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On the Robustness of Symmetric Systems*

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Abstract

This note concerns the robustness of LTI symmetric systems under symmetric or diagonal perturbations. It is shown that for symmetric systems the robust stability condition for dynamic perturbations given by the small gain theorem is also necessary if the perturbation matrix is assumed to be diagonal, but the robust stability condition for parametric perturbations given by the real stability radius is no longer necessary if the perturbation matrix is merely assumed to be symmetric.

1 Introduction

Systems with symmetric transfer matrices often occur in real world applications. For example, consider a space structure with collocated sensors and actuators described by

$$M\ddot{q} + D\dot{q} + Kq = Lu, \quad y = L^T q \tag{1}$$

where the mass matrix M, damping matrix D, and stiffness matrix K are all real symmetric matrices. The transfer matrix of this space structure,

$$P(s) = L^T(Ms^2 + Ds + K)^{-1}L,$$

is clearly symmetric, i.e., $P^T = P$. Such systems are called symmetric systems.

In the control of symmetric systems, symmetric controllers offer some advantages and are often used [6, 5]. In this case, the stability and performance robustness analysis of the closed loop system will often become the analysis of the $\mathcal{RH}_\infty$ (real rational matrices bounded in $\mathrm{Re}(s) > 0$) invertibility of $I - \Delta G$, where G is a fixed symmetric $\mathcal{RH}_\infty$ transfer matrix and Δ is an uncertain $\mathcal{RH}_\infty$ transfer matrix, representing dynamic perturbation, or an uncertain real matrix, representing parametric perturbation. For example, assume that a symmetric stabilizing feedback controller C is used to control the space structure in (1); if P is subject to an additive perturbation $P \to P + \Delta$, where Δ is an uncertain $\mathcal{RH}_\infty$ transfer matrix, then $G = C(I - PC)^{-1}$ which is symmetric; or if the only perturbation comes from an uncertain damping matrix $D \to D + \Delta$, where Δ is an uncertain real matrix, then $G(s) = [Ms^2 + Ds + K - LC(s)L^T]^{-1}s$ which is also symmetric. In general, perturbations in the space structure, such as that caused by gyroscopic force and the imprecise collocation of sensor and actuator characteristics, can lead to asymmetric Δ. For computing the stability margins of a system when asymmetric Δ is possible, the following formulas, due to [1] and [7] respectively, are instrumental:

$$\inf\{\|\Delta\|_\infty : \Delta \in \mathcal{RH}_\infty, (I - \Delta G)^{-1} \notin \mathcal{RH}_\infty\} = \|G\|_\infty^{-1}. \tag{2}$$

and

$$\inf\{\sigma_1(\Delta) : \Delta \text{ is a real matrix}, (I - \Delta G)^{-1} \notin \mathcal{RH}_\infty\} = \left\{ \sup_{\omega \in \mathbb{R} \cup \{\infty\}} \mu_{\mathbb{R}}[G(j\omega)] \right\}^{-1} \tag{3}$$

where, for complex matrix A,

$$\mu_{\mathbb{R}}(A) = \inf_{\gamma \in (0,1]} \sigma_2 \left(\begin{bmatrix} \mathrm{Re}\, A & -\gamma \,\mathrm{Im}\, A \\ \gamma^{-1} \mathrm{Im}\, A & \mathrm{Re}\, A \end{bmatrix} \right).$$

Here $\sigma_i(\cdot)$ is used to denote the i-th singular value assuming nonincreasing order.

However, if the only source of uncertainty in the space structure is the dynamics of the structure or the parameters of M, D, K matrices, then Δ will always be a symmetric matrix. This motivates us to study

$$\inf\{\|\Delta\|_\infty : \Delta \in \mathcal{RH}_\infty \text{ is symmetric}, (I - \Delta G)^{-1} \notin \mathcal{RH}_\infty\} \tag{4}$$

and

$$\inf\{\sigma_1(\Delta) : \Delta \text{ is real symmetric}, (I - \Delta G)^{-1} \notin \mathcal{RH}_\infty\}. \tag{5}$$

Clearly, (4) is greater than or equal to (2) and (5) is greater than or equal to (3). Our concerns are if (4) is equal to (2) and if (5) is equal to (3). We will also investigate

$$\inf\{\|\Delta\|_\infty : \Delta \in \mathcal{RH}_\infty \text{ is diagonal}, (I - \Delta G)^{-1} \notin \mathcal{RH}_\infty\} \tag{6}$$

and

$$\inf\{\sigma_1(\Delta) : \Delta \text{ is real diagonal}, (I - \Delta G)^{-1} \notin \mathcal{RH}_\infty\}. \tag{7}$$

(6) and (7) are important in their own right because of their obvious connections to robust stability under structured perturbations [8, 2]. They are studied in our context mainly because of their connection with (4) and (5). Since diagonal matrices are special symmetric matrices, it follows that if (6) is equal to (2), so is (4); and on the other hand if (4) is not equal to (2), neither is (6). A similar argument applies to the relationship between (3), (5), and (7). In Section 2 we show that (6) and hence (4) are equal to (2). In Section 3 we show that (5) and hence (7) can be greater than (3) in general.

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(6) has also been studied in [9, 10]. It was shown in [9] that (6) is equal to (2) under an unnatural assumption on the multiplicity of the singular values of $G(j\omega)$. After the present paper was submitted, [10] was published, which removed the assumption. Hence the result in [10] has priority over the present one; however, the proof in Section 2 is different from that in [10] and is based on the less well-known but very elegant Takagi factorization [4, pp. 204-205].

2 Dynamic Perturbation

Theorem *Let $G \in \mathcal{RH}_\infty$ and $G^T = G$. Then*

$$\inf\{\|\Delta\|_\infty\colon \Delta\in\mathcal{RH}_\infty \text{ is diagonal}, (I-\Delta G)^{-1}\notin\mathcal{RH}_\infty\} = \|G\|_\infty^{-1}.$$

PROOF It follows from (2) immediately that the left side is greater than or equal to the right side. What remains to be shown is the opposite inequality. Assume that G is m by m and that $\|G\|_\infty = \bar{\sigma}[G(j\omega_0)]$ for some $\omega_0 \in [0, \infty]$. Since $G(j\omega_0)$ is a complex symmetric matrix, it has a Takagi factorization

$$G(j\omega_0) = U\Sigma U^T$$

where $U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}$ is unitary and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$. Here $\sigma_1, \sigma_2, \ldots, \sigma_m$ are singular values of $G(j\omega_0)$, ordered nonincreasingly. Now write

$$u_1^T = \begin{bmatrix} u_{11}e^{-j\theta_1} & u_{12}e^{-j\theta_2} & \cdots & u_{1m}e^{-j\theta_m} \end{bmatrix}$$

such that u_{1i} are real and $\theta_i \in [0, \pi)$. Notice that if $\omega_0 = 0$ or ∞, then $\theta_i = 0$. Let $\delta_i = \frac{1}{\sigma_1}\alpha_i$, where α_i is an inner real rational function such that $\alpha_i(j\omega_0) = e^{j2\theta_i}$, and define

$$\Delta = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_m).$$

Then $\|\Delta\|_\infty = 1/\sigma_1 = \|G\|_\infty^{-1}$ and

$$\begin{aligned}[I - \Delta(j\omega_0)G(j\omega_0)]\bar{u}_1 &= [I - \Delta(j\omega_0)U\Sigma U^T]\bar{u}_1 \\ &= \bar{u}_1 - \operatorname{diag}(e^{j2\theta_1}, e^{j2\theta_2}, \ldots, e^{j2\theta_m})u_1 = 0\end{aligned}$$

where $\bar{u}_1$ means the complex conjugate of u_1. This implies that either $I - \Delta G$ is not invertible or $(I - \Delta G)^{-1}$ is not in $\mathcal{RH}_\infty$. □

3 Parametric Perturbation

In this section, we construct an example of symmetric G which gives

$$\inf\{\sigma_1(\Delta)\colon \Delta \text{ is real symmetric}, (I - \Delta G)^{-1}\notin\mathcal{RH}_\infty\} > \left\{\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \mu_{\mathbb{R}}[G(j\omega)]\right\}^{-1}.$$

Consider

$$G(s) = \begin{bmatrix} \frac{-2s(s-1)}{(s+1)^3} & 0 \\ 0 & \frac{-2s(s-1)}{(s+1)^3} \end{bmatrix}.$$

Clearly, G is symmetric. Since G is diagonal with repeated diagonal elements, it follows from [3] that $\mu_{\mathbb{R}}[G(j\omega)] = \sigma_1[G(j\omega)]$. Simple computation shows that the only maximizer of $\sigma_1[G(j\omega)]$ is $\omega = 1$ and $G(j1) = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$. Since the eigenvalues of a real symmetric matrix are always real, there exists no real symmetric matrix such that $\det[I - \Delta G(j1)] = 0$. This implies that

$$\begin{aligned}\inf\{\sigma_1(\Delta)\colon \Delta \in \mathbb{R}^{2\times 2}, \Delta^T{=}\Delta, \det[I{-}\Delta G(j1)] = 0\} \\ = \infty > \mu_{\mathbb{R}}{}^{-1}[G(j1)] = 1.\end{aligned}$$

At $\omega \neq 1$,

$$\begin{aligned}\inf\{\sigma_1(\Delta)\colon \Delta \in \mathbb{R}^{2\times 2}, \Delta^T{=}\Delta, \det[I{-}\Delta G(j\omega)] = 0\} \\ \geq \bar{\sigma}^{-1}[G(j\omega)] > \bar{\sigma}^{-1}[G(j1)] = 1.\end{aligned}$$

Hence

$$\begin{aligned}\inf\{\sigma_1(\Delta)\colon \Delta \in \mathbb{R}^{2\times 2}, \Delta^T{=}\Delta, (I - \Delta G)^{-1}\notin \mathcal{RH}_\infty\} \\ > 1 = \left\{\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \mu_{\mathbb{R}}[G(j\omega)]\right\}^{-1}.\end{aligned}$$

4 References

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