



On structured perturbation of Hermitian matrices¹

Tingshu Hu^{a,2}, Li Qiu^{b,*}

^a Department of Electronics and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

^b Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China

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Abstract

This paper concerns a quantity which is equal to the norm of the smallest structured perturbation to a Hermitian matrix that makes the perturbed matrix singular. This quantity of course then gives an indication on how much such structured perturbation the Hermitian matrix can tolerate before becoming singular. For some structures, this quantity can be computed explicitly. For some more general structures, only the lower bound on this quantity is given. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

For $M \in \mathbb{C}^{m \times n}$, the Schmidt–Mirsky theorem says (among other things) that

$$\inf \{ \|\Delta\| : \Delta \in \mathbb{C}^{n \times m}, \det(I + \Delta M) = 0 \} = \|M\|^{-1}. \quad (1)$$

* Corresponding author.

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² This author was with the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China.

Here and in the sequel, the matrix norm used is the spectral norm, i.e., the largest singular value.

Formula (1) has found wide applications in engineering problems, see for example [1,2]. However, in many applications, the perturbation Δ is often restricted to a subset of $\mathbb{C}^{n \times m}$. One interesting subset is $\mathbb{R}^{n \times m}$. It is shown in [3] that

$$\begin{aligned} & \inf \{ \|\Delta\| : \Delta \in \mathbb{R}^{n \times m}, \det(I + \Delta M) = 0 \} \\ &= \left\{ \inf_{\gamma \in (0,1]} \sigma_2 \begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \right\}^{-1}. \end{aligned}$$

In studying robust control of linear systems under structured perturbation, the following subset is of interest

$$\mathcal{X}_K := \left\{ \begin{bmatrix} \Delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta_K \end{bmatrix}, \Delta_i \in \mathbb{C}^{n_i \times m_i} \right\}.$$

It is shown in [4] that

$$\inf \{ \|\Delta\| : \Delta \in \mathcal{X}_K, \det(I + \Delta M) = 0 \} \geq \left[\inf_{D \in \mathcal{D}} \|D^{-1}MD\| \right]^{-1},$$

where

$$\mathcal{D} = \left\{ \begin{bmatrix} d_1 I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_K I \end{bmatrix} : d_i > 0 \right\}$$

and this inequality is actually an equality if $K \leq 3$.

Recently, it was discovered in [5] that the solution to certain robust performance problem for linear systems requires computing the following quantity for a given Hermitian matrix

$$M = \begin{bmatrix} S & N \\ N^* & R \end{bmatrix} \in \mathbb{C}^{(n+m) \times (n+m)}.$$

$$\psi(M) := \inf \left\{ \|\Delta\| : \Delta \in \mathbb{C}^{n \times m}, \det \left(I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right) = 0 \right\}. \tag{2}$$

In [5], a formula for computing $\psi(M)$ in the special case when $S \geq 0$ and $R \geq 0$ was obtained. In this paper, we will extend the formula to the general case.

We will also study the following generalized quantity.

$$\psi_K(M) := \inf \left\{ \|\Delta\| : \Delta \in \mathcal{X}_K, \det \left(I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right) = 0 \right\}. \tag{3}$$

A lower bound on $\psi_K(M)$ will be obtained. This lower bound will be shown to be equal to $\psi_K(M)$ if $K \leq 2$.

The paper is organized as follows. Section 2 studies $\psi(M)$. An explicit formula for $\psi(M)$ is obtained. Section 3 considers the general case. A lower bound on $\psi_K(M)$ is obtained. This lower bound, when specialized to the case when $K = 1$, gives the formula obtained in Section 2 and hence is equal to $\psi_K(M)$. Section 4 shows that the lower bound obtained in Section 3 is also equal to $\psi_K(M)$ when $K = 2$.

In the following, we define some notation used in this paper. For $X \in \mathbb{C}^{m \times n}$, the singular values of X are denoted by $\sigma_i(X)$, assuming nonincreasing order. The largest singular value of X is also denoted by $\bar{\sigma}(X)$. We always set $\|X\| = \bar{\sigma}(X)$. If X is Hermitian, then the eigenvalues of X are denoted by $\lambda_i(X)$, also assuming nonincreasing order, and the inertia of X is denoted by $\{\pi_+(X), \pi_0(X), \pi_-(X)\}$, representing the number of positive, zero, and negative eigenvalues of X , respectively.

2. Formula for computing $\psi(M)$

Note that for all $\gamma > 0$,

$$\begin{aligned} & \det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} \\ &= \det \left\{ I + \begin{bmatrix} \sqrt{\gamma}I & 0 \\ 0 & I/\sqrt{\gamma} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \begin{bmatrix} \sqrt{\gamma}I & 0 \\ 0 & I/\sqrt{\gamma} \end{bmatrix} \right\} \\ &= \det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix} \right\}. \end{aligned} \tag{4}$$

An immediate application of the Schmidt–Mirsky theorem gives

$$\psi(M) \geq \left\{ \inf_{\gamma > 0} \left\| \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix} \right\| \right\}^{-1}. \tag{5}$$

If

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then $\psi(M) = \infty$ but the right hand side of (5) is 1. This shows that the inequality (5) can be very loose.

Note that the inertia of

$$\begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix}$$

is invariant of γ , and is equal to the inertia of M .

Define:

$$r_+ := \begin{cases} \left\{ \inf_{\gamma > 0} \lambda_1 \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix} \right\}^{-1} & \text{if } \pi_+(M) > 0, \\ \infty & \text{if } \pi_+(M) = 0. \end{cases}$$

$$r_- := \begin{cases} \left\{ \inf_{\gamma > 0} \left(-\lambda_{m+n} \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix} \right) \right\}^{-1} & \text{if } \pi_-(M) > 0, \\ \infty & \text{if } \pi_-(M) = 0. \end{cases}$$

Theorem 1. $\psi(M) = \min\{r_+, r_-\}$.

Several lemmas are needed for the proof of Theorem 1.

Lemma 1 ([6], p. 149). *Let $F(\gamma) \in \mathbb{C}^{n \times n}$ be a Hermitian matrix function analytic on an open set $\Gamma \subset \mathbb{R}$. Then there exist a unitary matrix function $\tilde{V}(\gamma) = [\tilde{v}_1(\gamma), \dots, \tilde{v}_n(\gamma)] \in \mathbb{C}^{n \times n}$ and a diagonal matrix function $\tilde{\Lambda}(\gamma) = \text{diag}[\tilde{\lambda}_1(\gamma), \dots, \tilde{\lambda}_n(\gamma)] \in \mathbb{R}^{n \times n}$, both analytic on Γ , such that*

$$F(\gamma) = \tilde{V}(\gamma)\tilde{\Lambda}(\gamma)\tilde{V}^*(\gamma).$$

Furthermore,

$$\frac{d\tilde{\lambda}_i(\gamma)}{d\gamma} = \tilde{v}_i^*(\gamma) \frac{dF(\gamma)}{d\gamma} \tilde{v}_i(\gamma). \tag{6}$$

Lemma 2. *Let $F(\gamma) \in \mathbb{C}^{n \times n}$ be a Hermitian matrix function analytic on an open set $\Gamma \subset \mathbb{R}$. Let $\lambda_1(\gamma) \geq \lambda_2(\gamma) \geq \dots \geq \lambda_n(\gamma)$ be its ordered eigenvalues. If $\lambda_i(\gamma)$ has a local extremum at $\gamma_0 \in \Gamma$, then $F(\gamma_0)$ has an eigenvector $v \in \mathbb{C}^n$ corresponding to $\lambda_i(\gamma_0)$ such that $v^*(dF(\gamma_0)/d\gamma)v = 0$.*

Proof. If the multiplicity of $\lambda_i(\gamma_0)$ is one, then $\lambda_i(\gamma)$ is equal to some $\tilde{\lambda}_j(\gamma)$ given in Lemma 1 in an open neighborhood of γ_0 . Thus γ_0 is also a stationary point of $\tilde{\lambda}_j(\gamma)$. Let $\tilde{v}_j(\gamma)$ be the analytic eigenvector corresponding to $\tilde{\lambda}_j(\gamma)$. Then (6) gives:

$$\tilde{v}_j^*(\gamma_0) \frac{dF(\gamma_0)}{d\gamma} \tilde{v}_j(\gamma_0) = 0.$$

If instead the multiplicity of $\lambda_i(\gamma_0)$ is greater than one, then we can assume, without loss of generality, that in an open neighborhood of γ_0 , $\lambda_i(\gamma) = \tilde{\lambda}_{j_1}(\gamma)$ for $\gamma \leq \gamma_0$ and $\lambda_i(\gamma) = \tilde{\lambda}_{j_2}(\gamma)$ for $\gamma \geq \gamma_0$. If $j_1 = j_2$, then $\lambda_i(\gamma_0)$ must be a local extremum of $\tilde{\lambda}_{j_1}(\gamma)$, so we get the result by applying (6). Otherwise let $\tilde{v}_k(\gamma)$, $k = j_1, j_2$, be the analytic eigenvectors of $F(\gamma)$ corresponding to $\tilde{\lambda}_k(\gamma)$. Then (6) gives:

$$\frac{d\tilde{\lambda}_{j_1}(\gamma_0)}{d\gamma} = \tilde{v}_{j_1}^*(\gamma_0) \frac{dF(\gamma_0)}{d\gamma} \tilde{v}_{j_1}(\gamma_0),$$

$$\frac{d\tilde{\lambda}_{j_2}(\gamma_0)}{d\gamma} = \tilde{v}_{j_2}^*(\gamma_0) \frac{dF(\gamma_0)}{d\gamma} \tilde{v}_{j_2}(\gamma_0).$$

Put $v_\alpha = \alpha \tilde{v}_{j_1} + (1 - \alpha^2)^{1/2} \tilde{v}_{j_2}$ for $\alpha \in [0, 1]$. Then $v_\alpha(\gamma_0)$ is also a unit length eigenvector of $F(\gamma_0)$ corresponding to $\lambda_i(\gamma_0)$. Define

$$f(\alpha) = v_\alpha^*(\gamma_0) \frac{dF(\gamma_0)}{d\gamma} v_\alpha(\gamma_0).$$

Since γ_0 is a local extremum of $\lambda_i(\gamma)$, we must have

$$f(0)f(1) = \frac{d\tilde{\lambda}_{j_1}(\gamma_0)}{d\gamma} \frac{d\tilde{\lambda}_{j_2}(\gamma_0)}{d\gamma} \leq 0.$$

By continuity, $f(\alpha) = 0$ has a solution in $[0, 1]$. This proves the lemma. \square

Lemma 3 ([7], p. 203). *Let $M, \Delta \in \mathbb{C}^{n \times n}$ be Hermitian matrices. Denote the eigenvalues of M as $\eta_1 \geq \eta_2 \geq \dots \geq \eta_n$ and those of $M + \Delta$ as $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n$. Then*

$$|\xi_i - \eta_i| \leq \|\Delta\|$$

for $i = 1, 2, \dots, n$.

Lemma 4. *For $A \in \mathbb{C}^{n \times m}, B \in \mathbb{C}^{m \times n}$,*

$$n - \text{rank}[I + AB] = m - \text{rank}[I + BA].$$

Proof. It can be verified that

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} I + AB & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ B & I + BA \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}.$$

Hence

$$\text{rank} \begin{bmatrix} I + AB & 0 \\ B & I \end{bmatrix} = \text{rank} \begin{bmatrix} I & 0 \\ B & I + BA \end{bmatrix}.$$

This last equation implies that $\text{rank}(I + AB) + m = \text{rank}(I + BA) + n$, or equivalently $m - \text{rank}[I + BA] = n - \text{rank}[I + AB]$. \square

Lemma 5. *Let $F(\gamma) = A(\gamma) + \gamma^{-1}B \in \mathbb{C}^{n \times n}$ where $A(\gamma)$ is a Hermitian matrix function analytic on an open interval Γ around 0 and B is a constant Hermitian matrix with $\text{rank } B = r \leq n$. Assume that B has a spectral decomposition*

$$B = [V_1 \ V_2] \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1 \ V_2]^*,$$

where $A_1 \in \mathbb{R}^{r \times r}$ is diagonal and $[V_1 \ V_2]$ is unitary. Then as γ approaches 0, $\pi_+(B)$ eigenvalues of $F(\gamma)$ go to ∞ , $\pi_-(B)$ eigenvalues of $F(\gamma)$ to $-\infty$, and the rest to the eigenvalues of $V_2^*A(0)V_2$.

Proof. Without loss of generality, assume that an analytic spectral decomposition of $\gamma F(\gamma)$ is

$$\gamma F(\gamma) = [\tilde{V}_1(\gamma) \ \tilde{V}_2(\gamma)] \begin{bmatrix} \tilde{A}_1(\gamma) & 0 \\ 0 & \tilde{A}_2(\gamma) \end{bmatrix} [\tilde{V}_1(\gamma) \ \tilde{V}_2(\gamma)]^*$$

such that $\tilde{A}_1(0) = A_1$ and $\tilde{A}_2(0) = 0$. Then

$$F(\gamma) = [\tilde{V}_1(\gamma) \ \tilde{V}_2(\gamma)] \begin{bmatrix} \gamma^{-1}\tilde{A}_1(\gamma) & 0 \\ 0 & \gamma^{-1}\tilde{A}_2(\gamma) \end{bmatrix} [\tilde{V}_1(\gamma) \ \tilde{V}_2(\gamma)]^*.$$

Clearly as γ goes to 0, r eigenvalues of $F(\gamma)$ go to the diagonal elements of $\lim_{\gamma \rightarrow 0} \gamma^{-1}\tilde{A}_1(\gamma)$, $\pi_+(B)$ and $\pi_-(B)$ of which are equal to ∞ and $-\infty$, respectively, and the other eigenvalues of $F(\gamma)$ go to those of $\lim_{\gamma \rightarrow 0} \gamma^{-1}\tilde{A}_2(\gamma)$. Observe that:

$$\gamma^{-1}\tilde{A}_2(\gamma) = \tilde{V}_2^*(\gamma)F(\gamma)\tilde{V}_2(\gamma) = \tilde{V}_2^*(\gamma)A(\gamma)\tilde{V}_2(\gamma) + \gamma^{-1}\tilde{V}_2^*(\gamma)B\tilde{V}_2(\gamma),$$

$$B = \tilde{V}_1(0)\tilde{A}_1(0)\tilde{V}_1^*(0).$$

Hence

$$\gamma^{-1}\tilde{V}_2^*(\gamma)B\tilde{V}_2(\gamma) = \gamma^{-1}\tilde{V}_2^*(\gamma)\tilde{V}_1(0)\tilde{A}_1(0)\tilde{V}_1^*(0)\tilde{V}_2(\gamma).$$

Since $\tilde{V}_1^*(0)\tilde{V}_2(\gamma)$ is analytic and vanishes at $\gamma = 0$, it follows that

$$\lim_{\gamma \rightarrow 0} \gamma^{-1}\tilde{V}_2^*(\gamma)B\tilde{V}_2(\gamma) = 0.$$

This shows

$$\lim_{\gamma \rightarrow 0} \gamma^{-1}\tilde{A}_2(\gamma) = \tilde{V}_2^*(0)A(0)\tilde{V}_2(0).$$

Since V_2 and $\tilde{V}_2(0)$ are isometries with the same image, the result follows. \square

Proof of Theorem 1.

Denote

$$F(\gamma) = \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix}.$$

First we show that $\psi(M) \geq \min\{r_+, r_-\}$. To simplify notation, we abbreviate $\{\pi_+(M), \pi_0(M), \pi_-(M)\}$ to $\{\pi_+, \pi_0, \pi_-\}$. By Lemma 1 and the fact that the inertia of $F(\gamma)$ is a constant, there exist $V(\gamma) \in \mathbb{C}^{(m+n) \times (\pi_+ + \pi_-)}$ and $E(\gamma) \in \mathbb{C}^{(\pi_+ + \pi_-) \times (\pi_+ + \pi_-)}$, both analytic on $(0, \infty)$, such that

$F(\gamma) = V(\gamma)E(\gamma)V^*(\gamma)$, where $V^*(\gamma)V(\gamma) = I, E(\gamma)$ is Hermitian and the eigenvalues of $E(\gamma)$ are equal to the nonzero eigenvalues of $F(\gamma)$. Hence

$$\begin{aligned} \text{rank} \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} &= \text{rank} \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} F(\gamma) \right\} \\ &= \text{rank} \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} V(\gamma)E(\gamma)V^*(\gamma) \right\} \\ &= m + n - \pi_+ - \pi_- \\ &\quad + \text{rank} \left\{ I + V^*(\gamma) \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} V(\gamma)E(\gamma) \right\}. \end{aligned}$$

The last equality follows from Lemma 4. Denote

$$H(\gamma, \Delta) = E^{-1}(\gamma) + V^*(\gamma) \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} V(\gamma), \tag{7}$$

then $\text{rank } H(\gamma, \Delta)$ is independent of γ and

$$\det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0$$

if and only if $\det H(\gamma, \Delta) = 0$ for all $\gamma \in (0, \infty)$.

Since $E(\gamma)$ is analytic and nonsingular, and $V(\gamma)$ is analytic, it follows that for a fixed Δ , the eigenvalues of $H(\gamma, \Delta)$ are continuous in γ on $(0, \infty)$. Since $\text{rank } [H(\gamma, \Delta)]$ is independent of γ , we conclude that for a fixed Δ , the inertia of $H(\gamma, \Delta)$ is independent of γ . Consequently, this inertia can be denoted by $\{\pi_+^\Delta, \pi_0^\Delta, \pi_-^\Delta\}$.

Assume now that $\pi_+ > 0$ and $\pi_- > 0$. Then the eigenvalues of $H(\gamma, 0) = E^{-1}(\gamma)$ are

$$\lambda_{\pi_+}^{-1}[F(\gamma)] \geq \dots \geq \lambda_1^{-1}[F(\gamma)] > 0 > \lambda_{m+n}^{-1}[F(\gamma)] \geq \dots \geq \lambda_{\pi_+ + \pi_0 + 1}^{-1}[F(\gamma)].$$

If $\|\Delta\| < r_+ = \sup_{\gamma > 0} \lambda_1^{-1}[F(\gamma)]$, then there exists a γ_0 such that $\|\Delta\| < \lambda_1^{-1}[F(\gamma_0)]$. Since

$$\|V^*(\gamma_0) \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} V(\gamma_0)\| \leq \|\Delta\|,$$

it follows from Lemma 3 that $\pi_+^\Delta \geq \pi_+$. Similarly, if $\|\Delta\| < r_-$, then $\pi_-^\Delta \geq \pi_-$. Obviously these reasonings also hold if $\pi_+ = 0$ or $\pi_- = 0$.

Hence, if $\|\Delta\| < \min\{r_+, r_-\}$, then $\text{rank}[H(\gamma, \Delta)] = \pi_+^\Delta + \pi_-^\Delta \geq \pi_+ + \pi_-$. This forces $\text{rank}[H(\gamma, \Delta)] = \pi_+ + \pi_-$, i.e., $\det[H(\gamma, \Delta)] \neq 0$. This shows that $\psi(M) \geq \min\{r_+, r_-\}$.

Notice that the converse inequality $\psi(M) \leq \min\{r_+, r_-\}$ follows from the following claim:

Claim 1. (a) If $\lambda_i[F(\gamma)]$ has a local extremum λ_0 at $\gamma_0 \in (0, \infty)$, then $\psi(M) \leq |\lambda_0|^{-1}$. (b) If $\lim_{\gamma \rightarrow 0} \lambda_i[F(\gamma)] = \lambda_0$ or $\lim_{\gamma \rightarrow \infty} \lambda_i[F(\gamma)] = \lambda_0$ for some $|\lambda_0| < \infty$, then $\psi(M) \leq |\lambda_0|^{-1}$.

It remains to prove this claim. If $\lambda_0 = 0$, we certainly have $\psi(M) \leq |\lambda_0|^{-1}$. So we assume $|\lambda_0| > 0$.

First we show (a). Since λ_0 is a local extremum of $\lambda_i[F(\gamma)]$ at $\gamma_0 \in (0, \infty)$, it follows from Lemma 2 that there exists

$$v = \begin{bmatrix} x \\ y \end{bmatrix},$$

where $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ such that $F(\gamma_0)v = \lambda_0 v$, i.e.,

$$\gamma_0 Sx + Ny = \lambda_0 x, \tag{8}$$

$$N^*x + \frac{1}{\gamma_0} Ry = \lambda_0 y, \tag{9}$$

$$v^* \frac{dF(\gamma_0)}{d\gamma} v = x^* Sx - \frac{1}{\gamma_0^2} y^* Ry = 0.$$

Multiplying (8) and (9) from the left by x^* and y^* , respectively, subtracting the resulting equations, and noting that $x^* Ny$ must be real, we get

$$\lambda_0(x^*x - y^*y) = \gamma_0 x^* Sx - \frac{1}{\gamma_0} y^* Ry = 0,$$

which implies $x^*x = y^*y$. Let $\Delta = -\lambda_0^{-1} x y^* / x^* x$. Then it is easy to verify that $\|\Delta\| = |\lambda_0|^{-1}$, and

$$\left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} \begin{bmatrix} \gamma_0 S & N \\ N^* & R/\gamma_0 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \end{bmatrix} = 0,$$

which means that

$$\det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0.$$

By definition, we have $\psi(M) \leq |\lambda_0|^{-1}$.

Next we show (b). It suffices to do this for the case when $\lim_{\gamma \rightarrow 0} \lambda_i[F(\gamma)] = \lambda_0$. The proof for the other case can be obtained similarly by permuting the four blocks in $F(\gamma)$ and exchanging Δ and Δ^* .

First, suppose that R is nonsingular. By applying Lemma 5 with

$$A(\gamma) = \begin{bmatrix} \gamma S & N \\ N^* & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix},$$

we see that as $\gamma \rightarrow 0$, m eigenvalues of $F(\gamma)$ go to $\pm\infty$ and the others to zero. Hence λ_0 can only be zero. This leads to the trivial case. Therefore, we can assume that R is singular and R has spectral decomposition.

$$R = [U_1 \ U_2] \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} [U_1 \ U_2]^*,$$

where A is nonsingular. By Lemma 5, λ_0 is equal to an eigenvalue of

$$\begin{bmatrix} 0 & NU_2 \\ U_2^*N^* & 0 \end{bmatrix},$$

or equivalently, λ_0 is equal to a singular value of NU_2 . Therefore, there exist unit vectors x and y such that $NU_2x = \lambda_0y$. Let $\Delta = -(\lambda_0^{-1}U_2^*y^*)^*$. Then $\|\Delta\| = |\lambda_0|^{-1}$ and

$$\left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} \begin{bmatrix} S & N \\ N^* & R \end{bmatrix} \right\} \begin{bmatrix} 0 \\ U_2x \end{bmatrix} = 0,$$

which implies $\psi(M) \leq |\lambda_0|^{-1}$. \square

The computation of $\psi(M)$ involves two univariate minimizations. The following proposition says that any local minimization algorithm can be applied to find $\psi(M)$ accurately.

Proposition 1. (a) If $r_+ \leq r_-$, then $\lambda_1[F(\gamma)]$ is unimodal and any local infimum of $-\lambda_{m+n}[F(\gamma)]$ must be equal or smaller than $(r_+)^{-1}$. (b) If $r_- \leq r_+$, then $-\lambda_{m+n}[F(\gamma)]$ is unimodal and any local minimum of $\lambda_1[F(\gamma)]$ must be equal or smaller than $(r_-)^{-1}$.

Here, by unimodality, we mean that a function has only one local infimum.

Proof. Again write

$$F(\gamma) = \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix}.$$

Assume $r_+ \leq r_-$, then $\psi(M) = r_+$. Recall that $r_+ = \{\inf_{\gamma>0} \lambda_1[F(\gamma)]\}^{-1}$. Suppose that $\lambda_1[F(\gamma)]$ is not a unimodal function, then between any two local minima, $\lambda_1[F(\gamma)]$ must have a local maximum, say at $\gamma_1 \in (0, \infty)$, with $\lambda_1[F(\gamma_1)] > (r_+)^{-1}$. From the claim in the proof of Theorem 1, we have $\psi(M) \leq \lambda_1^{-1}[F(\gamma_1)] < r_+ = \psi(M)$, which is a contradiction. For the same reason, any local infimum of $-\lambda_{m+n}[F(\gamma)]$ must be equal to or smaller than $(r_+)^{-1}$. This proves (a).

Statement (b) follows from the same argument. \square

In general

$$\lambda_1 \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix} \quad \text{and} \quad -\lambda_{m+n} \begin{bmatrix} \gamma S & N \\ N^* & R/\gamma \end{bmatrix},$$

are not both unimodal. For example,

$$\lambda_1 \begin{bmatrix} \gamma & 3 \\ 3 & 1/\gamma \end{bmatrix}$$

is unimodal whereas

$$-\lambda_2 \begin{bmatrix} \gamma & 3 \\ 3 & 1/\gamma \end{bmatrix}$$

is not. Proposition 1 says that although local minimization algorithms cannot guarantee to find both r_+ and r_- , they can find $\psi(M)$ accurately.

3. Lower bound for $\psi_K(M)$

For a given matrix

$$M = \begin{bmatrix} S & N \\ N^* & R \end{bmatrix}$$

and a subspace \mathcal{X}_K of $\mathbb{C}^{n \times m}$:

$$\begin{aligned} \mathcal{X}_K &:= \{ \Delta \in \mathbb{C}^{n \times m} : \Delta \\ &= \text{block diag}[\Delta_1, \dots, \Delta_K], \Delta_i \in \mathbb{C}^{n_i \times m_i} \}. \end{aligned} \tag{10}$$

This section concerns

$$\psi_K(M) := \inf \left\{ \|\Delta\| : \Delta \in \mathcal{X}_K, \det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0 \right\}. \tag{11}$$

For $\gamma \in \mathbb{R}_+^K := \{[\gamma_1, \gamma_2, \dots, \gamma_K]^T : \gamma_i > 0\}$, define scaling matrix

$$D(\gamma) = \text{diag}[\sqrt{\gamma_1}I, \dots, \sqrt{\gamma_K}I, I/\sqrt{\gamma_1}, \dots, I/\sqrt{\gamma_K}].$$

It is easy to verify that for $\Delta \in \mathcal{X}_K$,

$$D(\gamma) \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} D(\gamma) = \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix}.$$

Thus,

$$\det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0$$

if and only if

$$\det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} D(\gamma)MD(\gamma) \right\} = 0.$$

Note that for all $\gamma \in \mathbb{R}_+^K$, $D(\gamma)MD(\gamma)$ have the same inertia. Similarly from Section 2, define:

$$r_+ := \begin{cases} \left\{ \inf_{\gamma \in \mathbb{R}_+^K} \lambda_1 [D(\gamma)MD(\gamma)] \right\}^{-1} & \text{if } \pi_+(M) > 0, \\ \infty & \text{if } \pi_+(M) = 0. \end{cases}$$

$$r_- := \begin{cases} \left\{ \inf_{\gamma \in \mathbb{R}_+^K} \{-\lambda_{n+m} [D(\gamma)MD(\gamma)]\} \right\}^{-1} & \text{if } \pi_-(M) > 0, \\ \infty & \text{if } \pi_-(M) = 0. \end{cases}$$

Theorem 2. $\psi_K(M) \geq \min\{r_+, r_-\}$.

Proof. Again abbreviate $\{\pi_+(M), \pi_0(M), \pi_-(M)\}$ to $\{\pi_+, \pi_0, \pi_-\}$. Since $\text{rank}(M) = \pi_+ + \pi_-$, it follows that M can be decomposed as $M = UAU^*$ where $A \in \mathbb{R}^{(\pi_+ + \pi_-) \times (\pi_+ + \pi_-)}$ is diagonal and nonsingular, $U \in \mathbb{C}^{(m+n) \times (\pi_+ + \pi_-)}$, and $U^*U = I$.

Let $X(\gamma) = D(\gamma)U$. Then $X(\gamma)$ has full column rank. By carrying out the Gram–Schmidt orthonormalization on the columns of $X(\gamma)$, we get $X(\gamma) = V(\gamma)R(\gamma)$ where $V^*(\gamma)V(\gamma) = I$ and $R(\gamma)$ is nonsingular. It is easy to see from the orthonormalization process that the maps from X to V and to R are analytic when X has full column rank. Hence, $V(\gamma)$ and $R(\gamma)$ are analytic in γ .

Let $E(\gamma) = R(\gamma)AR^*(\gamma)$, then $E(\gamma)$ is Hermitian, nonsingular and analytic. Furthermore

$$D(\gamma)MD(\gamma) = D(\gamma)UAU^*D(\gamma) = V(\gamma)E(\gamma)V^*(\gamma).$$

Since $V^*(\gamma)V(\gamma) = I$, the eigenvalues of $E(\gamma)$ are the same as the nonzero eigenvalues of $D(\gamma)MD(\gamma)$. The rest is completely analogous to the proof of $\psi(M) \geq \min\{r_+, r_-\}$ in Theorem 1, by replacing the scalar γ with the vector γ , $F(\gamma)$ with $D(\gamma)MD(\gamma)$, and $\psi(M)$ with $\psi_K(M)$. \square

Under some conditions, the inequality in Theorem 2 can be proved to be an equality. For example, suppose that $r_+ \leq r_-$ and $\inf_{\gamma \in \mathbb{R}_+^K} \lambda_1 [D(\gamma)MD(\gamma)]$ is attained at $\gamma_0 \in \mathbb{R}_+^K$. If $\lambda_1 [D(\gamma_0)MD(\gamma_0)]$ is a simple eigenvalue, then $\psi_K(M) = r_+$. A more general result is as follows.

Theorem 3. Suppose $\lambda_i [D(\gamma)MD(\gamma)]$ has a local extremum λ_0 at $\gamma_0 \in \mathbb{R}_+^K$, and λ_0 is a simple eigenvalue of $D(\gamma_0)MD(\gamma_0)$, then $\psi_K(M) \leq |\lambda_0|^{-1}$.

Proof. If $\lambda_0 = 0$, the assertion is obvious. Hence we assume that $\lambda_0 \neq 0$ in the following proof. Since λ_0 is a simple eigenvalue and is a local extremum, it

follows that $\lambda_i[D(\gamma)MD(\gamma)]$ must be analytic at γ_0 with $\partial\lambda_i[D(\gamma_0)MD(\gamma_0)]/\partial\gamma = 0$.

Let v be the eigenvector of $D(\gamma_0)MD(\gamma_0)$ corresponding to λ_0 , then $D(\gamma_0)MD(\gamma_0)v = \lambda_0v$ and

$$\frac{\partial\lambda_i[D(\gamma_0)MD(\gamma_0)]}{\partial\gamma_j} = v^* \left[\frac{\partial D(\gamma_0)}{\partial\gamma_j} MD(\gamma_0) + D(\gamma_0)M \frac{\partial D(\gamma_0)}{\partial\gamma_j} \right] v = 0, \quad j = 1, 2, \dots, K.$$

Partition v according to the structure of $D(\gamma)$, $v = [v_1^T, v_2^T, \dots, v_K^T, v_{K+1}^T, \dots, v_{2K}^T]^T$. Notice that

$$\begin{aligned} \frac{\partial D(\gamma_0)}{\partial\gamma_j} &= \frac{1}{2} \text{diag}[0, \dots, \gamma_{0j}^{-1/2}I, \dots, 0, 0, \dots, -\gamma_{0j}^{-3/2}I, \dots, 0] \\ &= \frac{1}{2\gamma_{0j}} \text{diag}[0, \dots, I, \dots, 0, 0, \dots, -I, \dots, 0]D(\gamma_0) \\ &= \frac{1}{2\gamma_{0j}} D(\gamma_0) \text{diag}[0, \dots, I, \dots, 0, 0, \dots, -I, \dots, 0], \end{aligned}$$

where γ_{0j} denotes the j th element of γ_0 , we get,

$$\frac{\partial\lambda_i[D(\gamma_0)MD(\gamma_0)]}{\partial\gamma_j} = \frac{\lambda_0}{\gamma_{0j}} (v_j^* v_j - v_{j+K}^* v_{j+K}) = 0, \quad j = 1, 2, \dots, K, \tag{12}$$

which implies $v_j^* v_j = v_{j+K}^* v_{j+K}$ for $j = 1, 2, \dots, K$.

Let $\Delta = \text{diag}[\Delta_1, \dots, \Delta_K]$ where

$$\Delta_j = \begin{cases} -\lambda_0^{-1} v_j v_{j+K}^* / \|v_j\|^2 & \text{if } \|v_j\| \neq 0, \\ 0 & \text{if } \|v_j\| = 0, \end{cases}$$

then $\|\Delta\| = |\lambda_0|^{-1}$ and

$$\left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} D(\gamma_0)MD(\gamma_0) \right\} v = v + \lambda_0 \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} v = \begin{bmatrix} v_1 + \lambda_0 \Delta_1 v_{K+1} \\ \vdots \\ v_K + \lambda_0 \Delta_K v_{2K} \\ v_{K+1} + \lambda_0 \Delta_1^* v_1 \\ \vdots \\ v_{2K} + \lambda_0 \Delta_K^* v_K \end{bmatrix} = 0.$$

This shows

$$\text{rank} \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} < n + m.$$

Therefore, $\psi_K(M) \leq |\lambda_0|^{-1}$. \square

Corollary 1. *Let λ_+ be a local infimum of $\lambda_1[D(\gamma)MD(\gamma)]$ at γ_+ and λ_- be a local infimum of $-\lambda_{m+n}[D(\gamma)MD(\gamma)]$ at γ_- , then $\psi_K(M) \geq \min\{(\lambda_+)^{-1}, (\lambda_-)^{-1}\}$. Suppose $\lambda_+ \geq \lambda_-$ and λ_+ is a simple eigenvalue of $\lambda_1[D(\gamma_+)MD(\gamma_+)]$, then $\psi_K(M) = (\lambda_+)^{-1}$. A similar result holds when $\lambda_+ \leq \lambda_-$.*

When Δ is a full block, $\psi(M)$ can be exactly computed by any one-dimensional local optimization method. When Δ is of block diagonal structure, a lower bound of $\psi_K(M)$ can be obtained by minimizing $\lambda_1[D(\gamma)MD(\gamma)]$ and $-\lambda_{m+n}[D(\gamma)MD(\gamma)]$. An interesting question that follows is if the global infima of $\lambda_1[D(\gamma)MD(\gamma)]$ and $-\lambda_{m+n}[D(\gamma)MD(\gamma)]$ are easily computable. The answer for the general case is not available yet, but under some conditions, $\min\{r_+, r_-\}$ can be reliably computed.

In some applications, the given Hermitian matrix

$$M = \begin{bmatrix} S & N \\ N^* & R \end{bmatrix}$$

satisfies $S, R \geq 0$. In this case, $r_+ \leq r_-$, and $\lambda_1[D(\gamma)MD(\gamma)]$ is unimodal.

Proposition 2. *Suppose $S, R \geq 0$. Then $\lambda_1[D(\gamma)MD(\gamma)] \geq -\lambda_{m+n}[D(\gamma)MD(\gamma)]$ and $r_+ \leq r_-$.*

Proof. Partition

$$D(\gamma)MD(\gamma) = \begin{bmatrix} F_{11}(\gamma) & F_{12}(\gamma) \\ F_{12}^*(\gamma) & F_{22}(\gamma) \end{bmatrix}$$

consistently with M and denote

$$F_1(\gamma) = \begin{bmatrix} F_{11}(\gamma) & 0 \\ 0 & F_{22}(\gamma) \end{bmatrix} \text{ and } F_2(\gamma) = \begin{bmatrix} 0 & F_{12}(\gamma) \\ F_{12}^*(\gamma) & 0 \end{bmatrix}.$$

Then $D(\gamma)MD(\gamma) = F_1(\gamma) + F_2(\gamma)$ and

$$\lambda_1[F_2(\gamma)] = -\lambda_{m+n}[F_2(\gamma)] = \|F_{12}(\gamma)\|.$$

Since $F_1(\gamma) \geq 0$, it follows from [8] Theorem I.3.14 that for $i = 1, 2, \dots, m+n$

$$\lambda_i[D(\gamma)MD(\gamma)] \geq \lambda_i[F_2(\gamma)].$$

Therefore,

$$\lambda_1[D(\gamma)MD(\gamma)] \geq \lambda_1[F_2(\gamma)] = -\lambda_{m+n}[F_2(\gamma)] \geq -\lambda_{m+n}[D(\gamma)MD(\gamma)],$$

which implies $r_+ \leq r_-$. \square

From Proposition 2, we only need to compute $r_+ = \{ \inf_{\gamma \in \mathbb{R}_+^K} \lambda_1 [D(\gamma)MD(\gamma)] \}^{-1}$. A continuous multivariate function is said to be unimodal if the inverse image of $(-\infty, y)$ is connected for all $y \in \mathbb{R}$.

Proposition 3. *If $S, R \geq 0$, then $\lambda_1 [D(\gamma)MD(\gamma)]$ is a unimodal function.*

Proof. The unimodality can be proved by showing that the following set

$$\Omega(\alpha) := \{ \gamma \in \mathbb{R}_+^K : \lambda_1 [D(\gamma)MD(\gamma)] \leq \alpha \}$$

is convex for all $\alpha > 0$.

For the simplicity of notation, we assume that each block of Δ is square. In this case, $D(\gamma)$ can be written as

$$D(\gamma) = \begin{bmatrix} \Gamma^{1/2} & 0 \\ 0 & \Gamma^{-1/2} \end{bmatrix},$$

where $\Gamma = \text{diag}[\gamma_1 I, \gamma_2 I, \dots, \gamma_K I]$. The following proof can be easily generalized to the case when the blocks are nonsquare.

Notice that $\lambda_1 [D(\gamma)MD(\gamma)] \leq \alpha$ if and only if

$$D(\gamma)MD(\gamma) = \begin{bmatrix} \Gamma^{1/2} S \Gamma^{1/2} & \Gamma^{1/2} N \Gamma^{-1/2} \\ \Gamma^{-1/2} N^* \Gamma^{1/2} & \Gamma^{-1/2} R \Gamma^{-1/2} \end{bmatrix} \leq \alpha I,$$

which in turn is equivalent to

$$\begin{bmatrix} \Gamma S \Gamma & \Gamma N \\ N^* \Gamma & R \end{bmatrix} \leq \alpha \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}.$$

Suppose that $\gamma_a, \gamma_b \in \Omega(\alpha)$, then:

$$\begin{bmatrix} \Gamma_a S \Gamma_a & \Gamma_a N \\ N^* \Gamma_a & R \end{bmatrix} \leq \alpha \begin{bmatrix} \Gamma_a & 0 \\ 0 & \Gamma_a \end{bmatrix}, \tag{13}$$

$$\begin{bmatrix} \Gamma_b S \Gamma_b & \Gamma_b N \\ N^* \Gamma_b & R \end{bmatrix} \leq \alpha \begin{bmatrix} \Gamma_b & 0 \\ 0 & \Gamma_b \end{bmatrix}. \tag{14}$$

Multiplying (13) with t and (14) with $1 - t$, $t \in [0, 1]$, and adding the two resulting inequalities, we have

$$\begin{bmatrix} t\Gamma_a S \Gamma_a + (1-t)\Gamma_b S \Gamma_b & [t\Gamma_a + (1-t)\Gamma_b]N \\ N^*[t\Gamma_a + (1-t)\Gamma_b] & R \end{bmatrix} \leq \alpha \begin{bmatrix} t\Gamma_a + (1-t)\Gamma_b & 0 \\ 0 & t\Gamma_a + (1-t)\Gamma_b \end{bmatrix}.$$

Let $X = S^{1/2} \Gamma_a, Y = S^{1/2} \Gamma_b$. Then from

$$\begin{aligned}
 & [tX + (1 - t)Y]^*[tX + (1 - t)Y] - [tX^*X + (1 - t)Y^*Y] \\
 & = -t(1 - t)(X - Y)^*(X - Y) \leq 0,
 \end{aligned}$$

we get

$$[t\Gamma_a + (1 - t)\Gamma_b]S[t\Gamma_a + (1 - t)\Gamma_b] \leq t\Gamma_a S\Gamma_a + (1 - t)\Gamma_b S\Gamma_b,$$

which implies that $t\gamma_a + (1 - t)\gamma_b \in \Omega(\alpha)$. Therefore, $\Omega(\alpha)$ is convex. It follows that $\lambda_1[D(\gamma)MD(\gamma)]$ is unimodal. \square

4. The two block case

In this section, we will show that the lower bound of $\psi_K(M)$ given by Theorem 3 is actually the exact value of $\psi_K(M)$ when $K = 2$.

For the sake of clarity, we replace $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ in $D(\gamma)$ with α and β , respectively, and replace $D(\gamma)$ with $D(\alpha, \beta) = \text{diag}[\alpha I, \beta I, (1/\alpha)I, (1/\beta)I]$. Denote $P(\alpha, \beta) = D(\alpha, \beta)MD(\alpha, \beta)$. Then:

$$\begin{aligned}
 r_+ &= \begin{cases} \{\inf_{\alpha, \beta > 0} \lambda_1[P(\alpha, \beta)]\}^{-1} & \text{if } \pi_+(M) > 0, \\ \infty & \text{if } \pi_+(M) = 0. \end{cases} \\
 r_- &= \begin{cases} \{\inf_{\alpha, \beta > 0} \{-\lambda_{m+n}[P(\alpha, \beta)]\}\}^{-1} & \text{if } \pi_-(M) > 0, \\ \infty & \text{if } \pi_-(M) = 0. \end{cases}
 \end{aligned}$$

Theorem 4. *When $K = 2$, $\psi_K(M) = \min\{r_+, r_-\}$. If $r_+ \leq r_-$, then any local infimum of $\lambda_1[P(\alpha, \beta)]$ is its global infimum. Furthermore, let λ_0 be a local infimum of $-\lambda_{m+n}[P(\alpha, \beta)]$, then $r_+ \leq \lambda_0^{-1}$. A similar result holds when $r_+ \geq r_-$.*

This theorem says that, in the two block case, the maximum of a local infimum of $\lambda_1[P(\alpha, \beta)]$ and a local infimum of $-\lambda_{m+n}[P(\alpha, \beta)]$ also exactly gives $\psi_K^{-1}(M)$. Thus $\psi_K(M)$ can be reliably computed by any reasonable nonlinear programming method. Before the proof of Theorem 4, we state a lemma which is a sort of extension of Lemma 5.

Lemma 6. *Let $F(\gamma) = A(\gamma) + \gamma^{-1}B + \gamma^{-2}C \in \mathbb{C}^{n \times n}$ where $A(\gamma)$ is a Hermitian matrix function analytic on an open interval Γ around 0 and B, C are constant Hermitian matrices. Assume that C has a spectral decomposition,*

$$C = [V_1 \ V_2] \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1 \ V_2]^*,$$

where $A_1 \in \mathbb{R}^{r \times r}$ is nonsingular and also assume that $V_2^* B V_2 = 0$. Then as γ approaches 0, $\pi_+(C)$ eigenvalues of $F(\gamma)$ go to ∞ , $\pi_-(C)$ eigenvalues of $F(\gamma)$ to $-\infty$, and the rest to the eigenvalues of $V_2^*[A(0) - B V_1 A_1^{-1} V_1^* B] V_2$.

Proof. Without loss of generality, assume that an analytic spectral decomposition of $\gamma^2 F(\gamma)$ is

$$\gamma^2 F(\gamma) = [\tilde{V}_1(\gamma) \quad \tilde{V}_2(\gamma)] \begin{bmatrix} \tilde{A}_1(\gamma) & 0 \\ 0 & \tilde{A}_2(\gamma) \end{bmatrix} [\tilde{V}_1(\gamma) \quad \tilde{V}_2(\gamma)]^*$$

such that $\tilde{A}_1(0) = A_1$ and $\tilde{A}_2(0) = 0$. Then

$$F(\gamma) = [\tilde{V}_1(\gamma) \quad \tilde{V}_2(\gamma)] \begin{bmatrix} \gamma^{-2} \tilde{A}_1(\gamma) & 0 \\ 0 & \gamma^{-2} \tilde{A}_2(\gamma) \end{bmatrix} [\tilde{V}_1(\gamma) \quad \tilde{V}_2(\gamma)]^*.$$

Clearly as γ goes to 0, r eigenvalues of $F(\gamma)$ go to the diagonal elements of $\lim_{\gamma \rightarrow 0} \gamma^{-2} \tilde{A}_1(\gamma)$, $\pi_+(C)$ and $\pi_-(C)$ of which are ∞ and $-\infty$, respectively, and the other eigenvalues of $F(\gamma)$ go to those of $\lim_{\gamma \rightarrow 0} \gamma^{-2} \tilde{A}_2(\gamma)$. Note that

$$\begin{aligned} \gamma^{-2} \tilde{A}_2(\gamma) &= \tilde{V}_2^*(\gamma) F(\gamma) \tilde{V}_2(\gamma) \\ &= \tilde{V}_2^*(\gamma) A(\gamma) \tilde{V}_2(\gamma) + \gamma^{-1} \tilde{V}_2^*(\gamma) B \tilde{V}_2(\gamma) + \gamma^{-2} \tilde{V}_2^*(\gamma) C \tilde{V}_2(\gamma). \end{aligned}$$

Since $\tilde{V}_2^*(0) B \tilde{V}_2(0) = 0$, it follows that

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \gamma^{-1} \tilde{V}_2^*(\gamma) B \tilde{V}_2(\gamma) &= \frac{d}{d\gamma} \tilde{V}_2^*(0) B \tilde{V}_2(0) \\ &= \dot{\tilde{V}}_2^*(0) B \tilde{V}_2(0) + \tilde{V}_2^*(0) B \dot{\tilde{V}}_2(0). \end{aligned}$$

Also since $C = \tilde{V}_1(0) \tilde{A}_1(0) \tilde{V}_1^*(0)$ and $\tilde{V}_1^*(0) \tilde{V}_2(0) = 0$, it follows that

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \gamma^{-2} \tilde{V}_2^*(\gamma) C \tilde{V}_2(\gamma) &= \frac{1}{2} \frac{d^2}{d\gamma^2} \tilde{V}_2^*(0) C \tilde{V}_2(0) \\ &= \frac{1}{2} \frac{d}{d\gamma} [\dot{\tilde{V}}_2^*(0) C \tilde{V}_2(0) + \tilde{V}_2^*(0) C \dot{\tilde{V}}_2(0)] \\ &= \frac{1}{2} [\ddot{\tilde{V}}_2^*(0) C \tilde{V}_2(0) + 2 \dot{\tilde{V}}_2^*(0) C \dot{\tilde{V}}_2(0) + \tilde{V}_2^*(0) C \ddot{\tilde{V}}_2(0)] \\ &= \dot{\tilde{V}}_2^*(0) C \dot{\tilde{V}}_2(0). \end{aligned}$$

The last equality follows from $C \tilde{V}_2(0) = 0$. Hence

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \gamma^{-2} \tilde{A}_2(\gamma) &= \tilde{V}_2^*(0) A(0) \tilde{V}_2(0) + \dot{\tilde{V}}_2^*(0) B \tilde{V}_2(0) + \tilde{V}_2^*(0) B \dot{\tilde{V}}_2(0) \\ &\quad + \dot{\tilde{V}}_2^*(0) C \dot{\tilde{V}}_2(0). \end{aligned}$$

Observe that

$$\gamma^2 F(\gamma) \tilde{V}_2(\gamma) = \tilde{V}_2(\gamma) \tilde{A}_2(\gamma).$$

Differentiating both sides, we obtain

$$\begin{aligned}
 & [2\gamma A(\gamma) + \gamma^2 \dot{A}(\gamma) + B] \tilde{V}_2(\gamma) + [\gamma^2 A(\gamma) + \gamma B + C] \dot{\tilde{V}}_2(\gamma) \\
 & = \dot{\tilde{V}}_2(\gamma) \tilde{A}_2(\gamma) + \tilde{V}_2(\gamma) \dot{\tilde{A}}_2(\gamma).
 \end{aligned}$$

Evaluating at $\gamma = 0$, we obtain

$$B\tilde{V}_2(0) + C\dot{\tilde{V}}_2(0) = \tilde{V}_2(0)\dot{\tilde{A}}_2(0).$$

Notice that

$$\begin{aligned}
 \dot{\tilde{A}}_2(0) & = \lim_{\gamma \rightarrow 0} \gamma^{-1} \tilde{A}_2(\gamma) = \lim_{\gamma \rightarrow 0} \tilde{V}_2^*(\gamma) [\gamma A(\gamma) + B + \gamma^{-1} C] \tilde{V}_2(\gamma) \\
 & = \tilde{V}_2^*(0) B \tilde{V}_2(0) + \lim_{\gamma \rightarrow 0} \gamma^{-1} \tilde{V}_2^*(\gamma) C \tilde{V}_2(\gamma) = 0.
 \end{aligned}$$

We then have

$$B\tilde{V}_2(0) + C\dot{\tilde{V}}_2(0) = 0.$$

From $C = \tilde{V}_1(0)\tilde{A}_1(0)\tilde{V}_1^*(0)$, we get

$$B\tilde{V}_2(0) + \tilde{V}_1(0)\tilde{A}_1(0)\tilde{V}_1^*(0)\dot{\tilde{V}}_2(0) = 0.$$

This leads to

$$\tilde{V}_1^*(0)\dot{\tilde{V}}_2(0) = -\tilde{A}_1^{-1}(0)\tilde{V}_1^*(0)B\tilde{V}_2(0).$$

Therefore,

$$\begin{aligned}
 \lim_{\gamma \rightarrow 0} \gamma^{-2} \tilde{A}_2(\gamma) & = \tilde{V}_2^*(0)A(0)\tilde{V}_2(0) + \tilde{V}_2^*(0)B\dot{\tilde{V}}_2(0) \\
 & = \tilde{V}_2^*(0)A(0)\tilde{V}_2(0) + \tilde{V}_2^*(0)B[\tilde{V}_1(0)\tilde{V}_1^*(0) + \tilde{V}_2(0)\tilde{V}_2^*(0)]\dot{\tilde{V}}_2(0) \\
 & = \tilde{V}_2^*(0)A(0)\tilde{V}_2(0) + \tilde{V}_2^*(0)B\tilde{V}_1(0)\tilde{V}_1^*(0)\dot{\tilde{V}}_2(0) \\
 & = \tilde{V}_2^*(0)[A(0) - B\tilde{V}_1(0)\tilde{A}_1^{-1}(0)\tilde{V}_1^*(0)B]\tilde{V}_2(0).
 \end{aligned}$$

Note that $\tilde{V}_1(0)\tilde{A}_1(0)\tilde{V}_1^*(0) = V_1 A_1 V_1^*$ and $\tilde{V}_2(0)$, V_2 are isometries with the same image. The result follows. \square

Proof of Theorem 4. It suffices to prove that if $\pi_+(M) > 0$ and λ_0 is a local infimum of $\lambda_1[P(\alpha, \beta)]$, then $\psi_K(M) \leq \lambda_0^{-1}$; and if $\pi_-(M) > 0$ and λ_0 is a local infimum of $-\lambda_{m+n}[P(\alpha, \beta)]$, then $\psi_K(M) \leq \lambda_0^{-1}$. By symmetry, we only need to prove the first statement. Typically, we have the following four cases:

- Case 1: The local infimum λ_0 is attained at $\alpha_0, \beta_0 \in (0, \infty)$.
- Case 2: $\lambda_0 = \lim_{\alpha \rightarrow \infty} \lambda_1[P(\alpha, k_0\alpha)]$, $k_0 \in (0, \infty)$.
- Case 3: $\lambda_0 = \lim_{\alpha \rightarrow \infty} \lambda_1[P(\alpha, \beta_0)]$, $\beta_0 \in (0, \infty)$.
- Case 4: $\lambda_0 = \lim_{\beta, \alpha/\beta \rightarrow \infty} \lambda_1[P(\alpha, \beta)]$.

Other cases such as α, β , and (or) $(\alpha/\beta) \rightarrow 0$ can be converted to one of the above cases by exchanging α with $1/\alpha$, β with $1/\beta$, α with β , or α with $1/\beta$, etc. and permuting

$$P(\alpha, \beta) = \begin{bmatrix} \alpha^2 S_1 & \alpha\beta S_2 & N_1 & \frac{\alpha}{\beta} N_2 \\ \alpha\beta S_2^* & \beta^2 S_3 & \frac{\beta}{\alpha} N_3 & N_4 \\ N_1^* & \frac{\beta}{\alpha} N_3^* & \frac{1}{\alpha^2} R_1 & \frac{1}{\alpha\beta} R_2 \\ \frac{\alpha}{\beta} N_2^* & N_4^* & \frac{1}{\alpha\beta} R_2^* & \frac{1}{\beta^2} R_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & \Delta_1 & 0 \\ 0 & 0 & 0 & \Delta_2 \\ \Delta_1^* & 0 & 0 & 0 \\ 0 & \Delta_2^* & 0 & 0 \end{bmatrix}$$

accordingly. Noticing the symmetry of the structure of $P(\alpha, \beta)$ with respect to $\alpha, \beta, 1/\alpha$ and $1/\beta$. For example, suppose $\lambda_0 = \lim_{\beta \rightarrow 0} \lambda_1[P(\alpha_0, \beta)]$. This case can be converted to case 3 by exchanging α with $1/\beta$, $1/\alpha$ with β , Δ_2 with Δ_1^* , Δ_2^* with Δ_1 , and permuting $P(\alpha, \beta)$ accordingly.

We can assume the above cases since from the following discussion, we see that the limits exist if they do not go to infinity.

The following proof is long and dry. The main idea is to construct a

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

by using the eigenvector of $P(\alpha, \beta)$ and to show that $\|\Delta\| = \lambda_0^{-1}$ and

$$\det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0$$

by exploiting the special properties of the local infima. This leads to $\psi_K(M) \leq \lambda_0^{-1}$.

Case 1 occurs generally as we shall see later from the conditions under which cases 2–4 occur. The proof for case 1 is relatively simple. Most part of the proof is dedicated to cases 2–4.

Case 1: The local infimum λ_0 is attained at $\alpha_0, \beta_0 \in (0, \infty)$.

Let the multiplicity of λ_0 be r and let the corresponding eigenspace be spanned by the columns of $V \in \mathbb{C}^{(m+n) \times r}$ with $V^*V = I$. Then in a neighborhood of α_0, β_0 , $\lambda_1[P(\alpha, \beta)]$ can be written as

$$\lambda_1[P(\alpha, \beta)] = \lambda_0 + \lambda_{\max} \left[(\alpha - \alpha_0)V^* \frac{\partial P(\alpha_0, \beta_0)}{\partial \alpha} V + (\beta - \beta_0)V^* \frac{\partial P(\alpha_0, \beta_0)}{\partial \beta} V \right] + O[(\alpha - \alpha_0)^2 + (\beta - \beta_0)^2]^{1/2}.$$

Denote

$$\nabla = \left\{ \left[\begin{array}{c} \eta^* V^* \frac{\partial P(\alpha_0, \beta_0)}{\partial \alpha} V \eta \\ \eta^* V^* \frac{\partial P(\alpha_0, \beta_0)}{\partial \beta} V \eta \end{array} \right] : \eta \in \mathbb{C}^r, \eta^* \eta = 1 \right\},$$

then by the corollary of [4], p. 246, ∇ is convex. By Theorem 2 and Lemma 4 of [4], pp. 245–246, λ_0 is a local infimum of $\lambda_1[P(\alpha, \beta)]$ if and only if $0 \in \nabla$. Which means that there exists a unit vector $v = V\eta \in \mathbb{C}^{m+n}$ such that $v^*(\partial P(\alpha_0, \beta_0)/\partial\alpha)v = v^*(\partial P(\alpha_0, \beta_0)/\partial\beta)v = 0$. Partition v in accordance with the structure of $D(\alpha, \beta)$ as

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Then $P(\alpha_0, \beta_0)v = \lambda_0 v$ and

$$v^* \frac{\partial P(\alpha_0, \beta_0)}{\partial\alpha} v = \frac{2\lambda_0}{\alpha_0} (v_1^* v_1 - v_3^* v_3) = 0,$$

$$v^* \frac{\partial P(\alpha_0, \beta_0)}{\partial\beta} v = \frac{2\lambda_0}{\beta_0} (v_2^* v_2 - v_4^* v_4) = 0$$

(cf. (12), notice the slight difference). We get $v_1^* v_1 = v_3^* v_3, v_2^* v_2 = v_4^* v_4$.

Let

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix},$$

where

$$\Delta_j = \begin{cases} -\lambda_0^{-1} v_j v_{j+2}^* / \|v_j\|^2 & \text{if } \|v_j\| \neq 0, \\ 0 & \text{if } \|v_j\| = 0. \end{cases}$$

Then $\|\Delta\| = \lambda_0^{-1}$ and

$$\left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} P(\alpha_0, \beta_0) \right\} v = 0.$$

Therefore, $\psi_k(M) \leq \lambda_0^{-1}$.

Case 2: $\lambda_0 = \lim_{\alpha \rightarrow \infty} \lambda_1[P(\alpha, k_0\alpha)], k_0 \in (0, \infty)$.

Notice that

$$P(\alpha, k\alpha) = \begin{bmatrix} \alpha I & 0 & 0 & 0 \\ 0 & k\alpha I & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} I & 0 \\ 0 & 0 & 0 & \frac{1}{k\alpha} I \end{bmatrix} \begin{bmatrix} S & N \\ N^* & R \end{bmatrix} \begin{bmatrix} \alpha I & 0 & 0 & 0 \\ 0 & k\alpha I & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} I & 0 \\ 0 & 0 & 0 & \frac{1}{k\alpha} I \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 S(k) & N(k) \\ N^*(k) & \frac{1}{\alpha^2} R(k) \end{bmatrix},$$

where

$$\begin{aligned}
 S(k) &= \begin{bmatrix} I & 0 \\ 0 & kI \end{bmatrix} S \begin{bmatrix} I & 0 \\ 0 & kI \end{bmatrix}, \\
 N(k) &= \begin{bmatrix} I & 0 \\ 0 & kI \end{bmatrix} N \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix}, \\
 R(k) &= \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} R \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix}.
 \end{aligned}$$

It follows from Lemma 5 with $\gamma = \alpha^{-2}$ that, for a given k , $\lim_{\alpha \rightarrow \infty} \lambda_1[P(\alpha, k\alpha)]$ exists if and only if $S(k) \leq 0$, or equivalently if and only if $S \leq 0$. It follows that if $\lim_{\alpha \rightarrow \infty} \lambda_1[P(\alpha, k\alpha)]$ exists for one k , then the limit exists for all $k \in (0, \infty)$.

Let $\lambda(k) = \lim_{\alpha \rightarrow \infty} \lambda_1[P(\alpha, k\alpha)]$. Then it is easy to see that $\lambda(k)$ has a local infimum λ_0 at k_0 . By Lemma 5, $\lambda(k) = \|N^*(k)U(k)\|$ where the columns of $U(k)$ form an orthonormal basis of the kernel of $S(k)$.

Let

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

be an isometry onto the kernel of S . Then

$$U(k) = \begin{bmatrix} U_1 \\ \frac{1}{k}U_2 \end{bmatrix} \left(U_1^*U_1 + \frac{1}{k^2}U_2^*U_2 \right)^{-1/2},$$

hence

$$\lambda(k) = \|N^*(k)U(k)\| = \left\| \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} N^* \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \left(U_1^*U_1 + \frac{1}{k^2}U_2^*U_2 \right)^{-1/2} \right\|.$$

Denote

$$\begin{aligned}
 F(k) &= N^*(k)U(k)[N^*(k)U(k)]^* \\
 &= \left\{ \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} N^* \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \left(U_1^*U_1 + \frac{1}{k^2}U_2^*U_2 \right)^{-1} \begin{bmatrix} U_1^* & U_2^* \end{bmatrix} N \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} \right\}.
 \end{aligned}$$

Then $\lambda^2(k) = \lambda_1[F(k)]$.

It is clear that $\lambda^2(k)$ has a local infimum λ_0^2 at k_0 . The following is to construct a desired

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

satisfying $\|\Delta\| = \lambda_0^{-1}$ and

$$\det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0$$

by applying this property.

By Lemma 2, there exists a vector

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

such that $F(k_0)z = \lambda^2(k_0)z$ and $z^*(dF(k_0)/dk)z = 0$, which implies,

$$\begin{aligned} & [z_1^* \quad z_2^*] \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{k_0^2}I \end{bmatrix} N^* \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \left(U_1^* U_1 + \frac{1}{k_0^2} U_2^* U_2 \right)^{-1} [U_1^* \quad U_2^*] \\ & N \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k_0}I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \frac{2}{k_0^3} [z_1^* \quad z_2^*] \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k_0}I \end{bmatrix} N^* \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \left(U_1^* U_1 + \frac{1}{k_0^2} U_2^* U_2 \right)^{-1} \\ & U_2^* U_2 \left(U_1^* U_1 + \frac{1}{k_0^2} U_2^* U_2 \right)^{-1} [U_1^* \quad U_2^*] N \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k_0}I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ & + [z_1^* \quad z_2^*] \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k_0}I \end{bmatrix} N^* \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \left(U_1^* U_1 + \frac{1}{k_0^2} U_2^* U_2 \right)^{-1} [U_1^* \quad U_2^*] \\ & N \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{k_0^2}I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0. \end{aligned}$$

Let

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_0^{-1} \begin{bmatrix} U_1 \\ \frac{1}{k_0}U_2 \end{bmatrix} \left(U_1^* U_1 + \frac{1}{k_0^2} U_2^* U_2 \right)^{-1} [U_1^* \quad U_2^*] N \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k_0}I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Then the previous equality is equivalent to

$$\begin{aligned} & -\frac{\lambda_0^2}{k_0} [z_1^* z_2^*] \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \frac{2\lambda_0^2}{k_0} x_2^* x_2 - \frac{\lambda_0^2}{k_0} [z_1^* z_2^*] \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ & = \frac{2\lambda_0^2}{k_0} (x_2^* x_2 - z_2^* z_2) = 0, \end{aligned}$$

notice that

$$\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{k^2}I \end{bmatrix} = -\frac{1}{k} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} = -\frac{1}{k} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

and $F(k_0)z = \lambda_0^2 z$. Thus $z_2^* z_2 = x_2^* x_2$. It is easy to verify that $x_1^* x_1 + x_2^* x_2 = z_1^* z_1 + z_2^* z_2$. Therefore $z_1^* z_1 = x_1^* x_1$.

Let

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix},$$

where

$$\Delta_i = \begin{cases} -\lambda_0^{-1} x_i z_i^* / (z_i^* z_i) & \text{if } z_i \neq 0, \\ 0 & \text{if } z_i = 0. \end{cases}$$

Then $\|\Delta\| = \lambda_0^{-1}$. The following is to show that

$$\det \left\{ I + \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} M \right\} = 0.$$

Since

$$S \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0,$$

we get

$$S(k_0) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \tag{15}$$

Furthermore,

$$\begin{aligned} & N^*(k_0) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \lambda_0^{-1} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k_0} I \end{bmatrix} N^* \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \left(U_1^* U_1 + \frac{1}{k_0^2} U_2^* U_2 \right)^{-1} [U_1^* \quad U_2^*] N \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k_0} I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \lambda_0^{-1} F(k_0) z = \lambda_0 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \end{aligned} \tag{16}$$

Denote

$$\Delta_a = \begin{bmatrix} 0 & \Delta \\ \Delta^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \Delta_1 & 0 \\ 0 & 0 & 0 & \Delta_2 \\ \Delta_1^* & 0 & 0 & 0 \\ 0 & \Delta_2^* & 0 & 0 \end{bmatrix},$$

then from (15) and (16),

$$\begin{aligned} [I + \Delta_a P(1, k_0)] \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} &= \left\{ I + \Delta_a \begin{bmatrix} S(k_0) & N(k_0) \\ N^*(k_0) & R(k_0) \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \Delta_a \begin{bmatrix} 0 \\ 0 \\ \lambda_0 z_1 \\ \lambda_0 z_2 \end{bmatrix} = 0. \end{aligned}$$

Hence $\det(I + \Delta_a M) = \det[I + \Delta_a P(1, k_0)] = 0$. Which implies $\psi_K(M) \leq \lambda_0^{-1}$.

Case 3: $\lambda_0 = \lim_{\alpha \rightarrow \infty} \lambda_1[P(\alpha, \beta_0)]$, $\beta_0 \in (0, \infty)$.

Let

$$J = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \Delta_b = J^T \Delta_a J = \begin{bmatrix} 0 & \Delta_1 & 0 & 0 \\ \Delta_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_2 \\ 0 & 0 & \Delta_2^* & 0 \end{bmatrix}.$$

Partition S, R, N according to the dimensions of Δ_1 and Δ_2 , we get

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2^* & S_3 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix}.$$

Denote

$$P_2(\alpha, \beta) = J^T P(\alpha, \beta) J = \begin{bmatrix} \alpha^2 S_1 & N_1 & \alpha \beta S_2 & \frac{\alpha}{\beta} N_2 \\ N_1^* & \frac{1}{\alpha^2} R_1 & \frac{\beta}{\alpha} N_3^* & \frac{1}{\alpha \beta} R_2 \\ \alpha \beta S_2^* & \frac{\beta}{\alpha} N_3 & \beta^2 S_3 & N_4 \\ \frac{\alpha}{\beta} N_2^* & \frac{1}{\alpha \beta} R_2^* & N_4^* & \frac{1}{\beta^2} R_3 \end{bmatrix},$$

then

$$\begin{aligned} \det(I + \Delta_a M) &= \det[I + \Delta_a P(\alpha, \beta)] \\ &= \det[I + \Delta_b P_2(\alpha, \beta)] \end{aligned}$$

and $\lambda_1[P_2(\alpha, \beta)] = \lambda_1[P(\alpha, \beta)]$. Since $\lim_{\alpha \rightarrow \infty} \lambda_1[P_2(\alpha, \beta_0)] = \lambda_0 < \infty$, we must have $S_1 \leq 0$. For simplicity, suppose that

$$S_1 = \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

$S_{11} < 0$ (if not so, a unitary similarity transformation can be applied without changing the value of $\psi_K(M)$ and $\lambda_1[P_2(\alpha, \beta)]$), and N_1, S_2, N_2 are partitioned accordingly, such that $P_2(\alpha, \beta)$ takes the form,

$$P_2(\alpha, \beta) = \begin{bmatrix} \alpha^2 S_{11} & 0 & N_{11} & \alpha \beta S_{21} & \frac{\alpha}{\beta} N_{21} \\ 0 & 0 & N_{12} & \alpha \beta S_{22} & \frac{\alpha}{\beta} N_{22} \\ N_{11}^* & N_{12}^* & \frac{1}{\alpha^2} R_1 & \frac{\beta}{\alpha} N_3^* & \frac{1}{\alpha \beta} R_2 \\ \alpha \beta S_{21}^* & \alpha \beta S_{22}^* & \frac{\beta}{\alpha} N_3 & \beta^2 S_3 & N_4 \\ \frac{\alpha}{\beta} N_{21}^* & \frac{\alpha}{\beta} N_{22}^* & \frac{1}{\alpha \beta} R_2^* & N_4^* & \frac{1}{\beta^2} R_3 \end{bmatrix}.$$

We conclude that $S_{22} = 0, N_{22} = 0$. If not so, the largest eigenvalue of the 4×4 blocks at the lower-right corner of $P_2(\alpha, \beta)$ will go to infinity as $\alpha \rightarrow \infty$ for each β . This is a contradiction. So we have,

$$P_2(\alpha, \beta) = \begin{bmatrix} \alpha^2 S_{11} & 0 & N_{11} & \alpha\beta S_{21} & \frac{\alpha}{\beta} N_{21} \\ 0 & 0 & N_{12} & 0 & 0 \\ N_{11}^* & N_{12}^* & \frac{1}{\alpha^2} R_1 & \frac{\beta}{\alpha} N_3^* & \frac{1}{\alpha\beta} R_2 \\ \alpha\beta S_{21}^* & 0 & \frac{\beta}{\alpha} N_3 & \beta^2 S_3 & N_4 \\ \frac{\alpha}{\beta} N_{21}^* & 0 & \frac{1}{\alpha\beta} R_2^* & N_4^* & \frac{1}{\beta^2} R_3 \end{bmatrix}. \tag{17}$$

It follows from Lemma 6 with $\gamma = \alpha^{-1}$ that, as $\alpha \rightarrow \infty$, the finite eigenvalues of $P_2(\alpha, \beta)$ approach the eigenvalues of

$$Q(\beta) := \begin{bmatrix} 0 & N_{12} & 0 & 0 \\ N_{12}^* & 0 & 0 & 0 \\ 0 & 0 & \beta^2(S_3 - S_{21}^* S_{11}^{-1} S_{21}) & N_4 - S_{21}^* S_{11}^{-1} N_{21} \\ 0 & 0 & N_4^* - N_{21}^* S_{11}^{-1} S_{21} & \frac{1}{\beta^2}(R_3 - N_{21}^* S_{11}^{-1} N_{21}) \end{bmatrix}.$$

Denote

$$Q_1(\beta) = \begin{bmatrix} \beta^2(S_3 - S_{21}^* S_{11}^{-1} S_{21}) & N_4 - S_{21}^* S_{11}^{-1} N_{21} \\ N_4^* - N_{21}^* S_{11}^{-1} S_{21} & \frac{1}{\beta^2}(R_3 - N_{21}^* S_{11}^{-1} N_{21}) \end{bmatrix}$$

and $\eta_0 = \inf_{\beta>0} \lambda_1[Q_1(\beta)]$. Since λ_0 is a local infimum of $\lambda_1[P_2(\alpha, \beta)]$, it must be a local infimum of $\lambda_1[Q(\beta)]$. There are two possibilities,

1. $\lambda_0 = \|N_{12}\| \geq \eta_0$,
2. $\lambda_0 = \eta_0 > \|N_{12}\|$, and η_0 is an infimum of $\lambda_1[Q_1(\beta)]$ at β_0 .

In the following, we show $\psi_K(M) \leq \lambda_0^{-1}$ in either case.

1. $\lambda_0 = \|N_{12}\| \geq \eta_0$. Then there exist vectors x, y of unit length such that $N_{12}^* x = \lambda_0 y$. Let

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

where

$$\Delta_1 = - \begin{bmatrix} 0 \\ \lambda_0^{-1} x y^* \end{bmatrix}$$

and $\Delta_2 = 0$, it is easy to verify that $\|\Delta\| = \lambda_0^{-1}$ and

$$\left\{ I + \begin{bmatrix} 0 & \Delta_1 & 0 & 0 \\ \Delta_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_2 \\ 0 & 0 & \Delta_2^* & 0 \end{bmatrix} \begin{bmatrix} S_{11} & 0 & N_{11} & S_{21} & N_{21} \\ 0 & 0 & N_{12} & 0 & 0 \\ N_{11}^* & N_{12}^* & R_1 & N_3^* & R_2 \\ S_{21}^* & 0 & N_3 & S_3 & N_4 \\ N_{21}^* & 0 & R_2^* & N_4^* & R_3 \end{bmatrix} \right\} \begin{bmatrix} 0 \\ x \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

Hence $\det(I + \Delta_a M) = \det(I + \Delta_b J^T M J) = 0$, which shows that $\psi_K(M) \leq \lambda_0^{-1}$.

2. $\lambda_0 = \eta_0 > \|N_{12}\|$, and η_0 is an infimum of $\lambda_1[Q_1(\beta)]$ at β_0 . Since λ_0 is a local infimum of $\lambda_1[Q_1(\beta)]$ at β_0 , it follows from Lemma 2 that there exists a vector

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

of unit length such that:

$$Q_1(\beta_0) \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_0 \begin{bmatrix} x \\ y \end{bmatrix},$$

$$[x^* \ y^*] \frac{dQ_1(\beta_0)}{d\beta} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{2\lambda_0}{\beta_0} (x^*x - y^*y) = 0.$$

(similar to (12)). Hence $x^*x = y^*y$.

For simplicity, assume that λ_0 is a simple eigenvalue of $Q_1(\beta_0)$. Then for α large enough, $\lambda_1[P_2(\alpha, \beta_0)]$ is a simple eigenvalue of $P_2(\alpha, \beta_0)$, with a corresponding analytic eigenvector of unit length. Denote this eigenvector by

$$x(\alpha) = \begin{bmatrix} x_1(\alpha) \\ x_2(\alpha) \\ x_3(\alpha) \\ x_4(\alpha) \\ x_5(\alpha) \end{bmatrix},$$

partitioned in accordance with $P_2(\alpha, \beta)$ in (17). In the following, the limit of $x_i(\alpha)$ as $\alpha \rightarrow \infty$ will be used to construct a desired Δ .

Since $\lambda_0 = \lim_{\alpha \rightarrow \infty} \lambda_1[P_2(\alpha, \beta_0)]$ is a local infimum of $\lambda_1[P_2(\alpha, \beta_0)]$, so there exists a positive number α_1 such that when $\alpha > \alpha_1$, $d\lambda_1[P_2(\alpha, \beta_0)]/d\alpha \leq 0$, which implies

$$x^*(\alpha) \frac{dP_2(\alpha, \beta_0)}{d\alpha} x(\alpha) = \frac{2\lambda_1[P_2(\alpha, \beta_0)]}{\alpha} [x_1^*(\alpha)x_1(\alpha) + x_2^*(\alpha)x_2(\alpha) - x_3^*(\alpha)x_3(\alpha)] \leq 0. \tag{18}$$

From $P_2(\alpha, \beta_0)x(\alpha) = \lambda_1[P_2(\alpha, \beta_0)]x(\alpha)$ and $\lim_{\alpha \rightarrow \infty} \lambda_1[P_2(\alpha, \beta_0)] = \lambda_0$, one can obtain

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \alpha x_1(\alpha) &= -S_{11}^{-1} \left(\beta_0 S_{21} x + \frac{1}{\beta_0} N_{21} y \right) := \bar{x}_1, \\ \lim_{\alpha \rightarrow \infty} \alpha x_2(\alpha) &= N_{12} (\lambda_0^2 I - N_{12}^* N_{12})^{-1} [\beta_0 (N_3^* - N_{11}^* S_{11}^{-1} S_{21}) x \\ &\quad + \frac{1}{\beta_0} (R_2 - N_{11}^* S_{11}^{-1} N_{21}) y] := \bar{x}_2, \end{aligned}$$

$$\lim_{x \rightarrow \infty} \alpha x_3(\alpha) = \lambda_0(\lambda_0^2 I - N_{12}^* N_{12})^{-1} [\beta_0(N_3^* - N_{11}^* S_{11}^{-1} S_{21})x + \frac{1}{\beta_0} (R_2 - N_{11}^* S_{11}^{-1} N_{21})y] := \bar{x}_3,$$

$$\lim_{x \rightarrow \infty} x_4(\alpha) = x,$$

$$\lim_{x \rightarrow \infty} x_5(\alpha) = y.$$

From (18), we have

$$\bar{x}_1^* \bar{x}_1 + \bar{x}_2^* \bar{x}_2 \leq \bar{x}_3^* \bar{x}_3. \tag{19}$$

Let

$$\Delta_1 = -\lambda_0^{-1} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \bar{x}_3^* / (\bar{x}_3^* \bar{x}_3),$$

$\Delta_2 = -\lambda_0^{-1} x y^* / (y^* y)$, then from (19) and $x^* x = y^* y$, we get $\|\Delta_1\| \leq \|\Delta_2\| = \lambda_0^{-1}$. Furthermore, it can be verified that

$$[I + \Delta_b P_2(1, \beta_0)] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ 0 \\ x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ 0 \\ x \\ y \end{bmatrix} + \begin{bmatrix} 0 & \Delta_1 & 0 & 0 \\ \Delta_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_2 \\ 0 & 0 & \Delta_2^* & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \lambda_0 \bar{x}_3 \\ \lambda_0 x \\ \lambda_0 y \end{bmatrix} = 0.$$

Hence $\psi_K(M) \leq \lambda_0^{-1}$.

Case 4: $\lambda_0 = \lim_{\beta, \alpha/\beta \rightarrow \infty} \lambda_1[P(\alpha, \beta)]$.

Since $\alpha/\beta \rightarrow \infty$, $P_2(\alpha, \beta)$ takes the same form as in (17). And as $\alpha \rightarrow \infty$, the finite eigenvalues approach the eigenvalues of $Q(\beta)$. If $\eta_0 \leq \|N_{12}\|$, it can be treated as in case 3. Hence we assume that $\eta_0 > \|N_{12}\|$ and $\lambda_0 = \eta_0 = \lim_{\beta \rightarrow \infty} \lambda_1[Q_1(\beta)]$. By Lemma 5,

$$S_3 - S_{21}^* S_{11}^{-1} S_{21} \leq 0, \quad \lambda_0 = \|(N_4^* - N_{21}^* S_{11}^{-1} S_{21})U\|,$$

where the columns of U form an orthonormal basis of the kernel of $S_3 - S_{21}^* S_{11}^{-1} S_{21}$. For simplicity, assume that $\lim_{\beta, \alpha/\beta \rightarrow \infty} \lambda_2[P_2(\alpha, \beta)] < \lambda_0$. Then for $\beta, \alpha/\beta$ large enough, $\lambda_1[P_2(\alpha, \beta)]$ is a simple eigenvalue of $P_2(\alpha, \beta)$, with a corresponding analytic eigenvector of unit length [8], p. 117. Denote this eigenvector by

$$x(\alpha, \beta) = \begin{bmatrix} x_1(\alpha, \beta) \\ x_2(\alpha, \beta) \\ x_3(\alpha, \beta) \\ x_4(\alpha, \beta) \\ x_5(\alpha, \beta) \end{bmatrix}.$$

Then as in case 3, there exists β_1, δ_1 such that when $\beta > \beta_1, \alpha/\beta > \delta_1, \partial\lambda_1[P_2(\alpha, \beta)]/\partial\alpha \leq 0$, and $\partial\lambda_1[P_2(\alpha, \beta)]/\partial\beta \leq 0$, which implies:

$$x_1^*(\alpha, \beta)x_1(\alpha, \beta) + x_2^*(\alpha, \beta)x_2(\alpha, \beta) - x_3^*(\alpha, \beta)x_3(\alpha, \beta) \leq 0, \tag{20}$$

$$x_4^*(\alpha, \beta)x_4(\alpha, \beta) - x_5^*(\alpha, \beta)x_5(\alpha, \beta) \leq 0. \tag{21}$$

From $P_2(\alpha, \beta)x(\alpha, \beta) = \lambda_1[P_2(\alpha, \beta)]x(\alpha, \beta)$, one can obtain:

$$\lim_{\beta \rightarrow \infty} \frac{\alpha}{\beta} x_1(\alpha, \beta) = -S_{11}^{-1}S_{21}x := \bar{x}_1,$$

$$\lim_{\beta \rightarrow \infty} \frac{\alpha}{\beta} x_2(\alpha, \beta) = N_{12}(\lambda_0^2 I - N_{12}^* N_{12})^{-1}(N_3^* - N_{11}^* S_{11}^{-1} S_{21})x := \bar{x}_2,$$

$$\lim_{\beta \rightarrow \infty} \frac{\alpha}{\beta} x_3(\alpha, \beta) = \lambda_0(\lambda_0^2 I - N_{12}^* N_{12})^{-1}(N_3^* - N_{11}^* S_{11}^{-1} S_{21})x := \bar{x}_3,$$

$$\lim_{\beta \rightarrow \infty} x_4(\alpha, \beta) = x,$$

$$\lim_{\beta \rightarrow \infty} x_5(\alpha, \beta) = y,$$

where x, y satisfy $x^*x + y^*y = 1$ and

$$(S_3 - S_{21}^* S_{11}^{-1} S_{21})x = 0, \quad (N_4^* - N_{21}^* S_{11}^{-1} S_{21})x = \lambda_0 y.$$

Hence $x = Uv$ for some v . Since $\lambda_0 = \|(N_4^* - N_{21}^* S_{11}^{-1} S_{21})U\|$, and $U^*U = I$, we must have $\|x\| \geq \|y\|$. By noticing (21), we get $x^*x = y^*y$.

From (20) we have

$$\bar{x}_1^* \bar{x}_1 + \bar{x}_2^* \bar{x}_2 \leq \bar{x}_3^* \bar{x}_3. \tag{22}$$

Let

$$\Delta_1 = -\lambda_0^{-1} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \bar{x}_3^* / (\bar{x}_3^* \bar{x}_3),$$

$\Delta_2 = -\lambda_0^{-1} xy^* / (y^*y)$, then from (22) and $x^*x = y^*y$, we get $\|\Delta_1\| \leq \|\Delta_2\| = \lambda_0^{-1}$. Furthermore, it can be verified that

$$[I + \Delta_b P_2(1, 1)] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ 0 \\ x \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ 0 \\ x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \Delta_1 & 0 & 0 \\ \Delta_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_2 \\ 0 & 0 & \Delta_2^* & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \lambda_0 \bar{x}_3 \\ 0 \\ \lambda_0 y \end{bmatrix} = 0.$$

Hence $\psi_K(M) \leq \lambda_0^{-1}$. \square

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