On the Computation of the Real Hurwitz-Stability Radius

Li Qiu, André L. Tits, and Yaguang Yang

***Abstract*— Recently Qiu *et al.* obtained a computationally attractive formula for the evaluation of the real stability radius. This formula involves a global maximization over frequency. Here, for the Hurwitz stability case, we show that the frequency range can be limited to a certain finite interval. Numerical experimentation suggests that this interval is often reasonably small.**

I. Introduction

For $k = 1, 2, \cdots$, let $\sigma_k(\cdot)$ denote the kth largest singular value of its matrix argument. The real (structured) Hurwitz-stability radius of a matrix triple $(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{p\times n}$, with A Hurwitz stable, is defined by (see [1])

$$r_{\mathbf{R}}(A, B, C) := \inf_{\Delta\in\mathbf{R}^{m\times p}} \{\sigma_1(\Delta): A + B\Delta C \text{ is not Hurwitz stable}\}.$$

Recently Qiu *et al.* [2] obtained a formula allowing efficient computation of $r_{\mathbf{R}}(A, B, C)$. Specifically they showed that

$$r_{\mathbf{R}}(A, B, C)^{-1} = \max_{\omega\in\mathbf{R}^+} \mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B] \quad (1)$$

where $\mathbf{R}^+ := \{\omega \in \mathbf{R}: \omega \geq 0\}$ and where, for any $M \in \mathbf{C}^{m\times p}$

$$\mu_{\mathbf{R}}(M) := \inf_{\gamma\in(0,1]} \sigma_2\left(\begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1}\operatorname{Im} M & \operatorname{Re} M\end{bmatrix}\right). \quad (2)$$

The computation of $\mu_{\mathbf{R}}(M)$ for a given M can be carried out at low computational cost as the univariate function to be minimized is unimodal (see [2]).

In this note, we obtain lower and upper bounds on the global maximizers in (1), computable at a small cost compared to that of performing the global maximization. Numerical experimentation suggests that these bounds are often reasonably close. Knowledge of such bounds simplifies the task of carrying out the numerical maximization.

II. A Finite Frequency Range

Given $(A, B, C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{p\times n}$ and an $\omega_0 \in \mathbf{R}^+$ such that $\mu_{\mathbf{R}}[C(j\omega_0 - A)^{-1}B] \neq 0$, let

$$\Omega := \{\omega \in \mathbf{R}^+: \sigma_1[C(j\omega I - A)^{-1}B] = \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]\}$$

and let

$$\omega_m := \begin{cases} \min\Omega & \text{if } \sigma_1[CA^{-1}B] < \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B] \\ 0 & \text{otherwise}\end{cases}$$

and

$$\omega_M := \max\Omega.$$

Since $\sigma_1[C(j\omega I - A)^{-1}B]$ is a continuous function of ω and since it vanishes as ω goes to infinity, it follows that

$$\sigma_1[C(j\omega I - A)^{-1}B] < \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B] \quad (3)$$

for all $\omega > \omega_M$. Also, if $\sigma_1[CA^{-1}B] < \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$, then (3) holds for all $\omega \in [0, \omega_m)$. Since for all ω

$$\begin{aligned}\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B] &\leq \sigma_2\left(\begin{bmatrix}\operatorname{Re} C(j\omega I - A)^{-1}B & -\operatorname{Im} C(j\omega I - A)^{-1}B \\ \operatorname{Im} C(j\omega I - A)^{-1}B & \operatorname{Re} C(j\omega I - A)^{-1}B\end{bmatrix}\right) \\ &= \sigma_1[C(j\omega I - A)^{-1}B]\end{aligned}$$

it follows that

$$\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B] < \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$$

for all $\omega \in \mathbf{R}^+\backslash[\omega_m, \omega_M]$. Therefore

$$\max_{\omega\in\mathbf{R}^+} \mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B] = \max_{\omega\in[\omega_m, \omega_M]} \mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]. \quad (4)$$

It turns out that ω_m and ω_M can be computed at low cost, so that (4) leads to a substantial reduction in the cost associated with computation of $r_{\mathbf{R}}(A, B, C)$. The idea is as follows. It is well known[1] that for a given $\sigma > 0$, ω satisfies

$$\sigma_k[C(j\omega I - A)^{-1}B] = \sigma$$

for some k if and only if $j\omega$ is an eigenvalue of

$$\begin{bmatrix} A & \sigma^{-2}BB' \\ -C'C & -A'\end{bmatrix}.$$

This shows that Ω is contained in the set consisting of the magnitudes of all imaginary eigenvalues of

$$\begin{bmatrix} A & \{\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]\}^{-2}BB' \\ -C'C & -A'\end{bmatrix}.$$

Furthermore, since

$$\begin{aligned}\sigma_k[C(j\omega I - A)^{-1}B] &\leq \sigma_1[C(j\omega I - A)^{-1}B] \\ &< \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]\end{aligned}$$

for all k and all $\omega \in \mathbf{R}^+\backslash[\omega_m, \omega_M]$, ω_M is the largest such magnitude and, if nonzero, ω_m is the smallest such magnitude.

In summary, the following theorem is obtained.

Theorem: Let $\omega_0 \in \mathbf{R}^+$ be such that $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B] \neq 0$, and let ω_M be the largest magnitude of imaginary eigenvalue of

$$\begin{bmatrix} A & \{\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]\}^{-2}BB' \\ -C'C & -A'\end{bmatrix}. \quad (5)$$

Further, let ω_m be the smallest such magnitude if $\sigma_1[CA^{-1}B] < \mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$, and zero otherwise. Then

$$r_{\mathbf{R}}(A, B, C)^{-1} = \max_{\omega\in[\omega_m, \omega_M]} \mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B].$$

For different ω_0, we will get different ω_m and ω_M. Since $\mu_{\mathbf{R}}(CA^{-1}B) = \sigma_1(CA^{-1}B)$, we can simply take $\omega_0 = 0$ whenever $CA^{-1}B \neq 0$. We can choose several ω_0, compute several corresponding ω_m and ω_M, and keep the smallest interval. We can also adjust ω_m and ω_M as the maximization in (1) progresses. There are extreme cases where $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B] = 0$ for all but a finite number of ω points. This occurs, for example, when $m = p = 1$. In such cases, our result is of no help.

[1] This result has been used by several authors in the computation of the "complex stability radius," or equivalently of the $\mathcal{H}_\infty$ norm (see [3]–[9]).

Manuscript received November 16, 1993; revised September 28, 1994 and February 15, 1995. This research was supported in part by Hong Kong's Research Grants Council Project HKUST552/94E and by NSF's Engineering Research Center NSFD-CDR-88-03012.

L. Qiu is with the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

A. L. Tits and Y. Yang are with the Department of Electrical Engineering and the Institute for Systems Research, University of Maryland at College Park, College Park, MD 20742 USA.

IEEE Log Number 9412925.

0018-9286/95$04.00 © 1995 IEEE

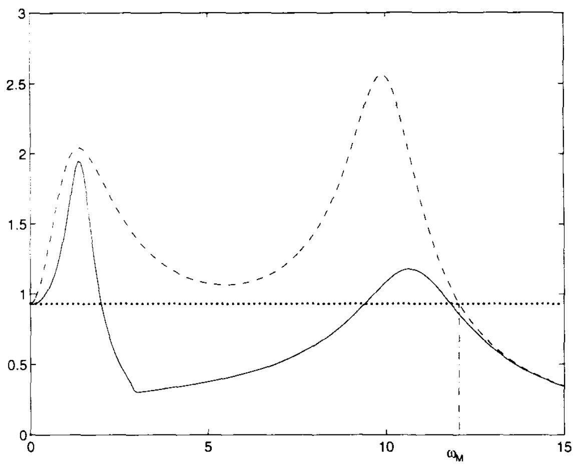

Fig. 1. For Example 1, plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and horizontal line at level $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$ with $\omega_0 = 0$ (dotted line).

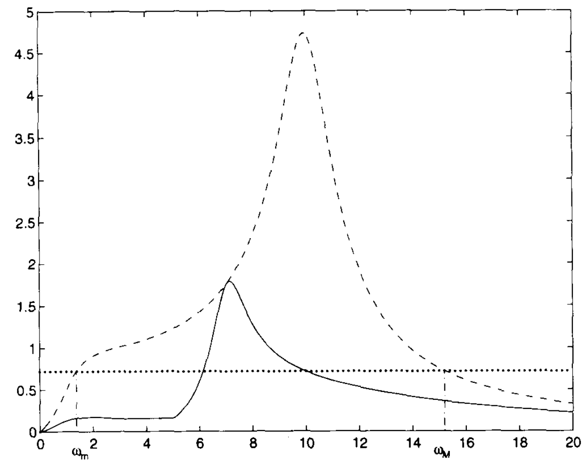

Fig. 2. For Example 2, plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and horizontal line at level $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$ with $\omega_0 = 10$ (dotted line).

III. Examples

Example 1: In [2], an example is examined in which

$$A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2190 & 0.9347 \\ 0.0470 & 0.3835 \\ 0.6789 & 0.5194 \\ 0.6793 & 0.8310 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0346 & 0.5297 & 0.0077 & 0.0668 \\ 0.0533 & 0.6711 & 0.3834 & 0.4175 \end{bmatrix}.$$

For this example, we choose $\omega_0 = 0$. Fig. 1 shows the plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), the plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and the horizontal line at level $\mu_{\mathbf{R}}(CA^{-1}B)$ (dotted line). The imaginary eigenvalues of (4) are $0, 0, \pm j12.0495$, which correspond to the intersections of the dashed line and the dotted line. Hence, $\omega_M = 12.0495$ (and $\omega_m = 0$). The actual maximizer in (1) is $\omega^* = 1.3000$.

Example 2: In this example, we consider the same A and B matrices as in Example 1

$$C = \begin{bmatrix} -0.6907 & -0.3244 & 0.4510 & 0.4630 \\ 0.6992 & -0.2259 & 0.2691 & 0.6226 \end{bmatrix}.$$

Since $CA^{-1}B = 0$ for this example, we cannot choose $\omega_0 = 0$. Let us choose $\omega_0 = 10$ instead. Fig. 2 shows the plot of $\mu_{\mathbf{R}}[C(j\omega I - A)^{-1}B]$ (solid line), the plot of $\sigma_1[C(j\omega I - A)^{-1}B]$ (dashed line), and the horizontal line at level $\mu_{\mathbf{R}}[C(j\omega_0 I - A)^{-1}B]$ with $\omega_0 = 10$ (dotted line). The imaginary eigenvalues of (5) are $\pm j1.3758, \pm j15.2012$, which correspond to the intersections of the dashed line and the dotted line. Hence, $\omega_m = 1.3758$ and $\omega_M = 15.2012$. The actual maximizer in (1) is $\omega^* = 7.1400$.

References

[1] D. Hinrichsen and A. J. Pritchard, "Stability radius for structured perturbations and the algebraic Riccati equation," *Syst. Contr. Lett.*, vol. 8, pp. 105–113, 1985.

[2] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, and J. C. Doyle, "A formula for computation of the real stability radius," *Automatica*, vol. 31, no. 6, pp. 879–890, 1995.

[3] R. Byers, "A bisection method for measuring the distance of a stable matrix to the unstable matrices," *SIAM J. Sci. Stat. Comput.*, vol. 9, pp. 875–881, 1988.

[4] S. Boyd, V. Balakrishnan, and P. Kabamba, "A bisection method for computing the $\mathcal{H}_\infty$-norm of a transfer matrix and related problems," *Math. Contr., Sig., Syst.*, vol. 2, pp. 207–219, 1989.

[5] D. Hinrichsen, B. Kelb, and A. Linnemann, "An algorithm for the computation of the structured stability radius," *Automatica*, vol. 25, pp. 771–775, 1989.

[6] G. Robel, "On computing the infinity norm," *IEEE Trans. Automat. Contr.*, vol. 34, no. 8, pp. 882–884, 1989.

[7] D. J. Clements and K. L. Teo, "Evaluation of the $\mathcal{H}_\infty$-norm," preprint, 1989.

[8] N. A. Bruinsma and M. Steinbuch, "A fast algorithm to compute the $\mathcal{H}_\infty$-norm of a transfer function matrix," *Syst. Contr. Lett.*, vol. 14, no. 5, pp. 287–293, 1990.

[9] S. Boyd and V. Balakrishnan, "A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its $\mathcal{L}_\infty$ norm," *Syst. Contr. Lett.*, vol. 15, pp. 1–7, 1990.

Changing Supply Functions in Input/State Stable Systems

Eduardo Sontag and Andrew Teel

***Abstract*—We consider the problem of characterizing possible supply functions for a given dissipative nonlinear system and provide a result which allows some freedom in the modification of such functions.**

I. Introduction

The "input-to-state stability" (ISS) property has been recently introduced in nonlinear systems analysis [4] and, together with close

Manuscript received January 5, 1995. This work was supported in part by U.S. Air Force Grant F49620-95-1-0101 and NSF Grant ECS-9309523.

E. Sontag is with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903 USA.

A. Teel is with the Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455 USA.

IEEE Log Number 9412920.

0018-9286/95$04.00 © 1995 IEEE