ON THE REAL STRUCTURED STABILITY RADIUS

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Abstract This paper presents a formula for the real structured stability radius with respect to an arbitrary stability region in the complex plane. This formula can be easily computed.

Keywords Stability; robustness; perturbation techniques; matrix analysis; linear systems.

1. INTRODUCTION

In many engineering applications it is required that a square matrix has all of its eigenvalues in a prescribed area in the complex plane. We will use the word stability to describe such an eigenvalue clustering property. Furthermore, it is often desired that the matrix should maintain this stability property when its elements are subject to certain perturbations. The real structured stability radius measures the ability of a matrix to preserve its stability under a certain class of real perturbations.

Let us partition the complex plane $\mathbf{C}$ into two disjoint subset $\mathbf{C}_g$ and $\mathbf{C}_b$, i.e., $\mathbf{C} = \mathbf{C}_g \dot\cup \mathbf{C}_b$, such that $\mathbf{C}_g$ is open. A matrix is said to be stable if its eigenvalues are contained in $\mathbf{C}_g$. Denote the singular values of $M \in \mathbf{C}^{p\times m}$, ordered non-increasingly, by $\sigma_i(M)$, $i = 1, 2, \ldots, \min\{p, m\}$. Also denote $\sigma_1(M)$ by $\overline{\sigma}(M)$. Let F be either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. Following (Hinrichsen and Pritchard, 1986b), we define the *(structured) stability radius* of a matrix triple $(A, B, C) \in \mathsf{F}^{n\times n} \times \mathsf{F}^{n\times m} \times \mathsf{F}^{p\times n}$ as

$$r_{\mathsf{F}}(A,B,C) := \inf\{\overline{\sigma}(\Delta) : \Delta \in \mathsf{F}^{m\times p} \text{ and } A + B\Delta C \text{ is unstable}\}.$$

We abbreviate $r_{\mathsf{F}}(A, I, I)$ by $r_{\mathsf{F}}(A)$ and call it the *(unstructured) stability radius* of A. For real (A, B, C), $r_{\mathbf{R}}(A, B, C)$ is called the *real* stability radius and for complex (A, B, C), $r_{\mathbf{C}}(A, B, C)$ is called the *complex* stability radius. The stability radius problem concerns the computation of $r_{\mathsf{F}}(A, B, C)$ when (A, B, C) is given.

Let $\partial\mathbf{C}_g$ denote the boundary of $\mathbf{C}_g$. By continuity, we can easily show that for stable A,

$$\begin{aligned}
&r_{\mathsf{F}}(A,B,C)\\
&= \inf\{\overline{\sigma}(\Delta) : \Delta \in \mathsf{F}^{m\times p} \text{ and } A + B\Delta C \text{ has an eigenvalue on } \partial\mathbf{C}_g\}\\
&= \inf_{s\in\partial\mathbf{C}_g} \inf\{\overline{\sigma}(\Delta) : \Delta \in \mathsf{F}^{m\times p} \text{ and } \det(sI - A - B\Delta C) = 0\}\\
&= \inf_{s\in\partial\mathbf{C}_g} \inf\{\overline{\sigma}(\Delta) : \Delta \in \mathsf{F}^{m\times p} \text{ and } \det[I - \Delta C(sI - A)^{-1}B] = 0\}.
\end{aligned}$$

Hence the key issue in the computation of the stability radius is to solve the following linear algebra problem: given $M \in \mathbf{C}^{p\times m}$, compute

$$\mu_{\mathsf{F}}(M) := \{\inf\{\overline{\sigma}(\Delta) : \Delta \in \mathsf{F}^{m\times p} \text{ and } \det(I - \Delta M) = 0\}\}^{-1}.$$

By using the fact that $\mu_{\mathbf{C}}(M) = \overline{\sigma}(M)$, (Hinrichsen and Pritchard, 1986b) obtained

$$r_{\mathbf{C}}(A,B,C) = \left\{\sup_{s\in\partial\mathbf{C}_g} \overline{\sigma}[C(sI - A)^{-1}B]\right\}^{-1}. \quad (1)$$

This paper concerns the computation of $r_{\mathbf{R}}$. It is easy to see that $r_{\mathbf{R}}(A, B, C) \geq r_{\mathbf{C}}(A, B, C)$ but the ratio $r_{\mathbf{R}}(A, B, C)/r_{\mathbf{C}}(A, B, C)$ can actually be arbitrarily large. As we have seen,

$$r_{\mathbf{R}}(A,B,C) = \left\{\sup_{s\in\partial\mathbf{C}_g} \mu_{\mathbf{R}}[C(sI - A)^{-1}B]\right\}^{-1}. \quad (2)$$

Our main result is as follows:

Main Result:

$$\mu_{\mathbf{R}}(M) = \inf_{\gamma\in(0,1]} \sigma_2 \begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1}\operatorname{Im} M & \operatorname{Re} M \end{bmatrix}. \quad (3)$$

The function to be minimized is a unimodal function on $(0, 1]$.

The stability radius problem, although not having been called so, has a long history in the mathematical literature. A theorem in (Rudin, 1973, p. 239) states that $r_{\mathbf{F}}(A) \geq \left\{\sup_{s\in\partial\mathbf{C}_g} \overline{\sigma}[(sI-A)^{-1}]\right\}^{-1}$. Various versions of this inequality have appeared in many text books. The fact that this inequality is actually an equality when $\mathbf{F} = \mathbf{C}$ follows from results such as (Golub and Van Loan, 1989, Theorem 2.5.2). For contributions to various aspects of the unstructured stability radius $r_{\mathbf{C}}(A)$, see (Van Loan, 1985; Hinrichsen and Pritchard, 1986a; Martin, 1987; Byers, 1988). The difference between $r_{\mathbf{R}}(A)$ and $r_{\mathbf{C}}(A)$ was first acknowledged in (Van Loan, 1985). With the recognition that some structural information on the perturbation matrix may be available, (Hinrichsen and Pritchard, 1986b) defined both complex and real structured stability radii, proved equality (1), and connected $r_{\mathbf{R}}(A)$ with Riccati equations.

Towards the end of 1980's, attention was focused on the real stability radius. Hinrichsen, Pritchard, and associates studied various properties of the real stability radius and surveyed their results in (Hinrichsen and Pritchard, 1990). Several lower bounds on $r_{\mathbf{R}}(A)$ were obtained in (Qiu and Davison, 1991) by using tensor product techniques. Conditions under which $r_{\mathbf{R}}(A)$ equals $r_{\mathbf{C}}(A)$ were investigated in (Lewkowicz, 1992).

A new lower bound on $r_{\mathbf{R}}(A)$, which is a specialization of the right hand sides of (2) and (3) to the case when $B = C = I$, was announced in (Qiu and Davison, 1992a) and (Qiu and Davison, 1992b). This lower bound was also conjectured to be actually equal to $r_{\mathbf{R}}(A)$. Our current study was sparked by this conjecture. Our main result stated above completely solves the general real structured stability radius problem. In particular, it shows that the conjecture by Qiu and Davison is indeed true.

The paper is organized in the following way. Section 2 gives a proof of the main result. The idea is to rewrite the mixed problem involving a complex matrix and a real constraint into a purely real problem. It is easy to prove inequality "$\leq$" in (3). To prove the opposite inequality, we construct a specific real Δ such that $I - \Delta M$ is singular and $[\overline{\sigma}(\Delta)]^{-1}$ is equal to the right hand side of (3). This will require several special properties of the singular vectors of the matrix on the right hand side of (3). Section 3 addresses the sensitivity of $\mu_{\mathbf{R}}(M)$ and some other computational issues regarding the construction of a smallest Δ and the minimization problem on the right hand side of (3). Section 4 presents several examples which illustrate different possible behaviors of the function on the right hand side of (3) at its minimum and also the extra sweep over $\partial\mathbf{C}_g$ needed for the real stability radius computation. Section 5 is the conclusion, in which we present an interesting generalization of the structured stability radius.

2. PROOF OF THE MAIN RESULT

Let $M \in \mathbf{C}^{p\times m}$ be given. Introduce $X = \operatorname{Re} M$ and $Y = \operatorname{Im} M$. The case when $Y = 0$ is trivial; we then have $\mu_{\mathbf{R}}(M) = \mu_{\mathbf{C}}(M) = \overline{\sigma}(X)$. Hence we assume $Y \neq 0$ in the following proof. For $\Delta \in \mathbf{R}^{m\times p}$ the matrix $I - \Delta M$ is singular if and only if there are $v_1, v_2 \in \mathbf{R}^m$ with $(v_1, v_2) \neq (0,0)$ such that

$$[I - \Delta(X + jY)](v_1 + jv_2) = 0. \tag{4}$$

An equivalent form of (4) is

$$\left(I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right) \begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix} = 0. \tag{5}$$

The advantage of (5) is that only real numbers are involved. Since $(v_1, v_2) \neq (0,0)$, the columns of $\begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix}$ are linearly independent, therefore

$$\operatorname{rank}\left(I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right) \leq 2m - 2. \tag{6}$$

To proceed, we need the following lemma, whose proof is left to the reader.

Lemma 1 *Let $E \in \mathbf{F}^{m\times p}$ and $F \in \mathbf{F}^{p\times m}$. Then for $i = 1, \ldots, \min\{p, m\}$,*

$$\inf\{\overline{\sigma}(E) : \operatorname{rank}(I_m - EF) \leq m - i\} = [\sigma_i(F)]^{-1}.$$

To achieve tightness, we apply Lemma 1 to a diagonally scaled version of (6). Let $\gamma \in \mathbf{R}\setminus\{0\}$. From (6) we get

$$\begin{aligned} &\operatorname{rank}\left(I - \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix}\right) \\ &\quad = \operatorname{rank}\left(I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix}\right) \\ &\quad \leq 2m - 2. \end{aligned} \tag{7}$$

Let us introduce a notation:

$$P(\gamma) = \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix}.$$

Lemma 1 and inequality (7) imply that

$$\overline{\sigma}(\Delta) = \overline{\sigma}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \geq \sigma_2^{-1}[P(\gamma)], \quad \forall \gamma \neq 0.$$

Consequently

$$\mu_{\mathbf{R}}(M) \leq \inf_{\gamma\neq 0} \sigma_2[P(\gamma)] = \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)].$$

Here, the search over γ has been restricted to $(0,1]$ using the fact that $P(\gamma), P(-\gamma)$ and $P(\gamma^{-1})$ all have the same singular values.

The rest of this section is devoted to the proof of the reverse inequality:

$$\mu_{\mathbf{R}}(M) \geq \inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)] =: \sigma^*,$$

which is significantly more difficult. We only need to prove this for the case when $\sigma^* > 0$. The proof is done by an explicit construction of a real Δ such that $I - \Delta M$ is singular and $\overline{\sigma}(\Delta) = \sigma^{*-1}$. Let us use $(\cdot)^\dagger$ to denote the Moore-Penrose generalized inverse. The following lemma, whose proof is left to the reader, is needed in the construction.

Lemma 2 *Suppose* $U \in \mathbf{R}^{p\times 2}$, $V \in \mathbf{R}^{m \times 2}$. *If* $\operatorname{rank} U = \operatorname{rank} \begin{bmatrix} U \\ V \end{bmatrix}$, *then the minimum*

$$\min\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{R}^{m\times p} \text{ and } \Delta U = V\}$$

is attained by $\Delta = VU^\dagger$*; otherwise there exists no* Δ *such that* $\Delta U = V$. *If* $U^T U = V^T V$, *then* $\overline{\sigma}(VU^\dagger) = 1$.

First we will treat the case when $\inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)]$ is attained for some $\gamma^* \in (0,1]$. Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a pair of left and right singular vectors of $P(\gamma^*)$ corresponding to σ^*, with $u_1, u_2 \in \mathbf{R}^p$ and $v_1, v_2 \in \mathbf{R}^m$, and set

$$\Delta = \sigma^{*-1}[v_1 \;\; v_2][u_1 \;\; u_2]^\dagger. \tag{8}$$

If u and v can be chosen so that

$$[u_1 \;\; u_2]^T[u_1 \;\; u_2] = [v_1 \;\; v_2]^T[v_1 \;\; v_2] \in \mathbf{R}^{2\times 2}, \tag{9}$$

then $\operatorname{rank}[u_1 \;\; u_2] = \operatorname{rank}\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$ and it follows from Lemma 2 that $\overline{\sigma}(\Delta) = \sigma^{*-1}$.

$$\begin{aligned}&[I - \Delta(X + jY)](v_1 + j\gamma^* v_2) \\ &= v_1 + j\gamma^* v_2 - \Delta\sigma^* u_1 - \Delta j\gamma^*\sigma^* u_2 = 0\end{aligned}$$

which means that $I - \Delta M$ is singular. Hence Δ given by (8) is the desired construction. What follows is a long elaboration which shows that the singular vectors u and v can always be chosen so that (9) is satisfied when $\gamma^* \in (0,1]$.

The proof for the case when $\inf_{\gamma\in(0,1]}\sigma_2[P(\gamma)]$ is attained only as $\gamma \to 0$, which occurs if and only if $\operatorname{rank}(Y) = 1$, is carried out in a different way, in which an explicit formula for $\mu_{\mathbf{R}}(M)$ (involving no minimization) and a more direct construction of Δ are available.

We start with several claims on the singular vectors of $P(\gamma)$. The first one is of a purely algebraic nature.

Claim 1 *Let* $\gamma \in \mathbf{R} \setminus \{-1, 0, 1\}$ *and let* $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *and* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *be a pair of singular vectors of* $P(\gamma)$ *corresponding to some nonzero singular value* σ. *Then* $u_1^T u_2 = v_1^T v_2$.

Proof The singular vectors satisfy

$$\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \sigma \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} X^T & \gamma^{-1} Y^T \\ -\gamma Y^T & X^T \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sigma \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{11}$$

The difference between $[u_2^T \;\; u_1^T]$ times (10) and $[v_2^T \;\; v_1^T]$ times (11) gives

$$(\gamma + \gamma^{-1})(u_1^T Y v_1 - u_2^T Y v_2) = 2\sigma(u_1^T u_2 - v_1^T v_2). \tag{12}$$

Similarly, the sum of $[u_2^T \;\; -u_1^T]$ times (10) and $[v_2^T \;\; -v_1^T]$ times (11) gives

$$(\gamma - \gamma^{-1})(u_1^T Y v_1 - u_2^T Y v_2) = 0. \tag{13}$$

Since $\sigma \neq 0$ and $\gamma \neq 0$ or ± 1, the claim follows from (12) and (13). □

The second claim concerns the singular vectors of $P(\gamma)$ corresponding to singular values at extrema. We need several lemmas.

Lemma 3 *Let* $F(\gamma) \in \mathbf{R}^{p\times m}$ *be an analytic matrix function on an open set* $\Gamma \subset \mathbf{R}$. *Then there exist an analytic diagonal matrix function* $\widetilde{\Sigma}(\gamma) = \operatorname{diag}(\tilde{\sigma}_1(\gamma), \ldots, \tilde{\sigma}_{\min\{p,m\}}(\gamma)) \in \mathbf{R}^{p\times m}$ *and analytic orthogonal matrix functions* $\widetilde{U}(\gamma) = [\tilde{u}_1(\gamma) \cdots \tilde{u}_p(\gamma)] \in \mathbf{R}^{p\times p}$ *and* $\widetilde{V}(\gamma) = [\tilde{v}_1(\gamma)\cdots\tilde{v}_m(\gamma)] \in \mathbf{R}^{m\times m}$, *all of which are defined on* Γ, *such that*

$$\widetilde{\Sigma}(\gamma) = \widetilde{U}^T(\gamma) F(\gamma) \widetilde{V}(\gamma).$$

Furthermore,

$$\frac{d\tilde{\sigma}_i(\gamma)}{d\gamma} = \tilde{u}_i^T(\gamma)\frac{dF(\gamma)}{d\gamma}\tilde{v}_i(\gamma), \;\; \forall i. \tag{14}$$

Proof The first statement follows from a similar result for symmetric matrices in (Kato, 1966, Section II.6.2). The second follows by differentiation and the orthogonality of $\widetilde{U}$ and $\widetilde{V}$. □

Apparently, $|\tilde{\sigma}_i(\gamma)|$, $i = 1, \ldots, \min\{p, m\}$, are singular values of $F(\gamma)$. However, they are not in any particular order. The kind of analytic singular value decomposition described in Lemma 3 is highly nonunique. Firstly, the order of $\tilde{\sigma}_i(\gamma)$ can be changed as long as the orders of $\tilde{u}_i(\gamma)$ and $\tilde{v}_i(\gamma)$ are changed accordingly. Secondly, $\tilde{\sigma}_i(\gamma)$ can be substituted with $-\tilde{\sigma}_i(\gamma)$ as long as one of $\tilde{u}_i(\gamma)$ or $\tilde{v}_i(\gamma)$ is substituted by $-\tilde{u}_i(\gamma)$ or $-\tilde{v}_i(\gamma)$. For some particular values of γ, we may have $\tilde{\sigma}_i(\gamma) = \tilde{\sigma}_j(\gamma)$

for $i \neq j$. However, if $\tilde{\sigma}_i(\gamma) = \tilde{\sigma}_j(\gamma)$ for all γ in an open interval, then $\tilde{\sigma}_i(\gamma) \equiv \tilde{\sigma}_j(\gamma)$.

In the following, we will also use the ordered singular values $\sigma_1(\gamma) \geq \cdots \geq \sigma_{\min\{p,m\}}(\gamma) \geq 0$ of $F(\gamma)$. The difference between $\tilde{\sigma}_i(\gamma)$ and $\sigma_i(\gamma)$ is that the former are analytic whereas the latter are generally not and the latter are nonnegative and ordered nonincreasingly whereas the former are generally not. Despite its lack of analyticity on the whole Γ, $\sigma_i(\gamma)$ is continuous and piecewise analytic.

Lemma 4 *Let $F(\gamma) \in \mathbf{R}^{p\times m}$ be an analytic matrix function on an open set $\Gamma \subset \mathbf{R}$. Let $\sigma_1(\gamma) \geq \cdots \geq \sigma_{\min\{p,m\}}(\gamma) \geq 0$ be its ordered singular values. If $\sigma_i(\gamma)$ has a nonzero local extremum at $\gamma^* \in \Gamma$, then there exists a pair of left and right singular vectors $u \in \mathbf{R}^p$ and $v \in \mathbf{R}^m$ of $F(\gamma^*)$ corresponding to $\sigma_i(\gamma^*)$ such that $u^T \frac{dF}{d\gamma}(\gamma^*)v = 0$.*

PROOF Without loss of generality, we can assume that $\tilde{\sigma}_j(\gamma^*) \geq 0$ for all $1 \leq j \leq \min\{p,m\}$. Assume first that $\sigma_i(\gamma)$ is equal to some $\tilde{\sigma}_j(\gamma)$ in an open neighborhood of γ^*. Then γ^* is also a stationary point of $\tilde{\sigma}_j(\gamma)$. Let $u_j(\gamma)$ and $v_j(\gamma)$ be the j-th column of $U(\gamma)$ and the j-th column of $V(\gamma)$ respectively. The lemma then follows since (14) gives

$$u_j^T(\gamma^*)\frac{dF}{d\gamma}(\gamma^*)v_j(\gamma^*) = 0.$$

If instead γ^* is one of the nonsmooth points of $\sigma_i(\gamma)$, then in an open neighborhood of γ^*, $\sigma_i(\gamma) = \tilde{\sigma}_{j_1}(\gamma)$ for $\gamma \leq \gamma^*$ and $\sigma_i(\gamma) = \tilde{\sigma}_{j_2}(\gamma)$ for $\gamma \geq \gamma^*$, where $j_1 \neq j_2$. Let $u_{j_1}(\gamma)$, $v_{j_1}(\gamma)$, $u_{j_2}(\gamma)$, and $v_{j_2}(\gamma)$ be the j_1-th column of $U(\gamma)$, the j_1-th column of $V(\gamma)$, the j_2-th column of $U(\gamma)$, and the j_2-th column of $V(\gamma)$ respectively. Then (14) gives

$$\begin{aligned}\frac{d\tilde{\sigma}_{j_1}}{d\gamma}(\gamma^*) &= u_{j_1}^T(\gamma^*)\frac{dF}{d\gamma}(\gamma^*)v_{j_1}(\gamma^*)\\ \frac{d\tilde{\sigma}_{j_2}}{d\gamma}(\gamma^*) &= u_{j_2}^T(\gamma^*)\frac{dF}{d\gamma}(\gamma^*)v_{j_2}(\gamma^*).\end{aligned}$$

Put $u_\alpha = \alpha u_{j_1} + (1-\alpha^2)^{1/2}u_{j_2}$ and $v_\alpha = \alpha v_{j_1} + (1-\alpha^2)^{1/2}v_{j_2}$ for $\alpha \in [0,1]$. Then $u_\alpha(\gamma^*)$ and $v_\alpha(\gamma^*)$ also form a pair of singular vectors of $F(\gamma^*)$ corresponding to the singular value $\sigma_i(\gamma^*)$. Define

$$f(\alpha) = u_\alpha^T(\gamma^*)\frac{dF}{d\gamma}(\gamma^*)v_\alpha(\gamma^*).$$

Since $\sigma_i(\gamma^*)$ is a local extremum, we must have $f(0)f(1) = \frac{d\tilde{\sigma}_{j_1}}{d\gamma}(\gamma^*)\frac{d\tilde{\sigma}_{j_2}}{d\gamma}(\gamma^*) \leq 0$, By continuity, $f(\alpha) = 0$ has a solution in $[0,1]$. This proves the lemma. □

For the matrix $P(\gamma)$, the singular vectors described in Lemma 4 satisfy some pleasant alignment conditions:

Claim 2 *Let $\gamma \in \mathbf{R}\setminus\{0\}$ and let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be a pair of singular vectors of $P(\gamma)$ corresponding to a nonzero singular value σ. If the extra condition $u^T\frac{dP}{d\gamma}(\gamma)v = 0$ is satisfied, then $u_1^Tu_1 = v_1^Tv_1$ and $u_2^Tu_2 = v_2^Tv_2$.*

PROOF The singular vectors satisfy

$$[u_1^T \;\; u_2^T]\begin{bmatrix} 0 & -Y \\ -\gamma^{-2}Y & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \tag{15}$$

$$\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \sigma\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} X^T & \gamma^{-1}Y^T \\ -\gamma Y^T & X^T \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sigma\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{17}$$

Equation (15) gives

$$u_1^TYv_2 + \gamma^{-2}u_2^TYv_1 = 0.$$

Multiplying (16) by $[u_1^T \;\; -u_2^T]$ from the left and (17) by $[v_1^T \;\; -v_2^T]$ from the left, we obtain

$$\begin{aligned}&u_1^TXv_1 - \gamma u_1^TYv_2 - \gamma^{-1}u_2^TYv_1 - u_2^TXv_2\\ &= u_1^TXv_1 - u_2^TXv_2 = \sigma(u_1^Tu_1 - u_2^Tu_2)\end{aligned}$$

and

$$\begin{aligned}&v_1^TX^Tu_1 - \gamma^{-1}v_1^TY^Tu_2 - \gamma v_2^TY^Tu_1 - v_2^TX^Tu_2\\ &= v_1^TX^Tu_1 - v_2^TX^Tu_2 = \sigma(v_1^Tv_1 - v_2^Tv_2).\end{aligned}$$

Since $\sigma > 0$, we get

$$u_1^Tu_1 - u_2^Tu_2 = v_1^Tv_1 - v_2^Tv_2.$$

Claim 2 now follows from $u_1^Tu_1 + u_2^Tu_2 = v_1^Tv_1 + v_2^Tv_2 = 1$. □

We are now ready to show

$$\mu_{\mathbf{R}}(M) \geq \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)] := \sigma^*.$$

We need to treat three different cases separately.

Case 1: $\sigma^* = \sigma_2[P(\gamma^*)]$ for some $\gamma^* \in (0,1)$.

Lemma 4, together with Claims 1 and 2, tells us that a pair of singular vectors $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ of $P(\gamma^*)$ corresponding to σ^* can be chosen to satisfy (9).

Case 2: $\sigma^* = \sigma_2[P(1)]$ but $\sigma^* < \sigma_2[P(\gamma)]$ for all $\gamma \in (0,1)$.

We have to treat this case separately since Claim 1 is not valid for $\gamma = 1$. We however know that the singular values of $P(1)$ are paired so that $\sigma_{2i-1}[P(1)] = \sigma_{2i}[P(1)] = \sigma_i(M)$ for all i. In particular, the largest and the second largest singular values of $P(1)$ are equal to σ^*. We need to consider two possibilities. The first possibility is that the multiplicity of the largest singular value is two. Without loss of generality, assume

$\sigma_1[P(1)] = \sigma_2[P(1)] = \tilde{\sigma}_1(1) = \tilde{\sigma}_2(1)$, where $\tilde{\sigma}_1(\gamma)$ and $\tilde{\sigma}_2(\gamma)$ are analytic singular values of $P(\gamma)$. Note also that if $\sigma(\gamma)$ is a singular value then so is $\sigma(\gamma^{-1})$. Since $\gamma = 1$ is a minimum of $\sigma_2(\gamma)$ which is equal to $\min\{\tilde{\sigma}_1(\gamma), \tilde{\sigma}_2(\gamma)\}$ locally around $\gamma = 1$, it follows that $\gamma = 1$ must be a local minimum of $\tilde{\sigma}_1(\gamma)$ and $\tilde{\sigma}_2(\gamma)$. Let $\begin{bmatrix} u_1(\gamma) \\ u_2(\gamma) \end{bmatrix}$ and $\begin{bmatrix} v_1(\gamma) \\ v_2(\gamma) \end{bmatrix}$ be a pair of analytic singular vectors of $P(\gamma)$ corresponding to $\tilde{\sigma}_2(\gamma)$. By Claim 1 we know that

$$u_1^T(\gamma)u_2(\gamma) = v_1^T(\gamma)v_2(\gamma) \qquad \gamma \neq 0, \pm 1.$$

By continuity, we must therefore have

$$u_1^T(1)u_2(1) = v_1^T(1)v_2(1).$$

Using the fact that $\frac{d\tilde{\sigma}_2}{d\gamma}(1) = 0$, we conclude from derivative relation (14) and Claim 2 that

$$\begin{aligned} u_1^T(1)u_1(1) &= v_1^T(1)v_1(1) \\ u_2^T(1)u_2(1) &= v_2^T(1)v_2(1). \end{aligned}$$

Putting $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1(1) \\ v_2(1) \end{bmatrix}$, it follows that (9) holds.

The second possibility is that the multiplicity of the largest singular value is greater than two. This means that the largest four singular values of $P(1)$ are equal to σ^*, i.e. The two (or more) largest singular values of $M = X + jY$ are equal to σ^*. This possibility is related to a problem considered in (Lewkowicz, 1992), on which our solution is based. Bring in a singular value decomposition

$$M = \sigma^*(\mu_1\nu_1^H + \mu_2\nu_2^H) + \sum_{i=3}^{\min\{p,m\}} \sigma_i(M)\mu_i\nu_i^H,$$

where $(\cdot)^H$ means conjugate transpose. Introduce

$$\begin{bmatrix} \widehat{\mu} & \overline{\mu} \\ \widehat{\nu} & \overline{\nu} \end{bmatrix} := \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix} \begin{bmatrix} z & -1 \\ 1 & \overline{z} \end{bmatrix} \frac{1}{\sqrt{1+|z|^2}} \tag{18}$$

for $z \in \mathbf{C}$. Then

$$M = \sigma^*(\widehat{\mu}\widehat{\nu}^H + \overline{\mu}\overline{\nu}^H) + \sum_{i=3}^{\min\{p,m\}} \sigma_i\mu_i\nu_i^H$$

is also a singular value decomposition. Algebraic manipulation shows

$$\widehat{\mu}^T\widehat{\mu} - \widehat{\nu}^T\widehat{\nu} = \frac{1}{1+|z|^2}q(z),$$

where $q(z)$ is a quadratic polynomial with highest term $(\mu_1^T\mu_1 - \nu_1^T\nu_1)z^2$. If $\mu_1^T\mu_1 - \nu_1^T\nu_1 = 0$, take $\begin{bmatrix} \widehat{\mu} \\ \widehat{\nu} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \nu_1 \end{bmatrix}$; otherwise choose $z \in \mathbf{C}$ such that $q(z) = 0$ and obtain $\begin{bmatrix} \widehat{\mu} \\ \widehat{\nu} \end{bmatrix}$ from (18). Finally set $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\,\widehat{\mu} \\ \mathrm{Im}\,\widehat{\mu} \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\,\widehat{\nu} \\ \mathrm{Im}\,\widehat{\nu} \end{bmatrix}$. Then simple algebra shows that $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ form a pair of left and right singular vectors of $P(1)$ corresponding to σ^* and condition (9), which is equivalent to $\widehat{\mu}^T\widehat{\mu} - \widehat{\nu}^T\widehat{\nu} = 0$, is satisfied.

Case 3: $\sigma^* = \lim_{\gamma\to 0}\sigma_2[P(\gamma)]$ but $\sigma^* < \sigma_2[P(\gamma)]$ for all $\gamma \in (0, 1]$.

We need a lemma in this case.

Lemma 5 *Let $F(\gamma) = G(\gamma) + \gamma^{-1}H \in \mathbf{R}^{p\times m}$, where $G(\gamma)$ is analytic on an open neighborhood Γ of 0 and H is a constant matrix with* $\mathrm{rank}\,(H) =:$ *$r < \min\{p, m\}$. Let $\sigma_1(\gamma) \geq \cdots \geq \sigma_{\min\{p,m\}}(\gamma) \geq 0$ be the ordered singular values of $F(\gamma)$ defined on $\Gamma \setminus \{0\}$. Assume a singular value decomposition of H is given by*

$$H = [U_1\ U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1\ V_2]^T$$

where $\Sigma_1 \in \mathbf{R}^{r\times r}$. Then

$$\lim_{\gamma\to 0}\sigma_{r+i}(\gamma) = \sigma_i[U_2^T G(0) V_2]$$

for $i = 1, \ldots, \min\{p, m\} - r$.

PROOF Without loss of generality, assume an analytic singular value decomposition of $\gamma F(\gamma)$ is

$$\begin{bmatrix} \widetilde{U}_1(\gamma) & \widetilde{U}_2(\gamma) \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_1(\gamma) & 0 \\ 0 & \widetilde{\Sigma}_2(\gamma) \end{bmatrix} \begin{bmatrix} \widetilde{V}_1(\gamma) & \widetilde{V}_2(\gamma) \end{bmatrix}^T$$

where $\widetilde{\Sigma}_1(0) \in \mathbf{R}^{r\times r}$ and $\widetilde{\Sigma}_2(0) = 0$. Then

$$\begin{aligned} \gamma^{-1}\widetilde{\Sigma}_2(\gamma) &= \widetilde{U}_2^T(\gamma)F(\gamma)\widetilde{V}_2(\gamma) \\ &= \widetilde{U}_2^T(\gamma)G(\gamma)\widetilde{V}_2(\gamma) + \gamma^{-1}\widetilde{U}_2^T(\gamma)H\widetilde{V}_2(\gamma), \end{aligned}$$

and

$$H = \begin{bmatrix} \widetilde{U}_1(0) & \widetilde{U}_2(0) \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_1(0) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{V}_1(0) & \widetilde{V}_2(0) \end{bmatrix}^T.$$

Since both $\widetilde{U}_2^T(\gamma)\widetilde{U}_1(0)$ and $\widetilde{V}_1^T(0)\widetilde{V}_2(\gamma)$ are analytic and vanishing at $\gamma = 0$, it follows that

$$\begin{aligned} &\lim_{\gamma\to 0}\gamma^{-1}\widetilde{U}_2^T(\gamma)H\widetilde{V}_2(\gamma) \\ &= \lim_{\gamma\to 0}\gamma^{-1}\widetilde{U}_2^T(\gamma)\widetilde{U}_1(0)\widetilde{\Sigma}_1(0)\widetilde{V}_1^T(0)\widetilde{V}_2(\gamma) = 0. \end{aligned}$$

Therefore

$$\lim_{\gamma\to 0}\gamma^{-1}\widetilde{\Sigma}_2(\gamma) = \widetilde{U}_2^T(0)G(0)\widetilde{V}_2(0).$$

Since $\gamma^{-1}\widetilde{\Sigma}_1(\gamma)$ goes to infinity as $\gamma \to 0$,

$$\begin{aligned} \lim_{\gamma\to 0}\sigma_{r+i}(\gamma) &= \lim_{\gamma\to 0}\sigma_i[\gamma^{-1}\widetilde{\Sigma}_2(\gamma)] \\ &= \sigma_i[\widetilde{U}_2^T(0)G(0)\widetilde{V}_2(0)] = \sigma_i[U_2^T G(0) V_2]. \end{aligned}$$

Notice that $\widetilde{U}_2(0)$ and $\widetilde{V}_2(0)$ can be replaced by U_2 and V_2 since they have the same ranges respectively. □

It follows from e.g. (Horn and Johnson, 1985, Theorem 7.3.9)) that $\sigma_2[P(\gamma)] \geq \sigma_2(\gamma^{-1}Y)$, so Case 3 is relevant only if $\operatorname{rank} Y = 1$. Let

$$G(\gamma) = \begin{bmatrix} X & -\gamma Y \\ 0 & X \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix},$$

and let a singular value decomposition of Y be

$$U^Y \Sigma^Y (V^Y)^T = [U_1^Y \; U_2^Y] \begin{bmatrix} \sigma_1(Y) & 0 \\ 0 & 0 \end{bmatrix} [V_1^Y \; V_2^Y]^T.$$

Then a singular value decomposition of H is

$$H = \begin{bmatrix} 0 & I \\ U^Y & 0 \end{bmatrix} \begin{bmatrix} \Sigma^Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^Y & 0 \\ 0 & I \end{bmatrix}^T.$$

Applying lemma 5, we obtain

$$\begin{aligned} \lim_{\gamma\to 0} \sigma_2[P(\gamma)] &= \overline{\sigma}\left(\begin{bmatrix} 0 & I \\ U_2^Y & 0 \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} V_2^Y & 0 \\ 0 & I \end{bmatrix} \right) \\ &= \max\{\overline{\sigma}[(U_2^Y)^T X], \overline{\sigma}(X V_2^Y)\}. \end{aligned}$$

Now we want to show that $\lim_{\gamma\to 0} \sigma_2[P(\gamma)] = \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)]$. If u and v are a pair of left and right singular vectors of $(U_2^Y)^T X$ corresponding to $\overline{\sigma}[(U_2^Y)^T X]$, then the choice $\Delta = -vu^T U_2^Y / \overline{\sigma}[(U_2^Y)^T X]$ satisfies $[I + \Delta(X+jY)]v = 0$ and $\overline{\sigma}(\Delta)^{-1} \geq \overline{\sigma}[(U_2^Y)^T X]$. Similarly, if u and v are a pair of left and right singular vectors of XV_2^Y corresponding to $\overline{\sigma}(XV_2^Y)$, then the choice $\Delta = -V_2^Y vu^T / \overline{\sigma}(XV_2^Y)$ satisfies $u^T[I + (X+jY)\Delta] = 0$ and $\overline{\sigma}(\Delta)^{-1} \geq \overline{\sigma}(XV_2^Y)$. Together this shows that $\max\{\overline{\sigma}[(U_2^Y)^T X], \overline{\sigma}(XV_2^Y)\} \leq \mu_{\mathbf{R}}(M)$, so

$$\begin{aligned} \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)] &\leq \lim_{\gamma\to 0} \sigma_2[P(\gamma)] \\ &= \max\{\overline{\sigma}[(U_2^Y)^T X], \overline{\sigma}(XV_2^Y)\} \\ &\leq \mu_{\mathbf{R}}(M) \leq \inf_{\gamma\in(0,1]} \sigma_2[P(\gamma)] \end{aligned}$$

and therefore the inequalities above can be replaced by equalities.

Note that if $\min\{p, m\} = 1$ then U_2^Y or V_2^Y will be empty. We define the largest singular value of an empty matrix to be zero.

We have completed the proof of the equality (3). Now suppose that $\sigma_2[P(\gamma)]$ has a local extremum (either minimum or maximum) $\gamma^{**} \in (0,1)$ such that $\sigma_2[P(\gamma^{**})] > \sigma^*$. Then using exactly the same arguments as in Case 1, one can construct a real Δ such that $I - \Delta M$ is singular and $\overline{\sigma}(\Delta) = \{\sigma_2[P(\gamma^{**})]\}^{-1} < \sigma^{*-1}$. This contradicts (3) and, therefore, can not happen. This shows that $\sigma_2[P(\gamma)]$ is a unimodal function on $(0,1]$.

To recap, we summarize what we have proved in this section in the following theorem:

Theorem 1 *If $X, Y \in \mathbf{R}^{p\times m}$, then*

$$\mu_{\mathbf{R}}(X+jY) = \inf_{\gamma\in(0,1]} \sigma_2 \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X \end{bmatrix} \qquad (19)$$

and the function to be minimized is a unimodal function on $(0,1]$. If $\operatorname{rank} Y = 1$ then

$$\begin{aligned} \mu_{\mathbf{R}}(X+jY) &= \lim_{\gamma\to 0} \sigma_2 \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X \end{bmatrix} \\ &= \max\{\overline{\sigma}[(U_2^Y)^T X], \overline{\sigma}(XV_2^Y)\}. \end{aligned}$$

where U_2^Y and V_2^Y come from any singular value decomposition of Y

$$Y = [U_1^Y \; U_2^Y] \begin{bmatrix} \sigma_1(Y) & 0 \\ 0 & 0 \end{bmatrix} [V_1^Y \; V_2^Y]^T.$$

3. COMPUTATIONAL ISSUES

In computations of the real stability radius it is of interest to know how sensitive $\mu_{\mathbf{R}}(M)$ is to disturbances in M. A simple bound on the relative error is given by the following easily proved result.

Proposition 1 *If $\operatorname{rank} \operatorname{Im} M > 1$, then for all $E \in \mathbf{C}^{p\times m}$ with $\overline{\sigma}(E) < \sigma_2(\operatorname{Im} M)$,*

$$\frac{|\mu_{\mathbf{R}}(M+E) - \mu_{\mathbf{R}}(M)|}{\mu_{\mathbf{R}}(M)} \leq \frac{\overline{\sigma}(E)}{\sigma_2(\operatorname{Im} M) - \overline{\sigma}(E)}.$$

The second computational issue concerns the numerical construction of a real Δ with $I - \Delta M$ singular and $\overline{\sigma}(\Delta) = [\mu_{\mathbf{R}}(M)]^{-1}$. The case when $\operatorname{Im} M = 0$ is trivial. For the case when $\operatorname{rank}(\operatorname{Im} M) = 1$, Section 2 gave an explicit construction. Hence, we only need to deal with the case when $\operatorname{rank}(\operatorname{Im} M) > 1$. In this case, there exist $\gamma^* \in (0,1]$ such that $\min_{\gamma\in(0,1]} \sigma_2[P(\gamma)] = \sigma_2[P(\gamma^*)]$.

To put this problem in a more general perspective, let us consider the setup in Lemma 4 and develop a computational procedure to find singular vectors u and v of $F(\gamma^*)$ corresponding to the extreme singular value $\sigma_i(\gamma^*)$ which satisfy $u^T \frac{dF}{d\gamma}(\gamma^*) v = 0$. From the proof of Lemma 4, we see that this is a problem only when γ^* is a nonsmooth point of $\sigma_i(\gamma)$. In this case, the multiplicity of the singular value $\sigma_i(\gamma^*)$, denoted by r, is greater than 1. A singular value decomposition of $F(\gamma^*)$ gives $U \in \mathbf{R}^{m\times r}$ and $V \in \mathbf{R}^{m\times r}$ such that

$$U^T F(\gamma^*) V = \sigma_i(\gamma^*) I_r.$$

Then Lemma 4 implies that there exists a unit length $w \in \mathbf{R}^r$ such that

$$w^T U^T \frac{dF}{d\gamma}(\gamma^*) V w = 0. \qquad (20)$$

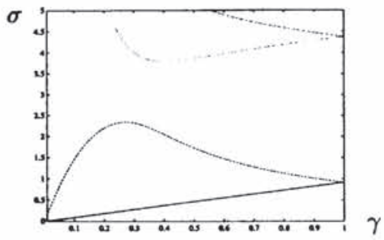

Figure 1 Singular values of $P(\gamma)$ for Example 1. Note that $\sigma_2[P(\gamma)]$ is smooth at its minimum.

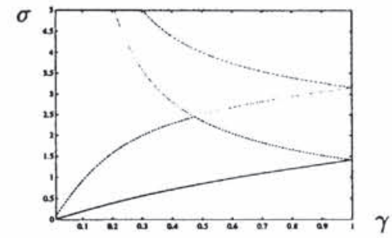

Figure 2 Singular values of $P(\gamma)$ for Example 2. Note that $\sigma_2[P(\gamma)]$ is nonsmooth at its minimum.

Let S be the symmetric part of $U^T \frac{dF}{d\gamma}(\gamma^*)V$. Then (20) is equivalent to $w^T S w = 0$. Carry out the Schur decomposition

$$S = W \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)\, W^T,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Then the existence of w implies $\lambda_1 \geq 0 \geq \lambda_r$. The case when $\lambda_1 = \lambda_r = 0$ is trivial; we can choose w arbitrarily. If $\lambda_1 \neq \lambda_r$, take

$$w = W[\,\sqrt{-\lambda_r}\ 0\ \cdots\ 0\ \sqrt{\lambda_1}\,]^T / \sqrt{\lambda_1 - \lambda_r}.$$

Then it is easy to check that $\|w\| = 1$ and $w^T S w = 0$. Therefore, a pair of desired u and v is given by $u = Uw$ and $v = Vw$ respectively.

The third computational issue concerns the minimization problem on the left hand side of (3). If $\operatorname{rank}(\operatorname{Im} M) \leq 1$, no search is needed. If $\operatorname{rank}(\operatorname{Im} M) > 1$, the infimum is actually a minimum. Since the function to be minimized is unimodal, any local minimum is a global minimum. Many standard search algorithms, such as golden section search, can be used with guaranteed convergence to the global minimum.

4. EXAMPLES

Example 1: Find $\mu_{\mathbb{R}}(M)$ for

$$M = \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix} + j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The singular values of $P(\gamma)$ are plotted in Fig. 1. We get $\gamma^* = 0.3996$ and $\mu_{\mathbb{R}}(M) = \sigma^* = 3.8042$. The singular vectors corresponding to σ^* are

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -0.6986 \\ 0.1095 \\ 0.4604 \\ -0.5367 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -0.4604 \\ -0.5367 \\ 0.6986 \\ 0.1095 \end{bmatrix}.$$

It is easily checked that (9) is satisfied. A real Δ with $I - \Delta M$ singular and $\overline{\sigma}(\Delta) = 3.8042^{-1}$ is given by

$$\Delta = \sigma^{*-1} [\, v_1\ v_2 \,][\, u_1\ u_2 \,]^\dagger = \begin{bmatrix} 0.1382 & -0.2236 \\ 0.2236 & 0.1382 \end{bmatrix}.$$

Example 2: Find $\mu_{\mathbb{R}}(M)$ for

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The singular values of $P(\gamma)$ are plotted in Fig. 2. We get $\gamma^* = 0.4745$ and $\mu_{\mathbb{R}}(M) = \sigma^* = 2.4495$. Since σ^* is a singular value of $P(\gamma)$ with multiplicity 2, we need to use the method given in Section 3 to obtain a pair of singular vectors satisfying (9). By doing this, we obtain

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -0.3501 \\ -0.6143 \\ 0.7038 \\ -0.0688 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0.0688 \\ -0.7038 \\ 0.6143 \\ 0.3501 \end{bmatrix}.$$

A real Δ with $I - \Delta M$ singular and $\overline{\sigma}(\Delta) = 2.4495^{-1}$ is given by

$$\Delta = \sigma^{*-1} [\, v_1\ v_2 \,][\, u_1\ u_2 \,]^\dagger = \begin{bmatrix} 0.3333 & -0.2357 \\ 0.2357 & 0.3333 \end{bmatrix}.$$

Example 3: Assume $\mathbb{C}_g = \{s \in \mathbb{C} : \operatorname{Re} s < 0\}$. Find $r_{\mathbb{R}}(A)$ for

$$A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix}.$$

We plot $\mu_{\mathbb{C}}[(j\omega I - A)^{-1}]$ and $\mu_{\mathbb{R}}[(j\omega I - A)^{-1}]$, computed by using golden section searchs, in Fig. 3. The maxima are 12.0912 and 6.5011 respectively. These maxima occur at $\omega = 9.9403$ and $\omega = 1.0515$ respectively. We get $r_{\mathbb{C}}(A) = 0.0823$ and $r_{\mathbb{R}}(A) = 0.1538$.

5. CONCLUDING DISCUSSION

This paper presents a formula for computation of the real stability radius. The basic problem is a pure linear algebra problem: Given a complex matrix M, find the smallest real matrix Δ such that $I - \Delta M$ is singular. Our main result reduces this problem to a minimization problem of a scalar unimodal function in a finite interval. Our proof gives a computationally efficient way to

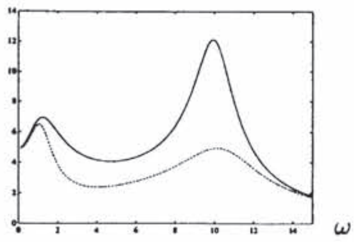

Figure 3 For Example 3, solid line is $\mu_{\mathbf{C}}[(j\omega I - A)^{-1}]$ and dashed line is $\mu_{\mathbf{R}}[(j\omega I - A)^{-1}]$.

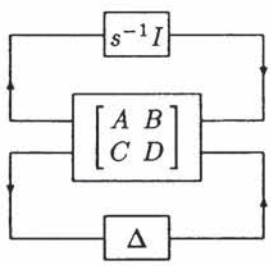

Figure 4 The stability radius generalized to linear fractional transformations

compute the real structured stability radius and a corresponding worst Δ.

For the unstructured stability radius, an alternative formula is available, which might be sometimes simpler to apply.

Corollary 1 *Assume $A \in \mathbf{R}^{n\times n}$ is stable. Then*

$$r_{\mathbf{R}}(A) = \min_{s\in\partial\mathbf{C}_g} \max_{\gamma\in(0,1]} \sigma_{2n-1}\begin{bmatrix} A - \operatorname{Re} sI & -\gamma \operatorname{Im} sI \\ \gamma^{-1}\operatorname{Im} sI & A - \operatorname{Re} sI \end{bmatrix}.$$

For each fixed $s \in \partial\mathbf{C}_g$, the function to be maximized is a quasiconcave function.

We leave it to the reader to derive this from (2) and (3). Note that it is justifiable to change the use of "sup" and "inf" to "max" and "min".

Finally, we notice that the definition of the structured stability radius given in Section 1 is not always general enough. In the stability robustness analysis of the linear feedback system shown in Fig. 4, the matrix whose stability is of concern depends on the perturbation Δ in a linear fractional way. This motivates a more general definition of the structured stability radius. For $(A, B, C, D) \in \mathbf{F}^{n\times n} \times \mathbf{F}^{n\times m} \times \mathbf{F}^{p\times n} \times \mathbf{F}^{p\times m}$, introduce

$$\begin{aligned} &r_{\mathbf{F}}(A,B,C,D) \\ &:= \inf\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{F}^{m\times p}, \det(I - \Delta D) = 0 \\ &\qquad \text{or } A + B(I-\Delta D)^{-1}\Delta C \text{ is unstable}\}. \end{aligned}$$

Then, assuming that $\mathbf{C}_b$ is unbounded, one can prove that for stable A

$$r_{\mathbf{F}}(A,B,C,D) = \left\{\sup_{s\in\partial\mathbf{C}_g} \mu_{\mathbf{F}}[D + C(sI-A)^{-1}B]\right\}^{-1}.$$

In the case when $\mathbf{C}_g = \{s \in \mathbf{C} : \operatorname{Re}(s) < 0\}$ or $\mathbf{C}_g = \{s \in \mathbf{C} : |s| < 1\}$, $r_{\mathbf{F}}(A,B,C,D)$ gives the smallest norm of a complex ($\mathbf{F} = \mathbf{C}$) or real ($\mathbf{F} = \mathbf{R}$) perturbation Δ which destabilizes the feedback system shown in Fig. 1.

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