase zeros can be considered to appear separately in two orthogonal channels and the performance limit is given by

$$\begin{aligned}E_{\mathrm{opt}} = 2\sum_{i=1}^{2}\left(\frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega}\right)\\ + \frac{2}{\omega}\sum_{i=1}^{2}\sin^2\left[\angle(z_i - j\omega)(z_i^* - j\omega)\right].\end{aligned}$$

In general, the performance limit is a convex combination of the two extreme cases depending on $\theta$.

It is worth mentioning that, if the plant has only one nonminimum phase zero $z_1$, the performance limit $E_{\mathrm{opt}}$ is given by

$$E_{\mathrm{opt}} = 2\left(\frac{1}{z_1 + j\omega} + \frac{1}{z_1 - j\omega}\right) + \frac{2}{\omega}\sin^2\left[2\angle(z_1 - j\omega)\right].$$

Notice that $z_1$ is a real number in this case. Then, we can obtain this result by straightforwardly following the proof of Theorem 3.

Theorem 3 also shows the potential difficulty in extending the result further to MIMO systems with more than two nonminimum phase zeros since the relative angles between each pair of the directional vectors associated with the nonminimum phase zeros will come into the picture. The number of such pairs grows combinatorially as the number of nonminimum phase zeros grows.

## V. Conclusion

In this note, the performance limitation of a feedback system in tracking a sinusoidal signal is studied under the assumption that the controller can only access the instantaneous value of the reference signal. This is in contrast to the previous study where the controller is assumed to have the complete information (past and future values) of the reference signal. A formula for the best achievable average tracking error, depending on the nonminimum phase zeros of the plant and their interactions with the reference frequency, is obtained for general SISO systems, with or without time delay. The worsening of the performance limitation due to the insufficient information is clearly shown. The study is also extended to a class of MIMO systems. It is shown that for MIMO systems, not only the plant nonminimum phase zeros but also the relative directions of the directional vectors associated with these zeros play a key role in the performance limitation. We believe that the results are significant in further understanding linear system structures and their effects on achievable control performances.

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# Analysis of a Second-Order Sliding-Mode Algorithm in Presence of Input Delays

Laura Levaggi and Elisabetta Punta

***Abstract*—In this note, a double integrator system under the action of a second-order sliding-mode control algorithm is considered, and the resulting closed-loop behavior in presence of an input delay is analysed. Due to the delay, in the limit the system trajectories are periodic. Whenever the control modulus is chosen to be constant, the amplitude and period of the resulting oscillations are fixed for any initial value. If the control behaves asymmetrically, it is shown that this is no more true, since the overall dynamical system can admit diverse limit cycles.**

***Index Terms*—Delay effects, discontinuous control, input delay, second-order systems, variable structure systems.**

## I. Introduction

In this note, it is investigated the closed-loop behavior of a double integrator under the action of a second-order sliding-mode algorithm in presence of an input delay.

When considering first-order sliding-mode control, the introduction of input delays results in delay-relay type dynamical systems. These have been analysed from a mathematical point of view in [1] and also in [2] and [3] from a different perspective. It has been shown that the delay induces oscillations around the sliding surface with finite limit frequency. Moreover, it has been proved that only zero limit frequency trajectories are stable.

Second-order systems with input delay and relay control have also been considered. In [4] it has been shown that the system's behaviour results in an oscillating motion and every trajectory has a finite limit frequency. However, the class of slowly oscillating solutions (those with zero limit frequency) is more complex in the second-order case. In particular, the stability of zero limit frequency modes is only proved under the uniqueness, up to time shifts, of slowly oscillating solutions. In [5]–[8], it has been moreover shown that for this class of differential systems bifurcations phenomena and even chaotic behaviour can happen.

We do not introduce here a new control algorithm in order to counteract the presence of the input delay and stabilize the considered system. This has been done for chain of integrators in [9] and [10] by means of continuous control laws, which guarantee global asymptotic stability. Here, we investigate the effect of an unknown (and disregarded) input delay on the behaviour of a closed-loop system resulting from the application of an existing second-order sliding-mode control algorithm [11]. This analysis should in fact answer the question whether, and up to which extent, the input delays can be disregarded in this control scheme. This issue is particularly interesting whenever relay type controllers are (or have to be) exploited and it is not possible to apply smooth feedbacks. Moreover, this analysis, which is conducted considering a quite simple case, constitutes the necessary basis in order to study more general systems with second-order sliding-mode and input delays. Particularly, the recently introduced second-order sliding-mode algorithms, the robustness of which have been studied and analysed with respect to uncertainties and disturbances of various nature, still need a systematic study of the input delay effects. In this sense the analysis proposed here for the unperturbed double integrator can be regarded as a first step in this direction and constitutes a theoretical novelty in the investigation of the second-order sliding-mode control methods.

We consider an unperturbed double integrator under the action of a control law designed according to the algorithm in [11] and analyse how the dynamical behaviour of the closed-loop system is affected by a disregarded input time delay. In this case, we can give an analytical description of the relevant terms governing the evolution of the closed-loop and thus we are able to study the asymptotic properties of the system trajectories. The presence of the delay deteriorates the performances of the closed-loop system. The origin of the phase plane is no longer reached, while the system's trajectories tend to a limit cycle. The new and interesting result is that, depending on the control parameters, different initial conditions may lead to different limit cycles. In the note we prove that the existence of a unique, globally attractive limit cycle is guaranteed by the application of a symmetric control law. If instead the control law is asymmetric, the limit behavior of the system's trajectories can depend on the initial value of the position. In the phase plane, the situation is represented by the presence of limit cycles, not necessarily centered in the origin. The method of the analysis is theoretical novel, since the investigation is performed taking into account not the frequencies of the oscillating motions in the limit, but the sequences of the generated singular points (points with zero velocity).

In Section II, we briefly describe the structure of the existent control algorithm. In Section III, we state our problem and study the effect of the input delay on the system evolution. When the modulus of the control is constant we prove that, in the limit, the system state is periodic.

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L. Levaggi is with the Department of Mathematics, University of Genova, 16126 Genoa, Italy (e-mail: levaggi@dima.unige.it).

E. Punta is with the Institute of Intelligent Systems for Automation, National Research Council of Italy (ISSIA-CNR), 16149 Genoa, Italy (e-mail: punta@ge.issia.cnr.it).

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