On a constrained tangential Hermite-Fejér interpolation problem

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Keywords: Hermite-Fejér interpolation, multirate systems, nest operator, matrix positive completion.

Abstract

In this paper we consider the tangential Hermite-Fejér interpolation problem with a structural constraint which requires the value of the interpolating functions at the origin to belong to a specified nest operator set. Necessary and sufficient conditions for the solvability of this problem are given based on the matrix positive completion. We also present a linear fractional transformation (LFT) characterization of all solutions.

1 Introduction

Analytic function interpolation problems have a very rich history in mathematics and there has been a large literature on this area, see recent books [1, 2, 3]. Many successful approaches have been proposed to solve the analytic function interpolation problem since the theory was first proposed at the beginning of this century. In particular, Sarason [4] encompassed different classical interpolation problems in a representation theorem of operators commuting with special contractions, which was later developed into a general framework built on the commutant lifting theorem [2]. On the other hand, using the realization method from the system theory, Ball et. al. presented another systematic way to deal with the interpolation of rational matrix functions [1]. Recently, Foias et. al. combined the commutant lifting theorem from operator theory and state space method from system theory to provide a unified approach for a more general setup by operator-valued functions with operator arguments [3].

The increasing research interest on analytic function interpolation theory is also partly due to its wide applications in a variety of engineering problems such as in control, circuit theory and digital filter design [5, 6, 7]. The Nevanlinna-Pick interpolation theory was first brought into system theory by Youla and Saito, who gave a circuit theoretical proof of the Pick criterion [8]. In the early stage of the development of $\mathcal{H}_\infty$ control theory, the analytic function interpolation theory played a fundamental role [9, 10, 11, 12]. A detailed review of this connection can be found in [6, 13]. Recently, the analytic function interpolation problems were used extensively in robust model validation and identification [14, 15, 16].

In this paper, we study the tangential Hermite-Fejér interpolation problem with a structural constraint which requires the value of the interpolating functions at the origin to belong to a specified nest operator set. We present the necessary and sufficient solvability conditions and the parameterization of all solutions explicitly. The interpolation and distance problems involving analytic functions with such structural constraints were first discussed in [6], but explicit solutions to the problem considered in this paper were not given there. Our study is motivated by the control and identification of multirate systems [17, 18, 19]. The constrained interpolation problem studied in this paper plays the same role to multirate systems as the unconstrained counterpart plays to single rate systems. The paper is organized as follows. The next section introduces some useful results about nest operators and nest algebra. The interpolation problem considered in this paper is proposed in section III. Section IV addresses the solvability condition and Section V provides the characterization of all solutions. Finally, the paper is concluded in Section VI.

2 Nest Operators and Nest Algebra

Let $\mathcal{U}$ be a finite dimensional Hilbert space. A nest in $\mathcal{U}$, denoted $\{\mathcal{U}_k\}$, is a chain of subspaces in $\mathcal{U}$, including $\{0\}$ and $\mathcal{U}$, with the nonincreasing ordering

$$\mathcal{U} = \mathcal{U}_0 \supseteq \mathcal{U}_1 \supseteq \cdots \supseteq \mathcal{U}_{l-1} \supseteq \mathcal{U}_l = \{0\}.$$

Let $\mathcal{U}$, $\mathcal{Y}$ be both finite dimensional vector spaces. Denote by $\mathcal{L}(\mathcal{U},\mathcal{Y})$ the set of linear operators $\mathcal{U} \to \mathcal{Y}$ and abbreviate it as $\mathcal{L}(\mathcal{U})$ if $\mathcal{U} = \mathcal{Y}$. Assume that $\mathcal{U}$ and $\mathcal{Y}$ are equipped, respectively, with nests $\{\mathcal{U}_k\}$ and $\{\mathcal{Y}_k\}$ which have the same number of subspaces, say, $l+1$ as above. A linear map $T \in \mathcal{L}(\mathcal{U},\mathcal{Y})$ is said to be a nest operator if

$$T\mathcal{U}_k \subseteq \mathcal{Y}_k, k = 0, 1, \ldots, l. \tag{1}$$

It is said to be a strict nest operator if

$$T\mathcal{U}_k \subseteq \mathcal{Y}_{k+1},\ k = 0, 1, \ldots, l-1. \tag{2}$$

Let $\Pi_{\mathcal{U}_k} : \mathcal{U} \to \mathcal{U}_k$ and $\Pi_{\mathcal{Y}_k} : \mathcal{Y} \to \mathcal{Y}_k$ be orthogonal projections. Then the condition (1) is equivalent to

$$(I - \Pi_{\mathcal{Y}_k})T\Pi_{\mathcal{U}_k} = 0,\ k = 0, \dots, l$$

and (2) is equivalent to

$$(I - \Pi_{\mathcal{Y}_{k+1}})T\Pi_{\mathcal{U}_k} = 0,\ k = 0, \dots, l-1.$$

The set of all nest operators (with given nests) is denoted $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ and abbreviated $\mathcal{N}(\{\mathcal{U}_k\})$ if $\{\mathcal{U}_k\} = \{\mathcal{Y}_k\}$; The set of all strict nest operators (with given nests) is denoted $\mathcal{N}_s(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ and abbreviated $\mathcal{N}_s(\{\mathcal{U}_k\})$ if $\{\mathcal{U}_k\} = \{\mathcal{Y}_k\}$. If we decompose the spaces $\mathcal{U}$ and $\mathcal{Y}$ in the following way:

$$\mathcal{U} = (\mathcal{U}_0 \ominus \mathcal{U}_1) \oplus (\mathcal{U}_1 \ominus \mathcal{U}_2) \oplus \cdots \oplus (\mathcal{U}_{l-1} \ominus \mathcal{U}_l) \quad (3)$$
$$\mathcal{Y} = (\mathcal{Y}_0 \ominus \mathcal{Y}_1) \oplus (\mathcal{Y}_1 \ominus \mathcal{Y}_2) \oplus \cdots \oplus (\mathcal{Y}_{l-1} \ominus \mathcal{Y}_l) \quad (4)$$

then a nest operator $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ has the following block lower triangular form

$$T = \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ T_{21} & T_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ T_{l1} & T_{l2} & \cdots & T_{ll} \end{bmatrix}. \quad (5)$$

Next, we list some useful lemmas in [20] without proof for completeness. These lemmas will be used in the later section frequently. For simplicity, we restrict our discussion to finite dimensional spaces although some of them hold in a more general case.

Lemma 1.
1) If $T_1 \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_k\}, \{\mathcal{Z}_k\})$, then $T_2T_1 \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Z}_k\})$.
2) If $T_1 \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ and $T_2 \in \mathcal{N}_s(\{\mathcal{Y}_k\}, \{\mathcal{Z}_k\})$, or if $T_1 \in \mathcal{N}_s(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ and $T_2 \in \mathcal{N}(\{\mathcal{Y}_k\}, \{\mathcal{Z}_k\})$, then $T_2T_1 \in \mathcal{N}_s(\{\mathcal{U}_k\}, \{\mathcal{Z}_k\})$.
3) $\mathcal{N}(\{\mathcal{U}_k\})$ forms an algebra, called a nest algebra.

Lemma 2.
1) If $T \in \mathcal{N}_s(\{\mathcal{U}_k\})$, then $I - T$ is always invertible.
2) If $T \in \mathcal{N}(\{\mathcal{U}_k\})$ and T is always invertible, then $T^{-1} \in \mathcal{N}_s(\{\mathcal{U}_k\})$.

Lemma 3 (Generalized QR Factorization).
Let $T \in \mathcal{L}(\mathcal{U})$.
1) There exist a unitary operator Q_1 on $\mathcal{U}$ and $R_1 \in \mathcal{N}(\{\mathcal{U}_k\})$ such that $T = Q_1R_1$.
2) There exist $R_2 \in \mathcal{N}(\{\mathcal{U}_k\})$ and a unitary operator Q_2 on $\mathcal{U}$ such that $T = R_2Q_2$.

Lemma 4 (Generalized Cholesky Factorization). *Let $T \in \mathcal{L}(\mathcal{U})$ and assume T is self-adjoint and nonnegative.*
1) There exists $C_1 \in \mathcal{N}(\{\mathcal{U}_k\})$ such that $T = C_1^*C_1$.
2) There exists $C_2 \in \mathcal{N}(\{\mathcal{U}_k\})$ such that $T = C_2C_2^*$.

The next two lemmas address the following matrix problem: for a given $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, characterize all $N \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that $\|T + N\| \le 1$. We need more notations. With $\mathcal{U}$ and $\mathcal{Y}$ as before, introduce two more finite dimensional Hilbert spaces $\mathcal{Z}$ and $\mathcal{W}$. A linear operator $T \in \mathcal{L}(\mathcal{U} \oplus \mathcal{Y}, \mathcal{Z} \oplus \mathcal{W})$ is partitioned as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

with $T_{11} \in \mathcal{L}(\mathcal{U}, \mathcal{Z}), T_{21} \in \mathcal{L}(\mathcal{U}, \mathcal{W})$, etc. For nests $\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}, \{\mathcal{Z}_k\}, \{\mathcal{W}_k\}$ in $\mathcal{U}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}$, respectively, all with $l + 1$ subspaces, the nests $\{\mathcal{U}_k \oplus \mathcal{Y}_k\}$ and $\{\mathcal{Z}_k \oplus \mathcal{W}_k\}$ are defined in the obvious way. Hence writing

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathcal{N}(\{\mathcal{U}_k \oplus \mathcal{Y}_k\}, \{\mathcal{Z}_k \oplus \mathcal{W}_k\})$$

means $T_{11} \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Z}_k\}), T_{21} \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{W}_k\})$, etc.

Lemma 5. *Let $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. The following statements are equivalent.*
1) There exists $N \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that $\|T+N\| \le 1$.
2) $\max_k \|(I - \Pi_{\mathcal{Y}_k})T\Pi_{\mathcal{U}_k}\| \le 1$.
3) there exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{N}(\{\mathcal{U}_k\} \oplus \{\mathcal{Y}_k\}, \{\mathcal{Y}_k\} \oplus \{\mathcal{U}_k\})$$

with P_{12} and P_{21} both invertible and $P_{22} \in \mathcal{N}_s(\{\mathcal{Y}_k\}, \{\mathcal{U}_k\})$ such that $\begin{bmatrix} T + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ is unitary.

A way to find P from T was given in [21].

Lemma 6. *Let $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and assume condition 3) in Lemma 5 is satisfied. Then the set of all $N \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that $\|T + N\| \le 1$ is given by*

$$\{N = \mathcal{F}(P, R) : R \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}) \text{ and } \|R\| \le 1\}.$$

3 Problem Statement

Let $\mathcal{X}_i, i = 1, \dots, n$, be finite dimensional Hilbert spaces. Also let $\mathcal{U}$ and $\mathcal{Y}$ be finite dimensional Hilbert spaces equipped with nests $\{\mathcal{U}_k\}$ and $\{\mathcal{Y}_k\}$ respectively. Let $U_{i,j}$ and $Y_{i,j}$, $j = 0, \dots, r_i - 1$, be linear bounded operators from $\mathcal{X}_i$ to $\mathcal{U}$ and from $\mathcal{X}_i$ to $\mathcal{Y}$ respectively. Let λ_i, $i = 1, \dots, n$, be n distinct complex numbers on the open unit disc $\mathbb{D}$. Denote $\mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ the Hardy class of all uniformly bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Denote by $\mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ the set of functions $\hat{G} \in \mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ satisfying $\hat{G}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. The tangential Hermite-Fejér interpolation problem with nest operator constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data λ_i, $U_{i,j}$, and $Y_{i,j}$ is to find (if possible) a function $\hat{G}$ in $\mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that $\|\hat{G}\|_\infty \le 1$, and

$$\sum_{k=0}^{j} \frac{1}{k!} \hat{G}^{(k)}(\lambda_i) U_{i,(j-k)} = Y_{i,j}$$

for $j = 0, \ldots, r_i - 1$ and $i = 1, \ldots, n$.

In particular, if $r_i = 1$ for all i, the problem then becomes the following constrained tangential Nevanlinna-Pick interpolation problem: Given linear operators U_i from $\mathcal{X}_i$ to $\mathcal{U}$, Y_i from $\mathcal{X}_i$ to $\mathcal{Y}$ and n distinct complex numbers λ_i on D, find (if possible) a function $\hat{G}(\lambda)$ in $\mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ satisfying $\|\hat{G}\|_\infty \leq 1$, and $\hat{G}(\lambda_i)U_i = Y_i$ for $i = 1, \ldots, n$.

On the other hand, if we take $n = 1$ and $\lambda_1 = 0$, the problem becomes the following constrained tangential Caratheodory-Fejér interpolation problem: Given linear operators U_j from $\mathcal{X}_1$ to $\mathcal{U}$ and Y_j from $\mathcal{X}_1$ to $\mathcal{Y}$ for $j = 0, \ldots, r-1$, find (if possible) a function $\hat{G}(\lambda) := \sum_{k=0}^{\infty} G_k \lambda^k$ in $\mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ satisfying $\|\hat{G}\|_\infty \leq 1$, and

$$\begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{r-1} \end{bmatrix} = \begin{bmatrix} G_0 & 0 & \cdots & 0 \\ G_1 & G_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ G_{r-1} & G_{r-2} & \cdots & G_0 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_{r-1} \end{bmatrix}.$$

For the constrained tangential Hermite-Fejér interpolation data, denote

$$\begin{aligned} Z_i &= \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}_{r_i \times r_i} \\ Z &= \operatorname{diag}(Z_1, \ldots, Z_n) && (6) \\ U_i &= \begin{bmatrix} U_{i,0} & U_{i,1} & \cdots & U_{i,r_i-1} \end{bmatrix} \\ U &= \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix} && (7) \\ Y_i &= \begin{bmatrix} Y_{i,0} & Y_{i,1} & \cdots & Y_{i,r_i-1} \end{bmatrix} \\ Y &= \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}. && (8) \end{aligned}$$

It is shown that $\hat{G}$ in $\mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ is a solution to the above Hermite-Fejér interpolation problem without nest operator constraint if and only if $\hat{G}$ is a solution to the so-called generalized Nevanlinna-Pick problem, which is defined by operator-valued function with operator arguments for the data Z, U, Y [3].

4 Solvability Conditions

Before giving the necessary and sufficient conditions of the constrained Hermite-Fejér interpolation problem, we need to state a result on matrix positive completion.

The matrix positive completion problem is as follows [22]: Given $B_{ij}, |j-i| \leq q$, satisfying $B_{ij} = B_{ji}^*$, find the remaining matrices $B_{ij}, |j-i| > q$, such that the block matrix $B = [B_{ij}]_{i,j=1}^n$ is positive semi-definite. The matrix positive problem was first proposed by Dym and Gohberg [22], who gave the following result:

Lemma 7. *The matrix positive completion problem has a solution if and only if*

$$\begin{bmatrix} B_{ii} & \cdots & B_{i,i+q} \\ \vdots & & \vdots \\ B_{i+q,i} & \cdots & B_{i+q,i+q} \end{bmatrix} \geq 0 \qquad (9)$$

for $i = 1, \ldots, n-q$.

Reference [23] gave a detailed discussion of such problem and presented an explicit description of the set of all solutions via a linear fractional map of which the coefficients are given in terms of the original data. However, Lemma 7 is enough for us. We are now in a position to give the main result of this section.

Theorem 1. *There exists a solution to the tangential Hermite-Fejér interpolation problem with nest operator constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data λ_i, $U_{i,j}$ and $Y_{i,j}$, where $i = 1, \ldots, n$ and $j = 0, \ldots, r_i - 1$, if and only if*

$$Q - \tilde{Q} + Y^* \Pi_{\mathcal{Y}_k} Y - U^* \Pi_{\mathcal{U}_k} U \geq 0 \qquad (10)$$

for all $k = 1, \ldots l$, where Q and $\tilde{Q}$ are respectively the unique solutions of Lyapunov equations

$$\begin{aligned} Q &= Z^* Q Z + U^* U && (11) \\ \tilde{Q} &= Z^* \tilde{Q} Z + Y^* Y && (12) \end{aligned}$$

where Z, U and Y are defined in (6-8).

Proof. The nest operator constraint on the interpolation function $\hat{G}$ can be viewed as an additional interpolation condition

$$\hat{G}(0) I = T$$

for some $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. Set $\lambda_0 = 0$, $U_0 = I$ and $Y_0 = T$. By the solvability condition of the standard Hermite-Fejér interpolation problem [3], the constrained interpolation problem has a solution if and only if there exists $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that

$$Q_a - \tilde{Q}_a \geq 0 \qquad (13)$$

where Q_a and $\tilde{Q}_a$ satisfy

$$\begin{aligned} Q_a &= \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix}^* Q_a \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix} \\ &\quad + \begin{bmatrix} I \\ U^* \end{bmatrix} \begin{bmatrix} I & U \end{bmatrix} && (14) \\ \tilde{Q}_a &= \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix}^* \tilde{Q}_a \begin{bmatrix} \lambda_0 I & 0 \\ 0 & Z \end{bmatrix} \\ &\quad + \begin{bmatrix} T^* \\ Y^* \end{bmatrix} \begin{bmatrix} T & Y \end{bmatrix}. && (15) \end{aligned}$$

It is easy to see from (14-15) that

$$Q_a = \begin{bmatrix} I & U \\ U^* & Q \end{bmatrix}, \quad \tilde{Q}_a = \begin{bmatrix} T^* T & T^* Y \\ Y^* T & \tilde{Q} \end{bmatrix}.$$

Substitute Q_a and $\tilde{Q}_a$ into the inequality (13), we have

$$\begin{bmatrix} I - T^*T & U - T^*Y \\ U^* - Y^*T & Q - \tilde{Q} \end{bmatrix} \geq 0. \qquad (16)$$

The left-hand side of (16) can be rewritten as

$$\begin{bmatrix} I & U \\ U^* & Q - \tilde{Q} + Y^*Y \end{bmatrix} - \begin{bmatrix} T^* \\ Y^* \end{bmatrix} \begin{bmatrix} T & Y \end{bmatrix}.$$

By Schur complement, (16) is equivalent to

$$\begin{bmatrix} I & U & T^* \\ U^* & Q - \tilde{Q} + Y^*Y & Y^* \\ T & Y & I \end{bmatrix} \geq 0. \qquad (17)$$

If we decompose the spaces as in (3-4), then a nest operator $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ has a block lower triangular form shown in (5). Therefore, the constrained Hermite-Fejér interpolation problem has a solution if and only if (17) holds for a block lower triangular matrix T. This is a matrix positive completion problem. By Lemma 7, there is a block lower triangular T for (17) if and only if

$$\begin{bmatrix} I & \Pi_{\mathcal{U}_k} U & 0 \\ (\Pi_{\mathcal{U}_k} U)^* & Q - \tilde{Q} + Y^*Y & (\Pi_{\mathcal{Y}_k^\perp} Y)^* \\ 0 & \Pi_{\mathcal{Y}_k^\perp} Y & I \end{bmatrix} \geq 0 \qquad (18)$$

for $k = 0, \ldots, l$, where

$$\begin{aligned} \Pi_{\mathcal{U}_k} U &= \begin{bmatrix} \Pi_{\mathcal{U}_k} U_1 & \cdots & \Pi_{\mathcal{U}_k} U_n \end{bmatrix} \\ \Pi_{\mathcal{Y}_k^\perp} Y &= \begin{bmatrix} \Pi_{\mathcal{Y}_k^\perp} Y_1 & \cdots & \Pi_{\mathcal{Y}_k^\perp} Y_n \end{bmatrix}. \end{aligned}$$

Note that

$$Y_i^* Y_j = Y_i^* \Pi_{\mathcal{Y}_k} Y_j + Y_i^* \Pi_{\mathcal{Y}_k^\perp} Y_j.$$

Using Schur complement twice, we can easily show that (18) is equivalent to

$$Q - \tilde{Q} + Y^* \Pi_{\mathcal{Y}_k} Y - U^* \Pi_{\mathcal{U}_k} U \geq 0 \qquad (19)$$

for $k = 0, \ldots, l$. We claim that inequalities (19) when $k = l$ implies the case when $k = 0$. In fact, when $k = 0$, inequality (19) gives

$$Q - \tilde{Q} + Y^*Y - U^*U \geq 0. \qquad (20)$$

Note that the inequality (20) is equivalent to

$$Z^*(Q - \tilde{Q})Z \geq 0.$$

When $k = l$, inequality (19) gives

$$Q - \tilde{Q} \geq 0. \qquad (21)$$

It is obvious that inequality (21) implies (20). $\square$

To verify the conditions in Theorem 1, we need to solve the Lyapunov equations (11-12). However, Q and $\tilde{Q}$ can be given directly from the original data in the case of the constrained Nevanlinna-Pick and Caratheodory-Fejér interpolation problems. We end this section by providing the explicit formula for the two special cases.

Corollary 1. *There exists a solution to the tangential Nevanlinna-Pick interpolation problem with nest operator constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data λ_i, U_i and Y_i, $i = 1, \ldots, n$, if and only if*

$$\left[\frac{U_\alpha^* U_\beta - Y_\alpha^* Y_\beta}{1 - \bar{\lambda}_\alpha \lambda_\beta} - U_\alpha^* \Pi_{\mathcal{U}_k} U_\beta + Y_\alpha^* \Pi_{\mathcal{Y}_k} Y_\beta \right]_{\alpha,\beta=0}^n \geq 0 \qquad (22)$$

for all $k = 1, \ldots, l$.

Proof. ¿From (6), we have Z=diag$(\lambda_1, \ldots, \lambda_n)$. It is easy to check that

$$Q = \left[\frac{U_\alpha^* U_\beta}{1 - \bar{\lambda}_\alpha \lambda_\beta} \right]_{\alpha,\beta=0}^n \text{ and } \tilde{Q} = \left[\frac{Y_\alpha^* Y_\beta}{1 - \bar{\lambda}_\alpha \lambda_\beta} \right]_{\alpha,\beta=0}^n$$

are solutions of the Lyapunov equations (11) and (12) respectively. The results then follows from Theorem 1 directly. $\square$

Corollary 2. *There exists a solution to the tangential Caratheodory-Fejér interpolation problem with nest operator constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data U_j and Y_j, $j = 0, \ldots, r-1$, if and only if*

$$T_U^* \Pi_{\mathcal{U}^{r-1} \oplus \mathcal{U}_k^\perp} T_U - T_Y^* \Pi_{\mathcal{Y}^{r-1} \oplus \mathcal{Y}_k^\perp} T_Y \geq 0 \qquad (23)$$

for all $k = 1, \ldots, l$, where

$$T_U := \begin{bmatrix} U_0 & 0 & \cdots & 0 \\ U_1 & U_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ U_{r-1} & U_{r-2} & \cdots & U_0 \end{bmatrix} \qquad (24)$$

$$T_Y := \begin{bmatrix} Y_0 & 0 & \cdots & 0 \\ Y_1 & Y_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ Y_{r-1} & Y_{r-2} & \cdots & Y_0 \end{bmatrix}. \qquad (25)$$

Proof. It follows from (6-8) that

$$\begin{aligned} Z &= \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \\ U &= \begin{bmatrix} U_0 & U_1 & \cdots & U_{r-1} \end{bmatrix} \\ Y &= \begin{bmatrix} Y_0 & Y_1 & \cdots & Y_{r-1} \end{bmatrix}. \end{aligned}$$

Hence Q can be computed by

$$\begin{aligned} Q &= \sum_{\alpha=0}^{\infty} Z^{*\alpha} U^* U Z^{\alpha} = \sum_{\alpha=0}^{r-1} Z^{*\alpha} U^* U Z^{\alpha} \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ U_0^* \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & U_0 \end{bmatrix} + \cdots + U^* U \\ &= \begin{bmatrix} 0 & 0 & \cdots & U_0^* \\ \vdots & \vdots & \iddots & \vdots \\ 0 & U_0^* & \cdots & U_{r-2}^* \\ U_0^* & U_1^* & \cdots & U_{r-1}^* \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} 0 & \cdots & 0 & U_0 \\ 0 & \cdots & U_0 & U_1 \\ \vdots & \iddots & \vdots & \vdots \\ U_0 & \cdots & U_{r-2} & U_{r-1} \end{bmatrix}. \end{aligned}$$

Similarly, we can get $\tilde{Q}$ as follows

$$\begin{aligned} \tilde{Q} &= \begin{bmatrix} 0 & 0 & \cdots & Y_0^* \\ \vdots & \vdots & \iddots & \vdots \\ 0 & Y_0^* & \cdots & Y_{r-2}^* \\ Y_0^* & Y_1^* & \cdots & Y_{r-1}^* \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} 0 & \cdots & 0 & Y_0 \\ 0 & \cdots & Y_0 & Y_1 \\ \vdots & \iddots & \vdots & \vdots \\ Y_0 & \cdots & Y_{r-2} & Y_{r-1} \end{bmatrix}. \end{aligned}$$

The condition (10) then becomes

$$Q - \tilde{Q} + Y^* \Pi_{\mathcal{Y}_k} Y - U^* \Pi_{\mathcal{U}_k} U \geq 0. \tag{26}$$

By pre- and post-multiplying inequality (26) by

$$\begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix},$$

we obtain another equivalent condition as follows:

$$\begin{aligned} & T_U^* T_U - T_Y^* T_Y \\ & \quad + \begin{bmatrix} Y_{r-1}^* \\ \vdots \\ Y_0^* \end{bmatrix} \Pi_{\mathcal{Y}_k} \begin{bmatrix} Y_{r-1} & \cdots & Y_0 \end{bmatrix} \\ & \quad - \begin{bmatrix} U_{r-1}^* \\ \vdots \\ U_0^* \end{bmatrix} \Pi_{\mathcal{U}_k} \begin{bmatrix} U_{r-1} & \cdots & U_0 \end{bmatrix} \geq 0 \end{aligned}$$

which is exactly (23). □

5 Parameterization of All Solutions

In this section, we will characterize all the solutions $\hat{G}$ for the constrained Hermite-Fejér interpolation problem. Since the characterization for the case without constraint has been given in [3], our strategy in solving this problem is then to choose, if possible, from this characterization all those satisfying the nest operator constraint.

The same notation is used as in section II. Besides, we need more notation. Given an operator K and two operator matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \ \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

the linear fractional transformation associated with P and K is denoted

$$\mathcal{F}(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

and the star product of P and Γ is defined as

$$\begin{aligned} P \star \Gamma &= \left[\begin{matrix} P_{11} + P_{12}\Gamma_{11}(I - P_{22}\Gamma_{11})^{-1} P_{21} \\ \Gamma_{21}(I - P_{22}\Gamma_{11})^{-1} P_{21} \end{matrix} \right. \\ &\qquad \left. \begin{matrix} P_{12}(I - \Gamma_{11} P_{22})^{-1} \Gamma_{12} \\ \Gamma_{21}(I - P_{22}\Gamma_{11})^{-1} P_{22}\Gamma_{12} + \Gamma_{22} \end{matrix} \right]. \end{aligned}$$

Here we assume that the operator manipulations are all compatible. With these definitions, we have

$$\mathcal{F}(P, \mathcal{F}(\Gamma, K)) = \mathcal{F}(P \star \Gamma, K).$$

Recall that an operator valued function Θ is two-sided inner if Θ is an inner function and $\Theta(e^{jw})$ is almost everywhere unitary. Assume that $Q - \tilde{Q} > 0$. Then by Theorem III 7.2 [3], there exist operators C from $\mathcal{U}$ to $\mathcal{X}$, and E on $\mathcal{U}$ such that the state space model $\{Z, C, U, E\}$ is controllable and observable and the transfer function

$$\Theta(\lambda) := E + \lambda U (I - \lambda Z)^{-1} C$$

is two-sided inner in $\mathcal{H}_\infty(\mathcal{U},\mathcal{U})$. Note that if $\Theta(\lambda)$ is a two-sided inner function in $\mathcal{H}_\infty(\mathcal{U},\mathcal{U})$ generated by $\{Z, U\}$, then $\Theta(\lambda)H$ is also a two-sided inner function generated by $\{Z, U\}$ for all unitary operator H on $\mathcal{U}$. It follows from Lemma 3 there is a special $\Theta(\lambda)$ such that $E^* \in \mathcal{N}(\{\mathcal{U}_k\})$. By Lemma 4, there exists $N \in \mathcal{N}(\{\mathcal{Y}_k\})$ and $S \in \mathcal{N}(\{\mathcal{U}_k\})$ satisfying

$$\begin{aligned} N^* N &= [I + Y (Q - \tilde{Q})^{-1} Y]^{-1} \\ S^* S &= [I + C^* \tilde{Q} (Q - \tilde{Q})^{-1} Q C]^{-1}. \end{aligned}$$

Also let M be the operator on $\mathcal{X}$ defined by

$$M = (Q - Z^* \tilde{Q} Z)^{-1} Z^* (Q - \tilde{Q}).$$

It is shown in Proposition V 1.7 [3] that M is stable. Define

$$\begin{aligned} \Phi_{11}(\lambda) &= Y (I - \lambda M)^{-1} (Q - Z^* \tilde{Q} Z)^{-1} U^* \\ \Phi_{12}(\lambda) &= N^{-1} - \lambda Y M (I - \lambda M)^{-1} (Q - \tilde{Q})^{-1} Y^* N^{-1} \\ \Phi_{21}(\lambda) &= S \Theta^*(\lambda) - S^{-1} C^* Q (I - \lambda M)^{-1} \\ &\quad \cdot (Q - \tilde{Q})^{-1} \tilde{Q} C \Theta^*(\lambda) \\ \Phi_{22}(\lambda) &= -\lambda S^{-1} C^* Q (I - \lambda M)^{-1} (Q - \tilde{Q})^{-1} Y^* N^{-1}. \end{aligned}$$

Denote $\Phi(\lambda) = \begin{bmatrix} \Phi_{11}(\lambda) & \Phi_{12}(\lambda) \\ \Phi_{21}(\lambda) & \Phi_{22}(\lambda) \end{bmatrix}$. It is shown in [3] that the set of all $\hat{G}$ solving the unconstrained interpolation problem is given by

$$\hat{G}(\lambda) = \mathcal{F}(\Phi(\lambda), \bar{R}(\lambda))$$

where $\bar{R}(\lambda)$ is a contractive analytic function in $\mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$. Obviously, the set of all solutions to the constrained Hermite-Fejé interpolation problem is

$$\{\hat{G}(\lambda) = \mathcal{F}(\Phi(\lambda), \bar{R}(\lambda)) : \hat{G}(0) \in \mathcal{N}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})\}. \tag{27}$$

It is easy to check that

$$\begin{aligned}
\Phi_{11}(0) &= Y(Q - Z^*\tilde{Q}Z)^{-1}U^* \\
\Phi_{12}(0) &= N^{-1} \\
\Phi_{21}(0) &= SE^* - S^{-1}C^*Q(Q-\tilde{Q})^{-1}\tilde{Q}CE^* \\
&= S[I - S^{-2}C^*Q(Q-\tilde{Q})^{-1}\tilde{Q}C]E^* \\
&= S[I - C^*Q(Q-\tilde{Q})^{-1}\tilde{Q}C]E^* \\
&= S^{-1}E^* \\
\Phi_{22}(0) &= 0.
\end{aligned}$$

Now assume the condition in Theorem 1 is satisfied. Then there is a $\bar{R}(\lambda)$ in $\mathcal{H}_\infty(\mathcal{U}, \mathcal{Y})$ such that

$$\hat{G}(0) = \Phi_{11}(0) + N^{-1}\bar{R}(0)S^{-1}E^* \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}).$$

That is,

$$\| - N\Phi_{11}(0)(S^{-1}E^*)^{-1} + N\hat{G}(0)(S^{-1}E^*)^{-1}\| \leq 1.$$

By Lemma 5, there exists

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathcal{N}(\{\mathcal{U}_k \oplus \mathcal{Y}_k\}, \{\mathcal{Y}_k \oplus \mathcal{U}_k\})$$

with $P_{22} \in \mathcal{N}_s(\{\mathcal{Y}_k\}, \{\mathcal{U}_k\})$ and P_{12} and P_{21} invertible such that

$$B = \begin{bmatrix} -N\Phi_{11}(0)(S^{-1}E^*)^{-1} + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary. Define $\Psi = \Phi \star B$. It is easy to check that $\Psi(0) \in \mathcal{N}(\{\mathcal{U}_k \oplus \mathcal{Y}_k\}, \{\mathcal{Y}_k \oplus \mathcal{U}_k\})$, $\Psi_{12}(0)$ and $\Psi_{21}(0)$ are invertible and $\Psi_{22}(0) \in \mathcal{N}_s(\{\mathcal{Y}_k\}, \{\mathcal{U}_k\})$. By setting $\bar{R} = \mathcal{F}(B, R)$, we have

$$\begin{aligned}
\hat{G} &= \mathcal{F}(\Phi, \bar{R}) = \mathcal{F}(\Phi, \mathcal{F}(B, R)) \\
&= \mathcal{F}(\Phi \star B, R) = \mathcal{F}(\Psi, R).
\end{aligned}$$

Note that $\hat{G}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ if and only if $R(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. Hence the set (27) can be rewritten as

$$\{\hat{G} = \mathcal{F}(\Psi, R) : R \in \mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}), \|R\| \leq 1\}.$$

This gives us the main result of this section:

Theorem 2. *Let λ_i, $U_{i,j}$ and $Y_{i,j}$, $i = 1, \ldots, n$ and $j = 0, \ldots, r_i - 1$, be the data for the constrained tangential Hermite-Fejér interpolation problem defined in section* III. *Also let Z, U, Y be defined in (6-8). Assume that $Q - \tilde{Q} > 0$ and the solvability condition (10) holds. Then the set of all interpolants $\hat{G}$ is given by*

$$\hat{G} = \mathcal{F}(\Psi, R)$$

where $R \in \mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ satisfies $\|R\| \leq 1$.

Proof. ¿From the assumption that $Q - \tilde{Q} > 0$ and (10) holds, we know by Theorem 1 that there exists $\hat{G} \in \mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ with $\|\hat{G}\|_\infty \leq 1$. The result then follows immediately from the above discussion. □

6 Conclusion

In this paper, we study the tangential Hermite-Fejér interpolation problem with a constraint requiring the value of the interpolants at the origin to belong to a specified nest operator set. The necessary and sufficient solvability conditions are given using the result on positive matrix completion. We also present the solvability conditions to two special cases called Nevanlinna-Pick and Carathéodory-Fejér interpolation problems, which are more easily to be used in engineering. Finally, all the solutions are given in terms of linear fractional transformation.

Acknowledgments

The authors are grateful to the Hong Kong Research Grants Council for its support.

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