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On the Computation of the Real Hurwitz-Stability Radius*

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Abstract

Recently Qiu et al. obtained a computationally attractive formula for the evaluation of the real stability radius. This formula involves a global maximization over frequency. Here, for the Hurwitz stability case, we show that the frequency range can be limited to a certain finite interval. Numerical experimentation suggests that this interval is often reasonably small.

1. Introduction

For $k = 1, 2, \ldots$, let $\sigma_k(\cdot)$ denote the kth largest singular value of its matrix argument. The real (structured) Hurwitz-stability radius of a matrix triple $(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{p\times n}$, with A Hurwitz stable, is defined by (see [1])

$$r_{\mathbf{R}}(A,B,C) := \inf_{\Delta\in\mathbf{R}^{m\times p}} \{\sigma_1(\Delta) : A + B\Delta C \text{ is not Hurwitz stable}\}.$$

Recently Qiu et al. [2] obtained a formula allowing efficient computation of $r_{\mathbf{R}}(A, B, C)$. Specifically they showed that

$$r_{\mathbf{R}}(A,B,C)^{-1} = \max_{\omega\in\mathbf{R}^+} \mu_{\mathbf{R}}[C(\mathrm{j}\omega I - A)^{-1}B] \quad (1)$$

where $\mathbf{R}^+ := \{\omega \in \mathbf{R} : \omega \geq 0\}$ and where, for any $M \in \mathbf{C}^{m\times p}$,

$$\mu_{\mathbf{R}}(M) := \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \Re eM & -\gamma\Im mM \\ \gamma^{-1}\Im mM & \Re eM \end{bmatrix}\right). \quad (2)$$

The computation of $\mu_{\mathbf{R}}(M)$ for a given M can be carried out at low computational cost as the univariate function to be minimized is unimodal.

In this note, we obtain lower and upper bounds on the global maximizers in (1), computable at a small cost compared to that of performing the global maximization. Numerical experimentation suggests that these bounds are often reasonably close. Knowledge of such bounds simplifies the task of carrying out the numerical maximization.

2. A Finite Frequency Range

Given $(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{p\times n}$ and an $\omega_0 \in \mathbf{R}^+$ such that $\mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 - A)^{-1}B] \neq 0$, let

$$\Omega := \left\{\omega \in \mathbf{R}^+ : \sigma_1[C(\mathrm{j}\omega I - A)^{-1}B] = \mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 I - A)^{-1}B]\right\}$$

and let

$$\omega_{\mathrm{m}} := \begin{cases} \min\Omega & \text{if } \sigma_1[CA^{-1}B] < \mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 I - A)^{-1}B] \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\omega_{\mathrm{M}} := \max\Omega.$$

Since $\sigma_1[C(\mathrm{j}\omega I - A)^{-1}B]$ is a continuous function of ω and since it vanishes as ω goes to infinity, it follows that

$$\sigma_1[C(\mathrm{j}\omega I - A)^{-1}B] < \mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 I - A)^{-1}B] \quad (3)$$

for all $\omega > \omega_{\mathrm{M}}$. Also, if $\sigma_1[CA^{-1}B] < \mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 I - A)^{-1} B]$, then (3) holds for all $\omega \in [0, \omega_{\mathrm{m}})$. Since, for all ω,

$$\begin{aligned} &\mu_{\mathbf{R}}[C(\mathrm{j}\omega I - A)^{-1}B] \\ &\leq \sigma_2\left(\begin{bmatrix} \Re eC(\mathrm{j}\omega I - A)^{-1}B & -\Im mC(\mathrm{j}\omega I - A)^{-1}B \\ \Im mC(\mathrm{j}\omega I - A)^{-1}B & \Re eC(\mathrm{j}\omega I - A)^{-1}B \end{bmatrix}\right) \\ &= \sigma_1[C(\mathrm{j}\omega I - A)^{-1}B], \end{aligned}$$

it follows that

$$\mu_{\mathbf{R}}[C(\mathrm{j}\omega I - A)^{-1}B] < \mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 I - A)^{-1}B]$$

for all $\omega \in \mathbf{R}^+ \backslash [\omega_{\mathrm{m}}, \omega_{\mathrm{M}}]$. Therefore

$$\max_{\omega\in\mathbf{R}^+} \mu_{\mathbf{R}}[C(\mathrm{j}\omega I - A)^{-1}B] = \max_{\omega\in[\omega_{\mathrm{m}},\omega_M]} \mu_{\mathbf{R}}[C(\mathrm{j}\omega I - A)^{-1}B]. \quad (4)$$

It turns out that ω_{m} and ω_{M} can be computed at low cost, so that (4) leads to a substantial reduction in the cost associated with computation of $r_{\mathbf{R}}(A, B, C)$. The idea is as follows. It is well-known (see, e.g., [3]) that for a given $\sigma > 0$, ω satisfies

$$\sigma_k[C(\mathrm{j}\omega I - A)^{-1}B] = \sigma$$

for some k if and only if $\mathrm{j}\omega$ is an eigenvalue of

$$\begin{bmatrix} A & \sigma^{-2}BB' \\ -C'C & -A' \end{bmatrix}.$$

This shows that Ω is contained in the set consisting of the magnitudes of all imaginary eigenvalues of

$$\begin{bmatrix} A & \{\mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 I - A)^{-1}B]\}^{-2}BB' \\ -C'C & -A' \end{bmatrix}.$$

Furthermore, since

$$\begin{aligned} \sigma_k[C(\mathrm{j}\omega I - A)^{-1}B] &\leq \sigma_1[C(\mathrm{j}\omega I - A)^{-1}B] \\ &< \mu_{\mathbf{R}}[C(\mathrm{j}\omega_0 I - A)^{-1}B] \end{aligned}$$

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for all k and all $\omega \in \mathbf{R}^+ \backslash [\omega_m, \omega_M]$, ω_M is the largest such magnitude and, if nonzero, ω_m is the smallest such magnitude.

On summary, the following theorem is obtained.

Theorem *Let $\omega_0 \in \mathbf{R}^+$ be such that $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1} B] \neq 0$, and let ω_M be the largest magnitude of imaginary eigenvalue of*

$$\begin{bmatrix} A & \{\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]\}^{-2}BB' \\ -C'C & -A' \end{bmatrix}. \quad (5)$$

Further, let ω_m be the smallest such magnitude if $\sigma_1[CA^{-1}B] < \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$, and zero otherwise. Then

$$r_{\mathbf{R}}(A,B,C)^{-1} = \max_{\omega \in [\omega_m, \omega_M]} \mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B].$$

For different ω_0, we will get different ω_m and ω_M. Since $\mu_{\mathbf{R}}(CA^{-1}B) = \sigma_1(CA^{-1}B)$, we can simply take $\omega_0 = 0$ whenever $CA^{-1}B \neq 0$. We can choose several ω_0, compute several corresponding ω_m and ω_M, and keep the smallest interval. We can also adjust ω_m and ω_M as the maximization in (1) progresses. There are extreme cases where $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B] = 0$ for all but finite number of ω points. This occurs, for example, when $m = p = 1$. In such cases, computing the real stability radius using (1) is numerically unstable; extra caution has to be taken.

3. Examples

Example 1: In [2], an example with the following (A, B, C) is examined:

$$A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2190 & 0.9347 \\ 0.0470 & 0.3835 \\ 0.6789 & 0.5194 \\ 0.6793 & 0.8310 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0346 & 0.5297 & 0.0077 & 0.0668 \\ 0.0533 & 0.6711 & 0.3834 & 0.4175 \end{bmatrix}.$$

For this example, we choose $\omega_0 = 0$. Figure 1 shows the plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), the plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and the horizontal line at level $\mu_{\mathbf{R}}(CA^{-1}B)$ (dotted line). The imaginary eigenvalues of (5) are $0, 0, \pm j12.0495$, which correspond to the intersections of the dashed line and the dotted line. Hence, $\omega_M = 12.0495$ (and $\omega_m = 0$). The actual maximizer in (1) is $\omega^* = 1.3000$.

Example 2: In this example, we consider the same A and B matrices as in Example 1 and a different C matrix:

$$C = \begin{bmatrix} -0.6907 & -0.3244 & 0.4510 & 0.4630 \\ 0.6992 & -0.2259 & 0.2691 & 0.6226 \end{bmatrix}.$$

Since $CA^{-1}B = 0$ for this example, we cannot choose $\omega_0 = 0$. Let us choose $\omega_0 = 10$ instead. Figure 2 shows the plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), the plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and the horizontal line at level $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$ with $\omega_0 = 10$ (dotted line). The imaginary eigenvalues of (5) are $\pm j1.3758, \pm j15.2012$, which correspond to the intersections of the dashed line and the dotted line. Hence, $\omega_m = 1.3758$ and $\omega_M = 15.2012$. The actual maximizer in (1) is $\omega^* = 7.1400$.

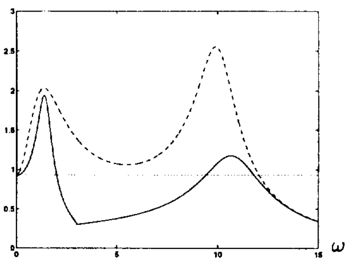

Figure 1: For Example 1, plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and horizontal line at level $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$ with $\omega_0 = 0$ (dotted line).

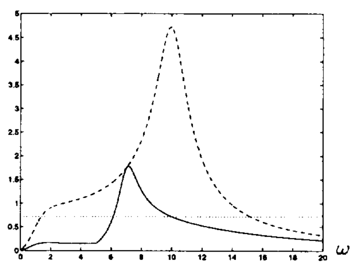

Figure 2: For Example 2, plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and horizontal line at level $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$ with $\omega_0 = 10$ (dotted line).

4. References

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