On Performance Limitations in Estimation

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Abstract. In this paper, we address the performance limitation issues in estimation problems. Our purpose is to explicitly relate the best achievable estimation errors with simple plant characteristics. In particular, we study performance limitations in achieving two objectives: estimating a signal from its version corrupted by a white noise and estimating a Brownian motion from its version distorted by an LTI system.

1 Introduction

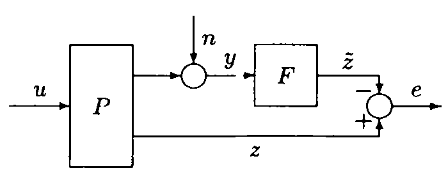

Fig. 1. A general estimation problem

A standard estimation problem can often be schematically shown by Fig. 1. Here $P = \begin{bmatrix} G \\ H \end{bmatrix}$ is an LTI plant, u is the input to the plant, n is the measurement noise, z is the signal to be estimated, y is the measured signal, $\tilde{z}$ is the estimate of z. Often u and n are modelled as stochastic processes with known means and covariances. We can assume, without loss of generality, that the means of the stochastic processes are zero. The objective is to design LTI filter F so that the steady state error variance

$$V = \lim_{t \to \infty} \boldsymbol{E}[e(t)'e(t)]$$

is small. Clearly, for V to be finite for nontrivial u and n, it is necessary that $F \in \mathcal{RH}_\infty$ and $H - FG \in \mathcal{RH}_\infty$. This condition is also necessary and sufficient for the error to be bounded for arbitrary initial conditions of P and F, i.e., for the filter to be a bounded error estimator (BEE). There is an extensive theory on the optimal design of the filter F to minimize V, see for example [1,2,6]. The optimal error variance is then given by

$$V^* = \inf_{F, H-FG \in \mathcal{RH}_\infty} V.$$

Our interest in this paper is not on how to find the optimal filter F, which is addressed by the standard optimal filtering theory. Rather, we are interested in relating V^* with some simple characteristics of the plant P in some important special cases. Since V^* gives a fundamental limitation in achieving certain performance objectives in filtering problems, the simple relationship between V^* and the plant characteristics, in addition to providing deep understanding and insightful knowledge on estimation problems, can be used to access the quality of different designs and to ascertain impossible design objectives before a design is carried out.

The variance V gives an overall measure on the size of the steady state estimation error. Sometimes, we may wish to focus on some detailed features of the error. For example we may wish to investigate the variance of the projection of the estimation error on certain direction. This variance then gives a measure of the error in a particular direction. Assume that $z(t), \tilde{z}(t), e(t) \in \mathbb{R}^m$. Let $\xi \in \mathbb{R}^m$ be a vector of unit length representing a direction in $\mathbb{R}^m$. Then the projection of $e(t)$ to the direction represented by ξ is given by $\xi' e(t)$ and its steady state variance is given by

$$V_\xi = \lim_{t \to \infty} \boldsymbol{E}[(\xi' e(t))^2].$$

The best achievable error in ξ direction is then given by

$$V_\xi^* = \inf_{F, H-FG \in \mathcal{RH}_\infty} V_\xi.$$

The optimal or near optimal filter in minimizing V_ξ in general depends on ξ. This very fact may limit the usefulness of V_ξ^*, since we are usually more interested in the directional error information under an optimal or near optimal filter designed for all directions, i.e., designed to minimize V. Let $\{F_k\}$ be a sequence of filters satisfying $F_k, H - F_k G \in \mathcal{RH}_\infty$ such that the corresponding sequence of errors $\{e_k\}$ satisfies

$$V = \lim_{k \to \infty} \lim_{t \to \infty} \boldsymbol{E}[e_k(t) e_k(t)'].$$

Then we are more interested in

$$V^*(\xi) = \lim_{k \to \infty} \boldsymbol{E}[(\xi' e_k(t))^2].$$

In this paper, we will also give the relationship between V_ξ^*, $V^*(\xi)$ and simple characteristics of the plant P for the same cases when that for V^* is considered.

The performance limitations in estimation have been studied recently in [4,5,9,10] in various settings. In [4,5,9], sensitivity and complimentary sensitivity functions of an estimation problem are defined and it is shown that they have to satisfy certain integral constraints independent of filter design. In [10], a time domain technique is used to study the performance limitations in some special cases when one of n and u is diminishingly small and the other one is either a white noise or a Brownian motion.

This paper addresses similar problems as in [10], but studies them from a pure input output point of view using frequency domain techniques. We also study them in more detail by providing directional information on the best errors. The results obtained are dual to those in [3,7] where the performance limitations of tracking and regulation problems are considered. The new investigation provides more insights into the performance limitations of estimation problems.

This paper is organized as follows: Section 2 provides background materials on transfer matrix factorizations which exhibit directional properties of each nonminimum phase zero and antistable pole. Section 3 relates the performance limitation in estimating a signal from its corrupted version by a white noise to the antistable modes, as well as their directional properties, of the signal. Section 4 relates the performance limitation in estimating a Brownian motion from its version distorted by an LTI system to the nonminimum phase zeros of the system, as well as their directional properties. Section 5 gives concluding remarks.

2 Preliminaries

Let G be a continuous time FDLTI system. We will use the same notation G to denote its transfer matrix. Assume that G is left invertible. The poles and zeros of G, including multiplicity, are defined according to its Smith-McMillan form. A zero of G is said to be nonminimum phase if it has positive real part. G is said to be minimum phase if it has no nonminimum phase zero; otherwise, it is said to be nonminimum phase. A pole of G is said to be antistable if it has a positive real part. G is said to be semistable if it has no antistable pole; otherwise strictly unstable.

Suppose that G is stable and z is a nonminimum phase zero of G. Then, there exists a vector u of unit length such that

$$G(z)u = 0.$$

We call u a (right or input) zero vector corresponding to the zero z. Let the nonminimum phase zeros of G be ordered as $z_1, z_2, \ldots, z_\nu$. Let also η_1 be a zero vector corresponding to z_1. Define

$$G_1(s) = I - \frac{2\,\mathrm{Re}\, z_1}{s + z_1^*}\eta_1\eta_1^*.$$

Note that G_1 is so constructed that it is inner, has only one zero at z_1 with η_1 as a zero vector. Now GG_1^{-1} has zeros $z_2, z_3, \ldots, z_\nu$. Find a zero vector η_2 corresponding to the zero z_2 of GG_1^{-1}, and define

$$G_2(s) = I - \frac{2\,\mathrm{Re}\, z_2}{s + z_2^*}\eta_2\eta_2^*.$$

It follows that $GG_1^{-1}G_2^{-1}$ has zeros $z_3, z_4, \ldots, z_\nu$. Continue this process until $\eta_1, \ldots, \eta_\nu$ and $G_1, \ldots, G_\nu$ are obtained. Then we have one vector corresponding to each nonminimum phase zero, and the procedure yields a factorization of G in the form of

$$G = G_0 G_\nu \cdots G_1, \tag{1}$$

where G_0 has no nonminimum phase zeros and

$$G_i(s) = I - \frac{2\,\mathrm{Re}\, z_i}{s + z_i^*}\eta_i\eta_i^*. \tag{2}$$

Since G_i is inner, has the only zero at z_i, and has η_i as a zero vector corresponding to z_i, it will be called a matrix Blaschke factor. Accordingly, the product

$$G_z = G_\nu \cdots G_1$$

will be called a matrix Blaschke product. The vectors $\eta_1, \ldots, \eta_\nu$ will be called zero Blaschke vectors of G corresponding to the nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$. Keep in mind that these vectors depend on the order of the nonminimum phase zeros. One might be concerned with the possible complex coefficients appearing in G_i when some of the nonminimum phase zeros are complex. However, if we order a pair of complex conjugate nonminimum phase zeros adjacently, then the corresponding pair of Blaschke factors will have complex conjugate coefficient and their product is then real rational and this also leads to real rational G_0.

The choice of G_i as in (2) seems *ad hoc* notwithstanding that G_i has to be unitary, have the only zero at z_i and have η_i as a zero vector corresponding to z_i. Another choice, among infinite many possible ones, is

$$G_i(s) = I - \frac{2\,\mathrm{Re}\, z_i}{z_i}\frac{s}{s + z_i^*}\eta_i\eta_i^*, \tag{3}$$

and if this choice is adopted, the same procedure can be used to find a factorization of the form (1). Of course, in this case the Blaschke vectors are not the same. We see that for the first choice $G_i(\infty) = I$, whereas for the second choice $G_i(0) = I$. We will use both choices in the following. For this purpose, we will call the factorization resulting from the first choice of Type I and that from the second choice of type II.

For an unstable G, there exist stable real rational matrix functions

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}, \begin{bmatrix} M & Y \\ N & X \end{bmatrix}$$

such that

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

and

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I.$$

This is called a doubly coprime factorization of G. Note that the nonminimum phase zeros of G are the nonminimum phase zeros of $\tilde{N}$ and the antistable poles of G are the nonminimum phase zeros of $\tilde{M}$. If we order the antistable poles of G as $p_1, p_2, \ldots, p_\mu$ and the nonminimum phase zeros of G as $z_1, z_2, \ldots, z_\nu$, then M and N can be factorized as

$$\tilde{M} = \tilde{M}_0 \tilde{M}_\mu \cdots \tilde{M}_1$$
$$\tilde{N} = \tilde{N}_0 \tilde{N}_\nu \cdots \tilde{N}_1$$

with

$$\tilde{M}_i(s) = I - \frac{2\,\mathrm{Re}\, p_i}{s + p_i^*} \zeta_i \zeta_i^*$$
$$\tilde{N}_i(s) = I - \frac{2\,\mathrm{Re}\, z_i}{z_i} \frac{s}{s + z_i^*} \eta_i \eta_i^*$$

where $\zeta_1, \zeta_2, \ldots, \zeta_\nu$ are zero Blaschke vectors of M and $\eta_1, \eta_2, \ldots, \eta_\nu$ are those of N. Here also $\tilde{N}_0$ and $\tilde{M}_0$ have no nonminimum phase zeros. Notice that we used type I factorization for $\tilde{M}$ and type II factorization for $\tilde{N}$. The reason for this choice is solely for the convenience of our analysis in the sequel.

Consequently, for any real rational matrix G with nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ and antistable poles $p_1, p_2, \ldots, p_\mu$, it can always be factorized to

$$G = G_p^{-1} G_0 G_z, \tag{4}$$

as shown in Fig. 2, where

$$G_p(s) = \prod_{i=1}^{\mu} \left[I - \frac{2\,\mathrm{Re}\, p_i}{s + p_i^*} \zeta_i \zeta_i^* \right]$$

$$G_z(s) = \prod_{i=1}^{\nu} \left[I - \frac{2\,\mathrm{Re}\, z_i}{z_i} \frac{s}{s + z_i^*} \eta_i \eta_i^* \right]$$

and G_0 is a real rational matrix with neither nonminimum phase zero nor antistable pole. Although coprime factorizations of G are not unique, this nonuniqueness does not affect factorization (4). Here $\eta_1, \eta_2, \ldots, \eta_\nu$ are called zero Blaschke vectors and $\zeta_1, \zeta_2, \ldots, \zeta_\nu$ pole Blaschke vectors of G.

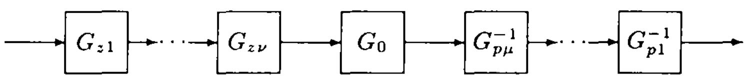

Fig. 2. Cascade factorization

3 Estimation under White Measurement Noise

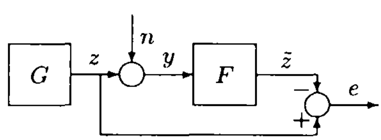

Fig. 3. Estimation under white measurement noise

Consider the estimation problem shown in Fig. 3. Here G is a given FDLTI plant, and n is a standard white noise. The purpose is to design a stable LTI filter F such that it generates an estimate $\tilde{z}$ of the true output z using the corrupted output y. This problem is clearly a special case of the general estimation problem stated in Sect. 1 with $P = \begin{bmatrix} G \\ G \end{bmatrix}$ and $u = 0$. The error of estimation is given by Fn. Since n is a standard white noise, the steady state variance of the error is given by

$$V = \|F\|_2^2$$

where $\|\cdot\|_2$ is the $\mathcal{H}_2$ norm. If we want V to be finite, we need to have $F(\infty) = 0$, in addition to $F, G - FG \in \mathcal{RH}_\infty$. Therefore

$$V^* = \inf_{F, G - FG \in \mathcal{RH}_\infty, F(\infty) = 0} \|F\|_2^2.$$

Let $G = \tilde{M}^{-1}\tilde{N}$ be a left coprime factorization of G. Then $F \in \mathcal{RH}_\infty$ and $G - FG = (I-F)G = (I-F)\tilde{M}^{-1}\tilde{N} \in \mathcal{RH}_\infty$ if and only if $I - F = Q\tilde{M}$ for some $Q \in \mathcal{RH}_\infty$. Therefore

$$V^* = \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}(\infty)=I} \|I - Q\tilde{M}\|_2^2.$$

Now assume that G has antistable poles $p_1, p_2, \ldots, p_\mu$ with $\zeta_1, \zeta_2, \ldots, \zeta_\mu$ be the corresponding pole Blaschke vectors of type I. Then $\tilde{M}$ has factorization

$$\tilde{M} = \tilde{M}_0 \tilde{M}_\mu \cdots \tilde{M}_1$$

where

$$\tilde{M}_i(s) = I - \frac{2\,\mathrm{Re}\, p_i}{s + p_i^*}\zeta_i\zeta_i^*.$$

Since $\tilde{M}_i(\infty) = I$, $i = 1, 2, \ldots, \mu$, it follows that $Q(\infty)\tilde{M}(\infty) = I$ is equivalent to $Q(\infty)\tilde{M}_0(\infty) = I$. Hence, by using the facts that $\tilde{M}_i$, $i = 1, 2, \ldots, \mu$, are unitary operators in $\mathcal{L}_2$ and that $\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I \in \mathcal{H}_2^\perp$ and $I - Q\tilde{M}_0 \in \mathcal{H}_2$, we obtain

$$\begin{aligned} V^* &= \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty)=I} \|I - Q\tilde{M}_0\tilde{M}_\mu \cdots \tilde{M}_1\|_2^2 \\ &= \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty)=I} \|\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I + I - Q\tilde{M}_0\|_2^2 \\ &= \|\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I\|_2^2 + \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty)=I} \|I - Q\tilde{M}_0\|_2^2. \end{aligned}$$

Since $\tilde{M}_0$ is co-inner with invertible $\tilde{M}_0(\infty)$, there exists a sequence $\{Q_k\} \in \mathcal{RH}_\infty$ with $Q_k(\infty)\tilde{M}_0(\infty) = I$ such that $\lim_{k\to\infty} \|I - Q\tilde{M}_0\| = 0$. This shows

$$\begin{aligned} V^* &= \|\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I\|_2^2 \\ &= \|\tilde{M}_2^{-1} \cdots \tilde{M}_\mu^{-1} - I + I - \tilde{M}_1\|_2^2 \\ &= \|\tilde{M}_2^{-1} \cdots \tilde{M}_\mu^{-1} - I\|_2^2 + \|I - \tilde{M}_1\|_2^2 \\ &= \sum_{i=1}^{\mu} \|I - \tilde{M}_i\|_2^2 \\ &= 2\sum_{i=1}^{\mu} p_i. \end{aligned}$$

Here the first equality follows from that $\tilde{M}_1$ is a unitary operator in $\mathcal{L}_2$, the second from that $\tilde{M}_2^{-1} \cdots \tilde{M}_\mu^{-1} - I \in \mathcal{H}_2^\perp$ and $I - \tilde{M}_1 \in \mathcal{H}_2$, the third from repeating the underlying procedure in the first and second equalities, and the last from straightforward computation. The above derivation shows that an arbitrarily near optimal Q can be chosen from the sequence $\{Q_k\}$. Therefore

$$V^*(\xi) = \lim_{k\to\infty} \|\xi'(I - Q_k\tilde{M}_0\tilde{M}_\mu\cdots\tilde{M}_1)\|_2^2.$$

The same reasoning as in the above derivation gives

$$V^*(\xi) = \sum_{i=1}^{\mu} \|\xi(I - \tilde{M}_i)\|_2^2 = 2\sum_{i=1}^{\mu} p_i \cos^2 \angle(\xi, \zeta_i).$$

The last equality follows from straightforward computation.

The directional steady state error variance with an arbitrary F is

$$V_\xi = \|\xi' F\|_2^2$$

and the optimal directional steady state error variance is

$$\begin{aligned} V_\xi^* &= \inf_{F,G-FG\in\mathcal{RH}_\infty} \|\xi' F\|_2^2 \\ &= \inf_{Q\in\mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty)=I} \|\xi'(I - Q\tilde{M}_0\tilde{M}_\mu\cdots\tilde{M}_1)\|_2^2. \end{aligned}$$

By following an almost identical derivation as the non-directional case, we can show that the same sequence $\{Q_k\}$ giving near optimal solutions there also gives near optimal solutions here for every $\xi \in \mathbb{R}^m$. Hence,

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\mu} p_i \cos^2 \angle(\xi, \zeta_i).$$

We have thus established the following theorem.

Theorem 1. *Let G's antistable poles be $p_1, p_2, \ldots, p_\mu$ with $\zeta_1, \zeta_2, \cdots, \zeta_\mu$ being the corresponding pole Blaschke vectors of type I. Then*

$$V^* = 2\sum_{i=1}^{\mu} p_i$$

and

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\mu} p_i \cos^2 \angle(\xi, \zeta_i).$$

This theorem says that to estimate a signal from its version corrupted by a standard while noise, the best achievable steady state error variance depends, in a simple way, only on the antistable modes of the signal to be estimated. The best achievable directional steady state error variance depends, in addition, on the directional characteristics of the antistable modes.

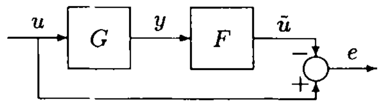

Fig. 4. Estimation of a Brownian motion process

4 Estimation of Brownian Motion

Consider the estimation problem shown in Fig. 4. Here G is a given FDLTI plant, u is the input to the plant which is assumed to be a Brownian motion process, i.e., the integral of a standard white noise, which can be used to model a slowly varying "constant". Assume that $G(0)$ is left invertible. The objective is to design an LTI filter F such that it measures the output of G and generates an estimate $\tilde{u}$ of u. This problem is clearly a special case of the general estimation problem stated in Sect. 1 with $P = \begin{bmatrix} G \\ I \end{bmatrix}$ and $n = 0$. The error of estimation is given by $(I - FG)u$. Since u is a Brownian process, the variance of the error is given by

$$V = \|(I - FG)U\|_2$$

where $U(s) = \frac{1}{s}I$ is the transfer matrix of m channels of integrators. If we want V to be finite, we need to have $I - F(0)G(0) = 0$, in addition to $F, I - FG \in \mathcal{RH}_\infty$. This requires $G(0)$ to be left invertible, which will be assumed. Equivalently, we need to have $F, FG \in \mathcal{H}_\infty$ and $F(0)G(0) = I$. Therefore,

$$V^* = \inf_{F, FG \in \mathcal{RH}_\infty, F(0)G(0)=I} \|(I - FG)U\|_2^2.$$

Let $G = \tilde{M}^{-1}\tilde{N}$ be a left coprime factorization of G. Then it is easy to see that $F, FG \in \mathcal{H}_\infty$ is equivalent to $F = Q\tilde{M}$ for some $Q \in \mathcal{H}_\infty$. Hence

$$V^* = \inf_{Q \in \mathcal{RH}_\infty, Q(0)\tilde{N}(0)=I} \|(I - Q\tilde{N})U\|_2^2.$$

Now let G have nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding input Blaschke vectors of type II. Then $\tilde{N}$ has factorizations

$$\tilde{N} = \tilde{N}_0\tilde{N}_\nu, \ldots, \tilde{N}_1$$

where

$$\tilde{N}_i = I - \frac{2\,\mathrm{Re}\, z_i}{z_i}\frac{s}{s + z_i^*}\eta_i\eta_i^*.$$

Since $\tilde{N}_i(\infty) = I$, $i = 1, 2, \ldots, \nu$, it follows that $Q(0)\tilde{N}(0) = I$ is equivalent to $Q(0)\tilde{N}_0(0) = I$. Hence, by using the facts that $\tilde{N}_i$, $i = 1, 2, \ldots, \nu$, are unitary operators in $\mathcal{L}_2$ and that $\tilde{N}_1^{-1}\cdots\tilde{N}_\nu^{-1}-I\in\mathcal{H}_2^\perp$ and $I-Q\tilde{N}_0\in\mathcal{H}_2$, we obtain

$$\begin{aligned}V^* &= \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I}\|(I-Q\tilde{N}_0\tilde{N}_\nu,\ldots,\tilde{N}_1)U\|_2^2\\ &= \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I}\|(\tilde{N}_1^{-1}\tilde{N}_2^{-1}\cdots\tilde{N}_\nu-I)U+(I-Q\tilde{N}_0)U\|_2^2\\ &= \|(\tilde{N}_1^{-1}\tilde{N}_2^{-1}\cdots\tilde{N}_\nu-I)U\|_2^2+\inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I}\|(I-Q\tilde{N}_0)U\|_2^2.\end{aligned}$$

Since $\tilde{N}_0$ is co-inner with invertible $\tilde{N}(0)$, there exists a sequence $\{Q_k\}\in\mathcal{RH}_\infty$ with $Q_k(0)\tilde{N}_0(0)=I$ such that $\lim_{k\to\infty}\|(I-Q\tilde{N}_0)U\|=0$. This shows

$$\begin{aligned}V^* &= \|(\tilde{N}_1^{-1}\cdots\tilde{N}_\nu^{-1}-I)U\|_2^2\\ &= \|(\tilde{N}_2^{-1}\cdots\tilde{N}_\nu^{-1}-I+I-\tilde{N}_1)U\|_2^2\\ &= \|(\tilde{N}_2^{-1}\cdots\tilde{N}_\nu^{-1}-I)U\|_2^2+\|(I-\tilde{M}_1)U\|_2^2\\ &= \sum_{i=1}^{\nu}\|(I-\tilde{N}_i)U\|_2^2\\ &= 2\sum_{i=1}^{\nu}\frac{1}{z_i}.\end{aligned}$$

Here the first equality follows from that $\tilde{N}_1$ is a unitary operator in $\mathcal{L}_2$, the second from that $(\tilde{N}_2^{-1}\cdots\tilde{N}_\nu^{-1}-I)U\in\mathcal{H}_2^\perp$ and $(I-\tilde{N}_1)U\in\mathcal{H}_2$, the third from repeating the underlying procedure in the first and second equalities, and the last from straightforward computation.

The above derivation shows that an arbitrarily near optimal Q can be chosen from the sequence $\{Q_k\}$. Therefore

$$V^*(\xi)=\lim_{k\to\infty}\|\xi'(I-Q_k\tilde{N}_0\tilde{N}_\nu\cdots\tilde{N}_1)U\|_2^2.$$

The same reasoning as in the above derivation gives

$$V^*(\xi)=\sum_{i=1}^{\mu}\|\xi(I-\tilde{N}_i)U\|_2^2=2\sum_{i=1}^{\mu}\frac{1}{z_i}\cos^2\angle(\xi,\eta_i).$$

The last equality follows from straightforward computation.

The directional steady state error variance with an arbitrary F is

$$V_\xi=\|\xi'(I-FG)U\|_2^2$$

and the optimal directional steady state error variance is

$$\begin{aligned}V_\xi^* &= \inf_{F,G-FG\in\mathcal{RH}_\infty}\|\xi'(I-FG)U\|_2^2\\ &= \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I}\|\xi'(I-Q\tilde{N}_0\tilde{N}_\nu\cdots\tilde{N}_1)U\|_2^2.\end{aligned}$$

By following an almost identical derivation as the non-directional case, we can show that the same sequence $\{Q_k\}$ giving near optimal solutions there also gives near optimal solutions here for every $\xi \in \mathbb{R}^m$. Hence,

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\nu} \frac{1}{z_i} \cos^2 \angle(\xi, \eta_i).$$

We have thus established the following theorem.

Theorem 2. *Let G's nonminimum phase zeros be $z_1, z_2, \cdots, z_\nu$ with $\eta_1, \eta_2, \cdots, \eta_\nu$ being the corresponding Blaschke vectors of type II, then*

$$V^* = 2\sum_{i=1}^{\nu} \frac{1}{z_i}$$

and

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\nu} \frac{1}{z_i} \cos^2 \angle(\xi, \eta_i).$$

This theorem says that to estimate a Brownian motion from its version distorted by an LTI system, the best achievable steady state error variance depends, in a simple way, only on the nonminimum phase zeros of the LTI system. The best achievable directional steady state variance depends, in addition, on the directional characteristics of the nonminimum phase zeros.

5 Concluding Remarks

This paper relates the performance limitations in two typical estimation problems to simple characteristics of the plants involved. By estimation problems we mean actually filtering problems here. The general estimation problems can include prediction and smoothing problems. We are now trying to extend the results in this paper to smoothing and prediction problems.

In the problem considered in Sect. 3, the noise is modelled by a white noise. In the problem considered in Sect. 4, the signal to be estimated is modelled as a Brownian motion. We are trying to extending our results to possibly other types of noises and signals.

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